INTRODUCTION TO GRAPH THEORY

SECOND EDITION (2001)

SOLUTION MANUAL

SUMMER 2005 VERSION

MATHEMATICS DEPARTMENT

UNIVERSITY OF ILLINOIS

NOTICE

This is the Summer 2005 version of the Instructor's Solution Manual for *Introduction to Graph Theory*, by Douglas B. West. A few solutions have been added or clarified since last year's version.

Also present is a (slightly edited) annotated syllabus for the one-semester course taught from this book at the University of Illinois.

This version of the Solution Manual contains solutions for 99.4% of the problems in Chapters 1–7 and 93% of the problems in Chapter 8. The author believes that only Problems 4.2.10, 7.1.36, 7.1.37, 7.2.39, 7.2.47, and 7.3.31 in Chapters 1–7 are lacking solutions here. There problems are too long or difficult for this text or use concepts not covered in the text; they will be deleted in the third edition.

The positions of solutions that have not yet been written into the files are occupied by the statements of the corresponding problems. These problems retain the (−), (!), (+), (∗) indicators. Also (•) is added to introduce the statements of problems without other indicators. Thus every problem whose solution is not included is marked by one of the indicators, for ease of identification.

The author hopes that the solutions contained herein will be useful to instructors. The level of detail in solutions varies. Instructors should feel free to write up solutions with more or less detail according to the needs of the class. Please do not leave solutions posted on the web.

Due to time limitations, the solutions have not been proofread or edited as carefully as the text, especially in Chapter 8. Please send corrections to *west@math.uiuc.edu*. The author thanks Fred Galvin in particular for contributing improvements or alternative solutions for many of the problems in the earlier chapters.

This will be the last version of the Solution Manual for the second edition of the text. The third edition will have many new problems, such as those posted at http://www.math.uiuc.edu/ west/igt/newprob.html . The effort to include all solutions will resume for the third edition. Possibly other pedagogical features may also be added later.

Inquiries may be sent to *west@math.uiuc.edu*. Meanwhile, the author apologizes for any inconvenience caused by the absence of some solutions.

Douglas B. West

Mathematics Department - University of Illinois

MATH 412

SYLLABUS FOR INSTRUCTORS

Text: West, *Introduction to Graph Theory,* second edition,
Prentice Hall, 2001.

Many students in this course see graph algorithms repeatedly in courses in computer science. Hence this course aims primarily to improve students' writing of proofs in discrete mathematics while learning about the structure of graphs. Some algorithms are presented along the way, and many of the proofs are constructive. The aspect of algorithms emphasized in CS courses is running time; in a mathematics course in graph theory from this book the algorithmic focus is on proving that the algorithms work.

Math 412 is intended as a rigorous course that challenges students to think. Homework and tests should require proofs, and most of the exercises in the text do so. The material is interesting, accessible, and applicable; most students who stick with the course will give it a fair amount of time and thought.

An important aspect of the course is the clear presentation of solutions, which involves careful writing. Many of the problems in the text have hints, either where the problem is posed or in Appendix C (or both). Producing a solution involves two main steps: finding a proof and properly writing it. It is generally beneficial to the learning process to provide "collaborative study sessions" in which students can discuss homework problems in small groups and an instructor or teaching assistant is available to answer questions and provide direction. Students should then write up clear and complete solutions on their own.

This course works best when students have had prior exposure to writing proofs, as in a "transition" course. Some students may need further explicit discussions of the structure of proofs. Such discussion appear in many texts, such as

D'Angelo and West, *Mathematical Thinking: Problem-Solving and Proofs*;
Eisenberg, *The Mathematical Method: A Transition to Advanced Mathematics*;
Fletcher/Patty, *Foundations of Higher Mathematics*;
Galovich, *Introduction to Mathematical Structures*;
Galovich, *Doing Mathematics: An Introduction to Proofs and Problem Solving*;
Solow, *How to Read and Do Proofs*.

Suggested Schedule

The subject matter for the course is the first seven chapters of the text, skipping most optional material. Modifications to this are discussed below. The 22 sections are allotted an average of slightly under two lectures each.

In the exercises, problems designated by $(-)$ are easier or shorter than most, often good for tests or for "warmup" before doing homework problems. Problems designated by $(+)$ are harder than most. Those designated by (!) are particularly instructive, entertaining, or important. Those designated by $(*)$ make use of optional material.

The semester at the University of Illinois has 43 fifty-minute lectures. The final two lectures are for optional topics, usually chosen by the students from topics in Chapter 8.

Chapter 1	Fundamental Concepts	8
Chapter 2	Trees and Distance	5.5
Chapter 3	Matchings and Factors	5.5
Chapter 4	Connectivity and Paths	6
Chapter 5	Graph Coloring	6
Chapter 6	Planar Graphs	5
Chapter 7	Edges and Cycles	5
*	Total	41

Optional Material

No later material requires material marked optional. The "optional" marking also suggests to students that the final examination will not cover that material.

The optional subsections on Disjoint Spanning Trees (Bridg-It) in Section 2.1 and Stable Matchings in Section 3.2 are always quite popular with the students. The planarity algorithm (without proof) in 6.2 is appealing to students, as is the notion of embedding graphs on the torus through Example 6.3.21. Our course usually includes these four items.

The discussion of f-factors in Section 3.3 is also very instructive and is covered when the class is proceeding on schedule. Other potential additions include the applications of Menger's Theorem at 4.2.24 or 4.2.25.

Other items marked optional generally should not be covered in class.

Additional text items not marked optional that can be skipped when behind schedule:

1.1: 31, 35	1.2: 16, 21–23	1.3: 24, 31–32	1.4: 1, 4, 7, 25–26
2.1: 8, 14–16	2.2: 13–19	2.3: 7–8	3.2: 4
4.1: 4–6	4.2: 20–21	5.1: 11, 22(proof)	5.3: 10–11, 16(proof)
6.1: 18–20, 28	6.3: 9–10, 13–15		7.2: 17

Comments

There are several underlying themes in the course, and mentioning these at appropriate moments helps establish continuity. These include
1) TONCAS (The Obvious Necessary Condition(s) are Also Sufficient).
2) Weak duality in dual maximation and minimization problems.
3) Proof techniques such as the use of extremality, the paradigm for inductive proofs of conditional statements, and the technique of transforming a problem into a previously solved problem.

Students sometimes find it strange that so many exercises concern the Petersen graph. This is not so much because of the importance of the Petersen graph itself, but rather because it is a small graph and yet has complex enough structure to permit many interesting exercises to be asked.

Chapter 1. In recent years, most students enter the course having been exposed to proof techniques, so reviewing these in the first five sections has become less necessary; remarks in class can emphasis techniques as reminders. To minimize confusion, digraphs should not be mentioned until section 1.4; students absorb the additional model more easily after becoming comfortable with the first.

1.1: p3-6 contain motivational examples as an overview of the course; this discussion should not extend past the first day no matter where it ends (the definitions are later repeated where needed). The material on the Petersen graph establishes its basic properties for use in later examples and exercises.

1.2: The definitions of path and cycle are intended to be intuitive; one shouldn't dwell on the heaviness of the notation for walks.

1.3: Although characterization of graphic sequences is a classical topic, some reviewers have questioned its importance. Nevertheless, here is a computation that students enjoy and can perform.

1.4: The examples are presented to motivate the model; these can be skipped to save time. The de Bruijn graph is a classical application. It is desirable to present it, but it takes a while to explain.

Chapter 2.

2.1: Characterization of trees is a good place to ask for input from the class, both in listing properties and in proving equivalence.

2.2: The inductive proof for the Prüfer correspondence seems to be easier for most students to grasp than the full bijective proof; it also illustrates the usual type of induction with trees. Most students in the class have seen determinants, but most have considerable difficulty understanding the proof of the Matrix Tree Theorem; given the time involved, it is best

just to state the result and give an example (the next edition will include a purely inductive proof that uses only determinant expansion, not the Cauchy-Binet Formula). Students find the material on graceful labelings enjoyable and illuminating; it doesn't take long, but also it isn't required. The material on branchings should certaily be skipped in this course.

2.3: Many students have seen rooted trees in computer science and find ordinary trees unnatural; Kruskal's algorithm should clarify the distinction. Many CS courses now cover the algorithms of Kruskal, Dijkstra, and Huffman; here cover Kruskal and perhaps Dijkstra (many students have seen the algorithm but not a proof of correctness), and skip Huffman.

Chapter 3.

3.1: Skip "Dominating Sets", but present the rest of the section.

3.2: Students find the Hungarian algorithm difficult; explicit examples need to be worked along with the theoretical discussion of the equality subgraph. "Stable Matchings" is very popular with students and should be presented unless far behind in schedule. Skip "Faster Bipartite Matching".

3.3: Present all of the subsection on Tutte's 1-factor Theorem. The material on f-factors is intellectually beautiful and leads to one proof of the Erdős-Gallai conditions, but it is not used again in the course and is an "aside". Skip everything on Edmonds' Blossom Algorithm: matching algorithms in general graphs are important algorithmically but would require too much time in this course; this is "recommended reading".

Chapter 4.

4.1: Students have trouble distinguishing "k-connected" from "connectivity k" and "bonds" from "edge cuts". Bonds are dual to cycles in the matroidal sense; there are hints of this in exercises and in Chapter 7, but the full duality cannot be explored before Chapter 8.

4.2: Students find this section a bit difficult. The proof of 4.2.10 is similar to that of 4.2.7, making it omittable, but the application in 4.2.14 is appealing. The details of 4.2.20-21 can be skipped or treated lightly, since the main issue is the local version of Menger's theorem. 4.2.24-25 are appealing applications that can be added; 4.2.5 (CSDR) is a fundamental result but takes a fair amount of effort.

4.3: The material on network flow is quite easy but can take a long time to present due to the overhead of defining new concepts. The basic idea of 4.3.13-15 should be presented without belaboring the details too much. 4.3.16 is a more appealing application that perhaps makes the point more effectively. Skip "Supplies and Demands".

Chapter 5.
5.1: If time is short, the proof of 5.1.22 (Brooks' Theorem) can be merely sketched.

5.2: Be sure to cover Turán's Theorem. Presentation of Dirac's Theorem in 5.2.20 is valuable as an application of the Fan Lemma (Menger's Theorem). The subsequent material has limited appeal to undergraduates.

5.3: The inclusion-exclusion formula for the chromatic polynomial is derived here (5.3.10) without using inclusion-exclusion, making it accessible to this class without prerequisite. However, this proof is difficult for students to follow in favor of the simple inclusion-exclusion proof, which will be optional since that formula is not prerequisite for the course. Hence this formula should be omitted unless students know inclusion-exclusion. Chordal graphs and perfect graphs are more important, but these can also be treated lightly if short of time. Skip "Counting Acyclic Orientations".

Chapter 6.
6.1: The technical definitions of objects in the plane should be treated very lightly. It is better to be informal here, without writing out formal definitions unless explicitly requested by students. Outerplanar graphs are useful as a much easier class on which to solve problems (exercises!) like those on planar graphs; 6.18-20 are fundamental observations about outerplanar graphs, but other items are more important if time is short. 6.1.28 (polyhedra) is an appealing application but can be skipped.

6.2: The preparatory material 6.2.4-7 on Kuratowski's Theorem can be presented lightly, leaving the annoying details as reading; the subsequent material on convex embedding of 3-connected graphs is much more interesting. Presentation of the planarity algorithm is appealing but optional; skip the proof that it works.

6.3: The four color problem is popular; for undergraduates, show the flaw in Kempe's proof (p271), but don't present the discharging rule unless ahead of schedule. Students find the concept of crossing number easy to grasp, but the results are fairly difficult; try to go as far as the recursive quartic lower bound for the complete graph. The edge bound and its geometric application are impressive but take too much time for undergraduates. The idea of embeddings on surfaces can be conveyed through the examples in 6.3.21 on the torus. Interested students can be advised to read the rest of this section.

Chapter 7.
7.1: The proof of Vizing's Theorem is one of the more difficult in the course, but undergraduates can gain follow it when it is presented with sufficient colored chalk. The proof can be skipped if short of time. Skip

"Characterization of Line Graphs", although if time and interest is plentiful the necessity of Krausz's condition can be explained.

7.2: Chvátal's theorem (7.2.13) is not as hard to present as it looks if the instructor has the statement and proof clearly in mind. Nevertheless, the proof is somewhat technical and can be skipped (the same can be said of 7.2.17). More appealing is the Chvátal–Erdős Theorem (7.2.19), which certainly should be presented. Skip "Cycles in Directed Graphs".

7.3: The theorems of Tait and Grinberg make a nice culmination to the required material of the course. Skip "Snarks" and "Flows and Cycle Covers". Nevertheless, these are lively topics that can be recommended for advanced students.

Chapter 8. If time permits, material from the first part of sections of Chapter 8 can be presented to give the students a glimpse of other topics. The best choices for conveying some understanding in a brief treatment are Section 8.3 (Ramsey Theory or Sperner's Lemma) and Section 8.5 (Random Graphs). Also possible are the Gossip Problem (or other items) from Section 8.4 and some of the optional material from earlier chapters. The first part of Section 8.1 (Perfect Graphs) may also be usable for this purpose if perfect graphs have been discussed in Section 5.3. Sections 8.2 and 8.6 require more investment in preliminary material and thus are less suitable for giving a "glimpse".

1.FUNDAMENTAL CONCEPTS

1.1. WHAT IS A GRAPH?

1.1.1. *Complete bipartite graphs and complete graphs.* The complete bipartite graph $K_{m,n}$ is a complete graph if and only if $m = n = 1$ or $\{m, n\} = \{1, 0\}$.

1.1.2. *Adjacency matrices and incidence matrices for a 3-vertex path.*

$$\begin{pmatrix}0&1&1\\1&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&1\\1&0\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\1&1\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\\1&1\end{pmatrix}$$

Adjacency matrices for a path and a cycle with six vertices.

$$\begin{pmatrix}0&1&0&0&0&0\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\0&0&0&0&1&0\end{pmatrix}\qquad\begin{pmatrix}0&1&0&0&0&1\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\1&0&0&0&1&0\end{pmatrix}$$

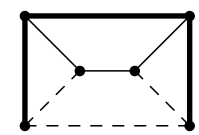
1.1.3. *Adjacency matrix for $K_{m,n}$.*

	m	n
m	0	1
n	1	0

1.1.4. $G \cong H$ *if and only if* $\overline{G} \cong \overline{H}$. If f is an isomorphism from G to H, then f is a vertex bijection preserving adjacency and nonadjacency, and hence f preserves non-adjacency and adjacency in $\overline{G}$ and is an isomorphism from $\overline{G}$ to $\overline{H}$. The same argument applies for the converse, since the complement of $\overline{G}$ is G.

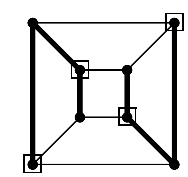
1.1.5. *If every vertex of a graph G has degree 2, then G is a cycle—FALSE.* Such a graph can be a disconnected graph with each component a cycle. (If infinite graphs are allowed, then the graph can be an infinite path.)

1.1.6. *The graph below decomposes into copies of P_4.*

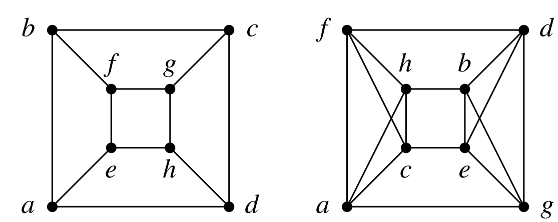


1.1.7. *A graph with more than six vertices of odd degree cannot be decomposed into three paths.* Every vertex of odd degree must be the endpoint of some path in a decomposition into paths. Three paths have only six endpoints.

1.1.8. *Decompositions of a graph.* The graph below decomposes into copies of $K_{1,3}$ with centers at the marked vertices. It decomposes into bold and solid copies of P_4 as shown.



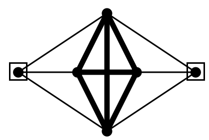
1.1.9. *A graph and its complement.* With vertices labeled as shown, two vertices are adjacent in the graph on the right if and only if they are not adjacent in the graph on the left.



1.1.10. *The complement of a simple disconnected graph must be connected—TRUE.* A disconnected graph G has vertices x, y that do not belong to a path. Hence$x$ and y are adjacent in $\overline{G}$. Furthermore, x and y have no common neighbor in G, since that would yield a path connecting them. Hence every vertex not in $\{x, y\}$ is adjacent in $\overline{G}$ to at least one of $\{x, y\}$. Hence every vertex can reach every other vertex in $\overline{G}$ using paths through $\{x, y\}$.

1.1.11. *Maximum clique and maximum independent set.* Since two vertices have degree 3 and there are only four other vertices, there is no clique of size 5. A complete subgraph with four vertices is shown in bold.

Since two vertices are adjacent to all others, an independent set containing either of them has only one vertex. Deleting them leaves P_4, in which the maximum size of an independent set is two, as marked.



1.1.12. *The Petersen graph.* The Petersen graph contains odd cycles, so it is not bipartite; for example, the vertices 12, 34, 51, 23, 45 form a 5-cycle.

The vertices 12, 13, 14, 15 form an independent set of size 4, since any two of these vertices have 1 as a common element and hence are nonadjacent. Visually, there is an independent set of size 4 marked on the drawing of the Petersen graph on the cover of the book. There are many ways to show that the graph has no larger independent set.

Proof 1. Two consecutive vertices on a cycle cannot both appear in an independent set, so every cycle contributes at most half its vertices. Since the vertex set is covered by two disjoint 5-cycles, every independent set has size at most 4.

Proof 2. Let ab be a vertex in an independent set S, where $a, b \in [5]$. We show that S has at most three additional vertices. The vertices not adjacent to ab are ac, bd, ae, bc, ad, be, and they form a cycle in that order. Hence at most half of them can be added to S.

1.1.13. *The graph with vertex set $\{0, 1\}^k$ and $x \leftrightarrow y$ when x and y differ in one place is bipartite.* The partite sets are determined by the parity of the number of 1's. Adjacent vertices have opposite parity. (This graph is the k-dimensional hypercube; see Section 1.3.)

1.1.14. *Cutting opposite corner squares from an eight by eight checkerboard leaves a subboard that cannot be partitioned into rectangles consisting of two adjacent unit squares.* 2-coloring the squares of a checkerboard so that adjacent squares have opposite colors shows that the graph having a vertex for each square and an edge for each pair of adjacent squares is bipartite. The squares at opposite corners have the same color; when these are deleted, there are 30 squares of that color and 32 of the other color. Each pair of adjacent squares has one of each color, so the remaining squares cannot be partitioned into sets of this type.

Generalization: the two partite sets of a bipartite graph cannot be matched up using pairwise-disjoint edges if the two partite sets have unequal sizes.

1.1.15. *Common graphs in four families:* $A = \{paths\}$, $B = \{cycles\}$, $C = \{complete\ graphs\}$, $D = \{bipartite\ graphs\}$.

$A \cap B = \varnothing$: In a cycle, the numbers of vertices and edges are equal, but this is false for a path.

$A \cap C = \{K_1, K_2\}$: To be a path, a graph must contain no cycle.

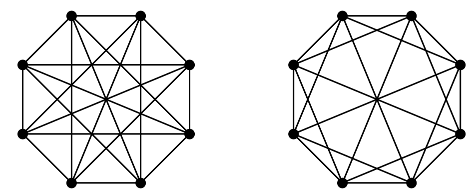
$A \cap D = A$: non-bipartite graphs have odd cycles.

$B \cap C = K_3$: Only when $n = 3$ does $\binom{n}{2} = n$.

$B \cap D = \{C_{2k} : k \geq 2\}$: odd cycles are not bipartite.

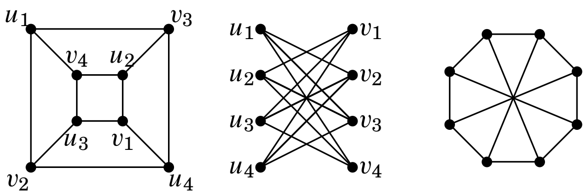
$C \cap D = \{K_1, K_2\}$: bipartite graphs cannot have triangles.

1.1.16. *The graphs below are not isomorphic.* The graph on the left has four cliques of size 4, and the graph on the right has only two. Alternatively, the complement of the graph on the left is disconnected (two 4-cycles), while the complement of the graph on the right is connected (one 8-cycle).



1.1.17. *There are exactly two isomorphism classes of 4-regular simple graphs with 7 vertices.* Simple graphs G and H are isomorphic if and only if their complements $\overline{G}$ and $\overline{H}$ are isomorphic, because an isomorphism $\phi \colon V(G) \to V(H)$ is also an isomorphism from $\overline{G}$ to $\overline{H}$, and vice versa. Hence it suffices to count the isomorphism classes of 2-regular simple graphs with 7 vertices. Every component of a finite 2-regular graph is a cycle. In a simple graph, each cycle has at least three vertices. Hence each class it determined by partitioning 7 into integers of size at least 3 to be the sizes of the cycles. The only two graphs that result are C_7 and $C_3 + C_4$ – a single cycle or two cycles of lengths three and four.

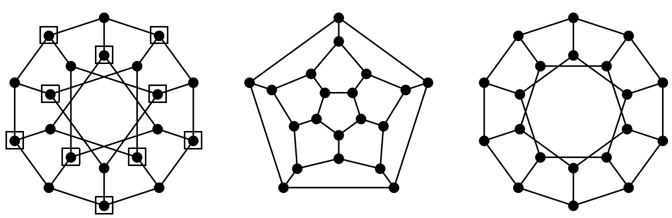
1.1.18. *Isomorphism.* Using the correspondence indicated below, the first two graphs are isomorphic; the graphs are bipartite, with $u_i \leftrightarrow v_j$ if and only if $i \neq j$. The third graph contains odd cycles and hence is not isomorphic to the others.



Visually, the first two graphs are Q_3 and the graph obtained by deleting four disjoint edges from $K_{4,4}$. In Q_3, each vertex is adjacent to the vertices whose names have opposite parity of the number of 1s, except for the complementary vertex. Hence Q_3 also has the structure of $K_{4,4}$ with four disjoint edges deleted; this proves isomorphism without specifying an explicit bijection.

1.1.19. *Isomorphism of graphs.* The rightmost two graphs below are isomorphic. The outside 10-cycle in the rightmost graph corresponds to the intermediate ring in the second graph. Pulling one of the inner 5-cycles of the rightmost graph out to the outside transforms the graph into the same drawing as the second graph.

The graph on the left is bipartite, as shown by marking one partite set. It cannot be isomorphic to the others, since they contain 5-cycles.



1.1.20. *Among the graphs below, the first (F) and third (H) are isomorphic, and the middle graph (G) is not isomorphic to either of these.*

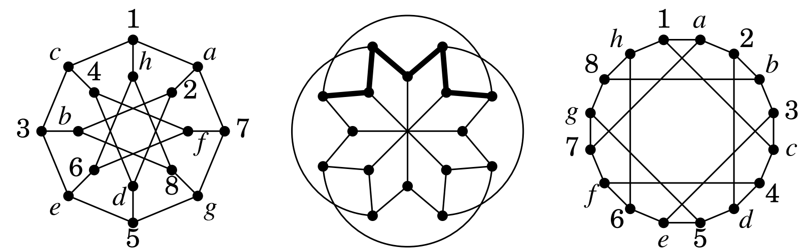
F and H are isomorphic. We exhibit an isomorphism (a bijection from $V(F)$ to $V(H)$ that preserves the adjacency relation). To do this, we name the vertices of F, write the name of each vertex of F on the corresponding vertex in H, and show that the names of the edges are the same in H and F. This proves that H is a way to redraw F. We have done this below using the first eight letters and the first eight natural numbers as names.

To prove quickly that the adjacency relation is preserved, observe that $1, a, 2, b, 3, c, 4, d, 5, e, 6, f, 7, g, 8, h$ is a cycle in both drawings, and the additional edges $1c, 2d, 3e, 4f, 5g, 6h, 7a, 8b$ are also the same in both drawings. Thus the two graphs have the same edges under this vertex correspondence. Equivalently, if we list the vertices in this specified order for

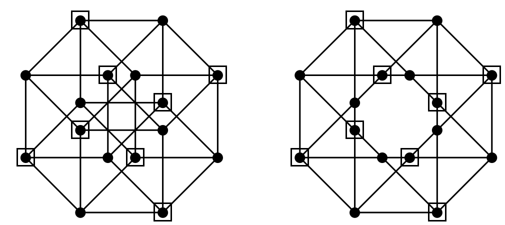
the two drawings, the two adjacency matrices are the same, but that is harder to verify.

G is not isomorphic to F or to H. In F and in H, the numbers form an independent set, as do the letters. Thus F and H are bipartite. The graph G cannot be bipartite, since it contains an odd cycle. The vertices above the horizontal axis of the picture induce a cycle of length 7.

It is also true that the middle graph contains a 4-cycle and the others do not, but it is harder to prove the absence of a 4-cycle than to prove the absence of an odd cycle.



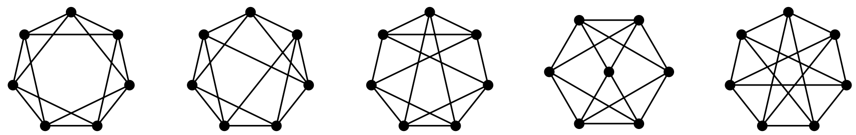
1.1.21. *Isomorphism.* Both graphs are bipartite, as shown below by marking one partite set. In the graph on the right, every vertex appears in eight 4-cycles. In the graph on the left, every vertex appears in only six 4-cycles (it is enough just to check one). Thus they are not isomorphic. Alternatively, for every vertex in the right graph there are five vertices having common neighbors with it, while in the left graph there are six such vertices.



1.1.22. *Isomorphism of explicit graphs.* Among the graphs below, $\{G_1, G_2, G_5\}$ are pairwise isomorphic. Also $G_3 \cong G_4$, and these are not isomorphic to any of the others. Thus there are exactly two isomorphism classes represented among these graphs.

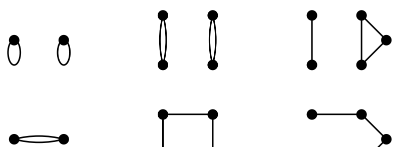
To prove these statements, one can present explicit bijections between vertex sets and verify that these preserve the adjacency relation (such as by displaying the adjacency matrix, for example). One can also make other structural arguments. For example, one can move the highest vertex in G_3 down into the middle of the picture to obtain G_4; from this one can list the desired bijection.

One can also recall that two graphs are isomorphic if and only if their complements are isomorphic. The complements of G_1, G_2, and G_5 are cycles of length 7, which are pairwise isomorphic. Each of $\overline{G}_3$ and $\overline{G}_4$ consists of two components that are cycles of lengths 3 and 4; these graphs are isomorphic to each other but not to a 7-cycle.



G_1 G_2 G_3 G_4 G_5

1.1.23. *Smallest pairs of nonisomorphic graphs with the same vertex degrees.* For multigraphs, loopless multigraphs, and simple graphs, the required numbers of vertices are 2,4,5; constructions for the upper bounds appear below. We must prove that these constructions are smallest.



a) general b) loopless c) simple

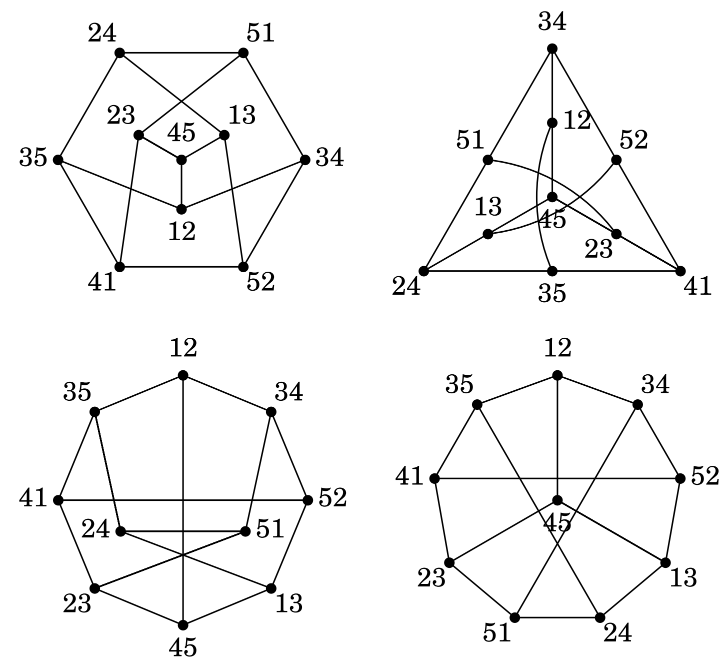
a) With 1 vertex, every edge is a loop, and the isomorphism class is determined by the number of edges, which is determined by the vertex degree. Hence nonisomorphic graphs with the same vertex degrees have at least two vertices.

b) Every loopless graph is a graph, so the answer for loopless graphs is at least 2. The isomorphism class of a loopless graph with two vertices is determined by the number of copies of the edge, which is determined by the vertex degrees. The isomorphism class of a loopless graph with three vertices is determined by the edge multiplicities. Let the three vertex degrees be a, b, c, and let the multiplicities of the opposite edges be x, y, z, where Since $a = y + z$, $b = x + z$, and $c = x + y$, we can solve for the multiplicities in terms of the degrees by $x = (b+c-a)/2$, $y = (a+c-b)/2$, and $z = (a+b-c)/2$. Hence the multiplicities are determined by the degrees, and all loopless graphs with vertex degrees a, b, c are pairwise isomorphic. Hence nonisomorphic loopless graphs with the same vertex degrees have at least four vertices.

c) Since a simple graph is a loopless graph, the answer for simple graphs is at least 4. There are 11 isomorphism classes of simple graphs with four vertices. For each of 0,1,5, or 6 edges, there is only one isomorphism class. For 2 edges, there are two isomorphism classes, but they have

different lists of vertex degrees (similarly for 4 edges). For 3 edges, the three isomorphism classes have degree lists 3111, 2220, and 2211, all different. Hence nonisomorphic simple graphs with the same vertex degrees must have at least five vertices.

1.1.24. *Isomorphisms for the Petersen graph.* Isomorphism is proved by giving an adjacency-preserving bijection between the vertex sets. For pictorial representations of graphs, this is equivalent to labeling the two graphs with the same vertex labels so that the adjacency relation is the same in both pictures; the labels correspond to a permutation of the rows and columns of the adjacency matrices to make them identical. The various drawings of the Petersen graph below illustrate its symmetries; the labelings indicate that these are all "the same" (unlabeled) graph. The number of isomorphisms from one graph to another is the same as the number of isomorphisms from the graph to itself.



1.1.25. *The Petersen graph has no cycle of length 7.* Suppose that the Petersen graph has a cycle C of length 7. Since any two vertices of C are connected by a path of length at most 3 on C, any additional edge with endpoints on C would create a cycle of length at most 4. Hence the third neighbor of each vertex on C is not on C.

Thus there are seven edges from $V(C)$ to the remaining three vertices. By the pigeonhole principle, one of the remaining vertices receives at least three of these edges. This vertex x not on C has three neighbors on C. For any three vertices on C, either two are adjacent or two have a common neighbor on C (again the pigeonhole principle applies). Using x, this completes a cycle of length at most 4. We have shown that the assumption of a 7-cycle leads to a contradiction.

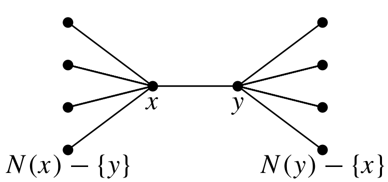
Alternative completion of proof. Let u be a vertex on C, and let v, w be the two vertices farthest from u on C. As argued earlier, no edges join vertices of C that are not consecutive on C. Thus u is not adjacent to v or w. Hence u, v have a common neighbor off C, as do u, w. Since u has only one neighbor off C, these two common neighbors are the same. The resulting vertex x is adjacent to all of u, v, w, and now x, v, w is a 3-cycle.

1.1.26. *A k-regular graph of girth four has at least $2k$ vertices, with equality only for $K_{k,k}$.* Let G be k-regular of girth four, and chose $xy \in E(G)$. Girth 4 implies that G is simple and that x and y have no common neighbors. Thus the neighborhoods $N(x)$ and $N(y)$ are disjoint sets of size k, which forces at least $2k$ vertices into G. Possibly there are others.

Note also that $N(x)$ and $N(y)$ are independent sets, since G has no triangle. If G has no vertices other than these, then the vertices in $N(x)$ can have neighbors only in $N(y)$. Since G is k-regular, every vertex of $N(x)$ must be adjacent to every vertex of $N(y)$. Thus G is isomorphic to $K_{k,k}$, with partite sets $N(x)$ and $N(y)$. In other words, there is only one such isomorphism class for each value of k.

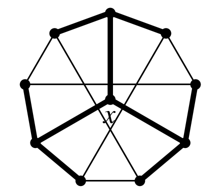
Comment. One can also start with a vertex x, choose y from among the k vertices in $N(x)$, and observe that $N(y)$ must have $k-1$ more vertices not in $N(x) \cup \{x\}$. The proof then proceeds as above.

(An alternative proof uses the methods of Section 1.3. A triangle-free simple graph with n vertices has at most $n^2/4$ edges. Since G is k-regular, this yields $n^2/4 \geq nk/2$, and hence $n \geq 2k$. Furthermore, equality holds in the edge bound only for $K_{n/2,n/2}$, so this is the only such graph with $2k$ vertices. (C. Pikscher))



1.1.27. *A simple graph of girth 5 in which every vertex has degree at least k has at least k^2+1 vertices, with equality achieveable when $k \in \{2, 3\}$.* Let G be k-regular of girth five. Let S be the set consisting of a vertex x and its neighbors. Since G has no cycle of length less than five, G is simple, and any two neighbors of x are nonadjacent and have no common neighbor other than x. Hence each $y \in S - \{x\}$ has at least $k-1$ neighbors that are not in S and not neighbors of any vertex in S. Hence G has at least $k(k-1)$ vertices outside S and at least $k+1$ vertices in S for at least k^2+1 altogether.

The 5-cycle achieves equality when $k=2$. For $k=3$, growing the graph symmetrically from x permits completing the graph by adding edges among the non-neighbors of x. The result is the Petersen graph. (Comment: For $k>3$, it is known that girth 5 with minimum degree k and exactly k^2+1 vertices is impossible, except for $k=7$ and possibly for $k=57$.)



1.1.28. *The Odd Graph has girth 6.* The Odd Graph O_k is the disjointness graph of the set of k-element subsets of $[2k+1]$.

Vertices with a common neighbor correspond to k-sets with $k-1$ common elements. Thus they have exactly one common neighbor, and O_k has no 4-cycle. Two vertices at distance 2 from a single vertex have at least $k-2$ common neighbors. For $k>2$, such vertices cannot be adjacent, and thus O_k has no 5-cycle when $k>2$. To form a 6-cycle when $k \geq 2$, let $A = \{2,\dots,k\}$, $B = \{k+2,\dots,2k\}$, $a=1$, $b=k+1$, $c=2k+1$. A 6-cycle is $A \cup \{a\}$, $B \cup \{b\}$, $A \cup \{c\}$, $B \cup \{a\}$, $A \cup \{b\}$, $B \cup \{c\}$.

The Odd Graph also is not bipartite. The successive elements $\{1,\dots,k\}$, $\{k+1,\dots,2k\}$, $\{2k+1,1,\dots,k-1\}$, . . ., $\{k+2,\dots,2k+1\}$ form an odd cycle.

1.1.29. *Among any 6 people, there are 3 mutual acquaintances or 3 mutual strangers.* Let G be the graph of the acquaintance relation, and let x be one of the people. Since x has 5 potential neighbors, x has at least 3 neighbors or at least 3 nonneighbors. By symmetry (if we complement G, we still have to prove the same statement), we may assume that x has at least 3 neighbors. If any pair of these people are acquainted, then with x we have 3 mutual acquaintances, but if no pair of neighbors of x is acquainted, then the neighbors of x are three mutual strangers.

1.1.30. *The number of edges incident to v_i is the ith diagonal entry in MM^T and in A^2.* In both MM^T and A^2 this is the sum of the squares of the entries in the ith row. For MM^T, this follows immediately from the definition of matrix multiplication and transposition, but for A^2 this uses the graph-theoretic fact that $A = A^T$; i.e. column i is the same as row i. Because G is simple, the entries of the matrix are all 0 or 1, so the sum of the squares in a row equals the number of 1s in the row. In M, the 1s in a row denote incident edges; in A they denote vertex neighbors. In either case, the number of 1s is the degree of the vertex.

If $i \neq j$, then the entry in position (i, j) of A^2 is the number of common neighbors of v_i and v_j. The matrix multiplication puts into position (i, j) the "product" of row i and column j; that is $\sum_{k=1}^{n} a_{i,k}a_{k,j}$. When G is simple, entries in A are 1 or 0, depending on whether the corresponding vertices are adjacent. Hence $a_{i,k}a_{k,j} = 1$ if v_k is a common neighbor of v_i and v_j; otherwise, the contribution is 0. Thus the number of contributions of 1 to entry (i, j) is the number of common neighbos of v_i and v_j.

If $i \neq j$, then the entry in position (i, j) of MM^T is the number of edges joining v_i and v_j (0 or 1 when G has no multiple edges). The ith row of M has 1s corresponding to the edges incident to v_i. The jth column of M^T is the same as the jth row of M, which has 1s corresponding to the edges incident to v_j. Summing the products of corresponding entries will contribute 1 for each edge incident to both v_i and v_j; 0 otherwise.

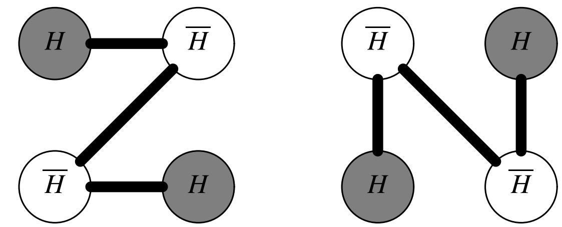
Comment. For graphs without loops, both arguments for (i, j) in general apply when $i = j$ to explain the diagonal entries.

1.1.31. *K_n decomposes into two isomorphic ("self-complementary") subgraphs if and only if n or $n-1$ is divisible by 4.*

a) The number of vertices in a self-complementary graph is congruent to 0 or 1 $(\mathrm{mod}\, 4)$. If G and $\overline{G}$ are isomorphic, then they have the same number of edges, but together they have $\binom{n}{2}$ edges (with none repeated), so the number of edges in each must be $n(n-1)/4$. Since this is an integer and the numbers n and $n-1$ are not both even, one of $\{n, n-1\}$ must be divisible by 4.

b) Construction of self-complementary graphs for all such n.

Proof 1 (explicit construction). We generalize the structure of the self-complementary graphs on 4 and 5 vertices, which are P_4 and C_5. For $n = 4k$, take four vertex sets of size k, say X_1, X_2, X_3, X_4, and join all vertices of X_i to those of X_{i+1}, for $i = 1, 2, 3$. To specify the rest of G, within these sets let X_1 and X_4 induce copies of a graph H with k vertices, and let X_2 and X_3 induce $\overline{H}$. (For example, H may be K_k.) In $\overline{G}$, both X_2 and X_3 induce H, while X_1 and X_4 induce $\overline{H}$, and the connections between sets are $X_2 \leftrightarrow X_4 \leftrightarrow X_1 \leftrightarrow X_3$. Thus relabeling the subsets defines an isomorphism between G and $\overline{G}$. (There are still other constructions for G.)



For $n = 4k + 1$, we add a vertex x to the graph constructed above. Join x to the $2k$ vertices in X_1 and X_4 to form G. The isomorphism showing that $G - x$ is self-complementary also works for G (with x mapped to itself), since this isomorphism maps $N_G(x) = X_1 \cup X_4$ to $N_{\overline{G}}(x) = X_2 \cup X_3$.

Proof 2 (inductive construction). If G is self-complementary, then let H_1 be the graph obtained from G and P_4 by joining the two ends of P_4 to all vertices of G. Let H_2 be the graph obtained from G and P_4 by joining the two center vertices of P_4 to all vertices of G. Both H_1 and H_2 are self-complementary. Using this with $G = K_1$ produces the two self-complementary graphs of order 5, namely C_5 and the bull.

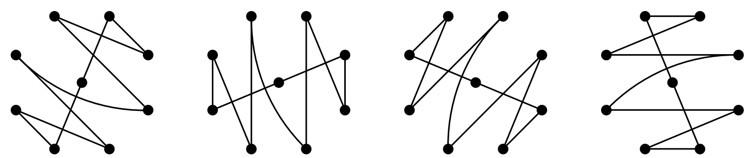
Self-complementary graphs with order divisible by 4 arise from repeated use of the above using $G = P_4$ as a starting point, and self-complementary graphs of order congruent to 1 modulo 4 arise from repeated use of the above using $G = K_1$ as a starting point. This construction produces many more self-complementary graphs than the explicit construction in Proof 1.

1.1.32. *$K_{m,n}$ decomposes into two isomorphic subgraphs if and only if m and n are not both odd.* The condition is necessary because the number of edges must be even. It is sufficient because $K_{m,n}$ decomposes into two copies of $K_{m,n/2}$ when n is even.

1.1.33. *Decomposition of complete graphs into cycles through all vertices.* View the vertex set of K_n as $\mathbb{Z}_n$, the values $0, \ldots, n-1$ in cyclic order. Since each vertex has degree $n-1$ and each cycle uses two edges at each vertex, the decomposition has $(n-1)/2$ cycles.

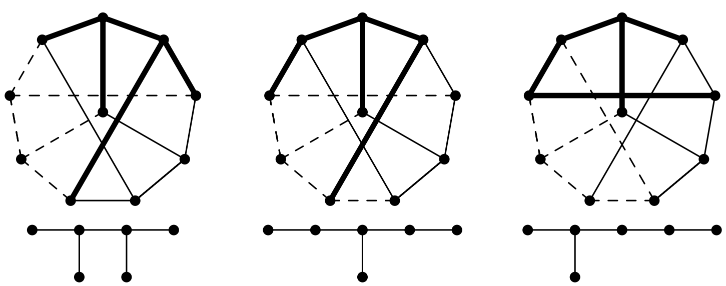
For $n = 5$ and $n = 7$, it suffices to use cycles formed by traversing the vertices with constant difference: $(0, 1, 2, 3, 4)$ and $(0, 2, 4, 1, 3)$ for $n = 5$ and $(0, 1, 2, 3, 4, 5, 6)$, $(0, 2, 4, 6, 1, 3, 5)$, and $(0, 3, 6, 2, 5, 1, 4)$ for $n = 7$.

This construction fails for $n = 9$ since the edges with difference 3 form three 3-cycles. The cyclically symmetric construction below treats the vertex set as $\mathbb{Z}_8$ together with one special vertex.



1.1.34. *Decomposition of the Petersen graph into copies of P_4.* Consider the drawing of the Petersen graph with an inner 5-cycle and outer 5-cycle. Each P_4 consists of one edge from each cycle and one cross edge joining them. Extend each cross edge e to a copy of P_4 by taking the edge on each of the two 5-cycles that goes in a clockwise direction from e. In this way, the edges on the outside 5-cycle are used in distinct copies of P_4, and the same holds for the edges on the inside 5-cycle.

Decomposition of the Petersen graph into three pairwise-isomorphic connected subgraphs. Three such decompositions are shown below. We restricted the search by seeking a decomposition that is unchanged by 120° rotations in a drawing of the Petersen graph with 3-fold rotational symmetry. The edges in this drawing form classes of size 3 that are unchanged under rotations of 120°; each subgraph in the decomposition uses exactly one edge from each class.



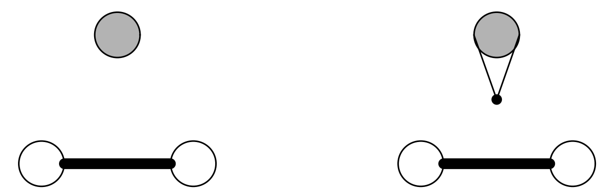
1.1.35. *K_n decomposes into three pairwise-isomorphic subgraphs if and only if $n+1$ is not divisible by 3.* The number of edges is $n(n-1)/2$. If $n+1$ is divisible by 3, then n and $n-1$ are not divisible by 3. Thus decomposition into three subgraphs of equal size is impossible in this case.

If $n+1$ is not divisible by 3, then $e(K_n)$ is divisible by 3, since n or $n-1$ is divisible by 3. We construct a decomposition into three subgraphs that are pairwise isomorphic (there are many such decompositions).

When n is a multiple of 3, we partition the vertex set into three subsets V_1, V_2, V_3 of equal size. Edges now have two types: within a set or joining two sets. Let the ith subgraph G_i consist of all the edges within V_i and all the edges joining the two other subsets. Each edge of K_n appears in exactly

one of these subgraphs, and each G_i is isomorphic to the disjoint union of $K_{n/3}$ and $K_{n/3,n/3}$.

When $n \equiv 1 \pmod 3$, consider one vertex w. Since $n-1$ is a multiple of 3, we can form the subgraphs G_i as above on the remaining $n-1$ vertices. Modify G_i to form H_i by joining w to every vertex of V_i. Each edge involving w has been added to exactly one of the three subgraphs. Each H_i is isomorphic to the disjoint union of $K_{1+(n-1)/3}$ and $K_{(n-1)/3,(n-1)/3}$.



1.1.36. *If K_n decomposes into triangles, then $n-1$ or $n-3$ is divisible by 6.* Such a decomposition requires that the degree of each vertex is even and the number of edges is divisible by 3. To have even degree, n must be odd. Also $n(n-1)/2$ is a multiple of 3, so 3 divides n or $n-1$. If 3 divides n and n is odd, then $n-3$ is divisible by 6. If 3 divides $n-1$ and n is odd, then $n-1$ is divisible by 6.

1.1.37. *A graph in which every vertex has degree 3 has no decomposition into paths with at least five vertices each.* Suppose that G has such a decomposition. Since every vertex has degree 3, each vertex is an endpoint of at least one of the paths and is an internal vertex on at most one of them. Since every path in the decomposition has two endpoints and has at least three internal vertices, we conclude that the number of paths in the decomposition is at least $n(G)/2$ and is at most $n(G)/3$, which is impossible.

Alternatively, let k be the number of paths. There are $2k$ endpoints of paths. On the other hand, since each internal vertex on a path in the decomposition must be an endpoint of some other path in the decomposition, there must be at least $3k$ endpoints of paths. The contradiction implies that there cannot be such a decomposition.

1.1.38. *A 3-regular graph G has a decomposition into claws if and only if G is bipartite.* When G is bipartite, we produce a decomposition into claws. We use all claws obtained by taking the three edges incident with a single vertex in the first partite set. Each claw uses all the edges incident to its central vertex. Since each edge has exactly one endpoint in the first partite set, each edge appears in exactly one of these claws.

When G has a decomposition into claws, we partition $V(G)$ into two independent sets. Let X be the set of centers of the claws in the decomposition. Since every vertex has degree 3, each claw in the decomposition uses all edges incident to its center. Since each edge is in at most one claw, this makes X an independent set. The remaining vertices also form an independent set, because every edge is in some claw in the decomposition, which means that one of its endpoints must be the center of that claw.

1.1.39. *Graphs that decompose K_6.*

Triangle—No. A graph decomposing into triangles must have even degree at each vertex. (This excludes all decompositions into cycles.)

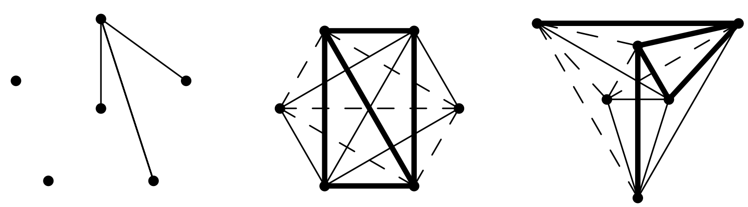
Paw, P_5—No. K_6 has 15 edges, but each paw or P_5 has four edges.

House, Bowtie, Dart–No. K_6 has 15 edges, but each house, bowtie, or dart has six edges.

Claw—Yes. Put five vertices 0, 1, 2, 3, 4 on a circle and the other vertex z in the center. For $i \in \{0, 1, 2, 3, 4\}$, use a claw with edges from i to $i+1$, $i+2$, and z. Each edge appears in exactly one of these claws.

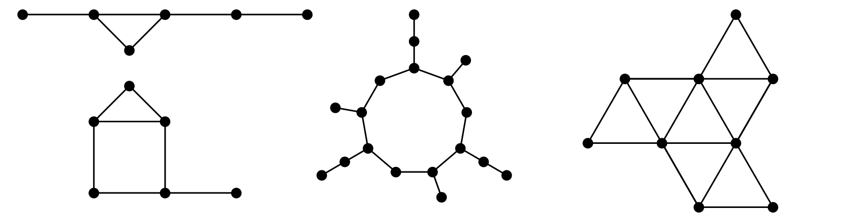
Kite—Yes. Put all six vertices on a circle. Each kite uses two opposite edges on the outside, one diagonal, and two opposite edges of "length" 2. Three rotations of the picture complete the decomposition.

Bull—Yes. The bull has five edges, so we need three bulls. Each bull uses degrees 3, 3, 2, 1, 1, 0 at the six vertices. Each bull misses one vertex, and each vertex of K_6 has five incident edges, so three of the vertices will receive degrees 3, 2, 0 from the three bulls, and the other three will receive degrees 3, 1, 1. Thus we use vertices of two types, which leads us to position them on the inside and outside as on the right below. The bold, solid, and dashed bulls obtained by rotation complete the decomposition.



1.1.40. *Automorphisms of P_n, C_n, and K_n.* A path can be left alone or flipped, a cycle can be rotated or flipped, and a complete graph can be permuted arbtrarily. The numbers of automorphisms are 2, $2n$, $n!$, respectively. Correspondingly, the numbers of distinct labelings using vertex set $[n]$ are $n!/2$, $(n-1)!/2$, 1, respectively. For P_n, these formulas require $n > 1$.

1.1.41. *Graphs with one and three automorphisms.* The two graphs on the left have six vertices and only the identity automorphism. The two graphs on the right have three automorphisms.



1.1.42. *The set of automorphisms of a graph G satisfies the following:*

a) The composition of two automorphisms is an automorphism.

b) The identity permutation is an automorphism.

c) The inverse of an automorphism is also an automorphism.

d) Composition of automorphisms satisfies the associative property.

The first three properties are essentially the same as the transitive, reflexive, and symmetric properties for the isomorphism relation; see the discussion of these in the text. The fourth property holds because composition of functions always satisfies the associative property (see the discussion of composition in Appendix A).

1.1.43. *Every automorphism of the Petersen graph maps the 5-cycle (12,34,51,23,45) into a 5-cycle with vertices ab, cd, ea, bc, de by a permutation of* $[5]$ *taking 1,2,3,4,5 to a, b, c, d, e, respectively.* Let σ denote the automorphism, and let the vertex ab be the image of the vertex 12 under σ. The image of 34 must be a pair disjoint from ab, so we may let $cd = \sigma(34)$. The third vertex must be disjoint from the second and share an element with the first. We may select a to be the common element in the first and third vertices. Similarly, we may select c to be the common element in the second and fourth vertices. Since nonadjacent vertices correspond to sets with a common element, the other element of the fourth vertex must be b, and the fifth vertex can't have a or b and must have d and e. Thus every 5-cycle must have this form and is the image of (12,34,51,23,45) under the specified permutation σ.

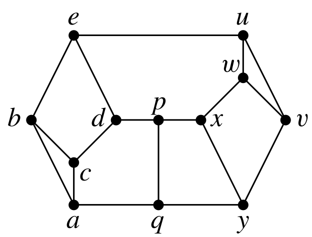
The Peterson graph has 120 automorphisms. Every permutation of [5] preserves the disjointness relation on 2-element subsets and therefore defines an automorphism of the Petersen graph. Thus there are at least 120 automorphism. To show that there are no others, consider an arbitrary automorphism σ. By the preceding paragraph, the 5-cycle C maps to some 5-cycle (ab, cd, ea, bc, de). This defines a permutation f mapping $1, 2, 3, 4, 5$ to a, b, c, d, e, respectively. It suffices to show that the other vertices must also have images under σ that are described by f.

The remaining vertices are pairs consisting of two nonconsecutive values modulo 5. By symmetry, it suffices to consider just one of them, say 24. The only vertex of C that 24 is adjacent to (disjoint from) is 51. Since

$\sigma(51) = ea$, and the only vertex not on (ab, cd, ea, bc, de) that is adjacent to ea is bd, we must have $\sigma(24) = bd$, as desired.

1.1.44. *For each pair of 3-edge paths $P = (u_0, u_1, u_2, u_3)$ and $Q = (v_0, v_1, v_2, v_3)$ in the Petersen graph, there is an automorphism of the Petersen graph that turns P into Q.* In the disjointness representation of the Petersen graph, suppose the pairs corresponding to the vertices of P are ab, cd, ef, g_h, respectively. Since consecutive pairs are disjoint and the edges are unordered pairs, we may write the pairs so that a, b, c, d, e are distinct, $f = a$, $g = b$, and $h = c$. Putting the vertex names of Q in the same format $AB, CD, EF, G \square H$, we chose the isomorphism generated by the permutation of [5] that turns a, b, c, d, e into A, B, C, D, E, respectively.

1.1.45. *A graph with 12 vertices in which every vertex has degree 3 and the only automorphism is the identity.*



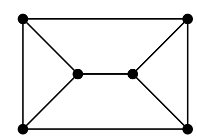
There are many ways to prove that an automorphism must fix all the vertices. The graph has only two triangles (abc and uvw). Now an automorphism must fix p, since is the only vertex having no neighbor on a triangle, and also e, since it is the only vertex with neighbors on both triangles. Now d is the unique common neighbor of p and e. The remaining vertices can be fixed iteratively in the same way, by finding each as the only unlabeled vertex with a specified neighborhood among the vertices already fixed. (This construction was provided by Luis Dissett, and the argument forbidding nontrivial automorphisms was shortened by Fred Galvin. Another such graph with three triangles was found by a student of Fred Galvin.)

1.1.46. *Vertex-transitivity and edge-transitivity.* The graph on the left in Exercise 1.1.21 is isomorphic to the 4-dimensional hypercube (see Section 1.3), which is vertex-transitive and edge-transitive via the permutation of coordinates. For the graph on the right, rotation and inside-out exchange takes care of vertex-transitivity. One further generating operation is needed to get edge-transitivity; the two bottom outside vertices can be switched with the two bottom inside vertices.

1.1.47. *Edge-transitive versus vertex-transitive. a) If G is obtained from K_n with $n \geq 4$ by replacing each edge of K_n with a path of two edges through a new vertex of degree 2, then G is edge-transitive but not vertex-transitive.* Every edge consists of an old vertex and a new vertex. The $n!$ permutations of old vertices yield automorphism. Let $x\&y$ denote the new vertex on the path replacing the old edge xy; note that $x\&y = y\&x$. The edge joining x and $x\&y$ is mapped to the edge joining u and $u\&v$ by any automorphism that maps x to u and y to v. The graph is not vertex-transitive, since $x\&y$ has degree 2, while x has degree $n - 1$.

b) If G is edge-transitive but not vertex-transitive and has no isolated vertices, then G is bipartite. Let uv be an arbitrary edge of G. Let S be the set of vertices to which u is mapped by automorphisms of G, and let T be the set of vertices to which v is mapped. Since G is edge-transitive and has no isolated vertex, $S \cup T = V(G)$. Since G is not vertex-transitive, $S \neq V(G)$. Together, these statements yield $S \cap T = \varnothing$, since the composition of two automorphisms is an automorphism. By edge-transitivity, every edge of G contains one vertex of S and one vertex of T. Since $S \cap T = \varnothing$, this implies that G is bipartite with vertex bipartition S, T.

c) The graph below is vertex-transitive but not edge-transitive. A composition of left-right reflections and vertical rotations can take each vertex to any other. The graph has some edges on triangles and some edges not on triangles, so it cannot be edge-transitive.



1.2. PATHS, CYCLES, AND TRAILS

1.2.1. *Statements about connection.*

a) Every disconnected graph has an isolated vertex—FALSE. A simple 4-vertex graph in which every vertex has degree 1 is disconnected and has no isolated vertex.

b) A graph is connected if and only if some vertex is connected to all other vertices—TRUE. A vertex is "connected to" another if they lie in a common path. If G is connected, then by definition each vertex is connected to every other. If some vertex x is connected to every other, then because a u, x-path and x, v-path together contain a u, v-path, every vertex is connected to every other, and G is connected.

c) The edge set of every closed trail can be partitioned into edge sets of cycles—TRUE. The vertices and edges of a closed trail form an even graph, and Proposition 1.2.27 applies.

d) If a maximal trail in a graph is not closed, then its endpoints have odd degree. If an endpoint v is different from the other endpoint, then the trail uses an odd number of edges incident to v. If v has even degree, then there remains an incident edge at v on which to extend the trail.

1.2.2. *Walks in K_4.*

a) K_4 has a walk that is not a trail; repeat an edge.

b) K_4 has a trail that is not closed and is not a path; traverse a triangle and then one additional edge.

c) The closed trails in K_4 that are not cycles are single vertices. A closed trail has even vertex degrees; in K_4 this requires degrees 2 or 0, which forbids connected nontrivial graphs that are not cycles. By convention, a single vertex forms a closed trail that is not a cycle.

1.2.3. *The non-coprimality graph with vertex set $\{1, \ldots, 15\}$.* Vertices 1,11,13 are isolated. The remainder induce a single component. It has a spanning path 7,14,10,5,15,3,9,12,8,6,4,2. Thus there are four components, and the maximal path length is 11.

1.2.4. *Effect on the adjacency and incidence matrices of deleting a vertex or edge.* Assume that the graph has no loops.

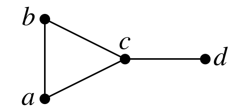
Consider the vertex ordering $v_1, \ldots, v_n$. Deleting edge v_iv_j simply deletes the corresponding column of the incidence matrix; in the adjacency matrix it reduces positions i, j and j, i by one.

Deleting a vertex v_i eliminates the ith row of the incidence matrix, and it also deletes the column for each edge incident to v_i. In the adjacency matrix, the ith row and ith column vanish, and there is no effect on the rest of the matrix.

1.2.5. *If v is a vertex in a connected graph G, then v has a neighbor in every component of $G - v$.* Since G is connected, the vertices in one component of $G - v$ must have paths in G to every other component of $G - v$, and a path can only leave a component of $G - v$ via v. Hence v has a neighbor in each component.

No cut-vertex has degree 1. If G is connected and $G - v$ has k components, then having a neighbor in each such component yields $d_G(v) \geq k$. If v is a cut-vertex, then $k \geq 2$, and hence $d_G(v) \geq 2$.

1.2.6. *The paw.* Maximal paths: acb, $abcd$, $bacd$ (two are maximum paths). Maximal cliques: abc, cd (one is a maximum clique). Maximal independent sets: c, bd, ad (two are maximum independent sets).



1.2.7. *A bipartite graph has a unique bipartition (except for interchanging the two partite sets) if and only if it is connected.* Let G be a bipartite graph. If u and v are vertices in distinct components, then there is a bipartition in which u and v are in the same partite set and another in which they are in opposite partite sets.

If G is connected, then from a fixed vertex u we can walk to all other vertices. A vertex v must be in the same partite set as u if there is a u, v-walk of even length, and it must be in the opposite set if there is a u, v-walk of odd length.

1.2.8. *The biclique $K_{m,n}$ is Eulerian if and only if m and n are both even or one of them is 0.* The graph is connected. It vertices have degrees m and n (if both are nonzero), which are all even if and only if m and n are both even. When m or n is 0, the graph has no edges and is Eulerian.

1.2.9. *The minimum number of trails that decompose the Petersen graph is 5.* The Petersen graph has exactly 10 vertices of odd degree, so it needs at least 5 trails, and Theorem 1.2.33 implies that five trails suffice.

The Petersen graph does have a decomposition into five paths. Given the drawing of the Petersen graph consisting of two disjoint 5-cycles and edges between them, form paths consisting of one edge from each cycle and one edge joining them.

1.2.10. *Statements about Eulerian graphs.*

a) Every Eulerian bipartite graph has an even number of edges—TRUE.

Proof 1. Every vertex has even degree. We can count the edges by summing the degrees of the vertices in one partite set; this counts every edge exactly once. Since the summands are all even, the total is also even.

Proof 2. Since every walk alternates between the partite sets, following an Eulerian circuit and ending at the initial vertex requires taking an even number of steps.

Proof 3. Every Eulerian graph has even vertex degrees and decomposes into cycles. In a bipartite graph, every cycle has even length. Hence the number of edges is a sum of even numbers.

b) Every Eulerian simple graph with an even number of vertices has an even number of edges—FALSE. The union of an even cycle and an odd cycle that share one vertex is an Eulerian graph with an even number of vertices and an odd number of edges.

1.2.11. *If G is an Eulerian graph with edges e, f that share a vertex, then G need not have an Eulerian circuit in which e, f appear consecutively.* If G consists of two edge-disjoint cycles sharing one common vertex v, then edges incident to v that belong to the same cycle cannot appear consecutively on an Eulerian circuit.

1.2.12. *Algorithm for Eulerian circuit.* We convert the proof by extremality to an iterative algorithm. Assume that G is a connected even graph. Initialize T to be a closed trail of length 0; a single vertex.

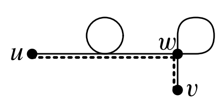
If T is not all of G, we traverse T to reach a vertex v on T that is incident to an edge e not in T. Beginning from v along e, traversing an arbitrary trail T' not using edges of T; eventually the trail cannot be extended. Since $G - E(T)$ is an even graph, this can only happen upon a return to the original vertex v, completing a closed trail. Splice T' into T by traversing T up to v, then following T', then the rest of T.

If this new trail includes all of $E(G)$, then it is an Eulerian circuit, and we stop. Otherwise, let this new trail be T and repeat the iterative step.

Since each successive trail is longer and G has finitely many edges, the procedure must terminate. It can only terminate when an Eulerian circuit has been found.

1.2.13. *Each u, v-walk contains a u, v-path.*

a) (induction). We use ordinary induction on the length l of the walk, proving the statement for all pairs of vertices. A u, v-walk of length 1 is a u, v-path of length 1; this provides the basis. For the induction step, suppose $l > 1$, and let W be a u, v-walk of length l; the induction hypothesis is that walks of length less than l contain paths linking their endpoints. If $u = v$, the desired path has length 0. If $u \neq v$, let wv be the last edge of W, and let W' be the u, w-walk obtained by deleting wv from W. Since W' has length $l - 1$, the induction hypothesis guarantees a u, w-path P in W'. If $w = v$, then P is the desired u, v-path. If $w \neq v$ and v is not on P, then we extend P by the edge wv to obtain a u, v-path. If $w \neq v$ and v is on P, then P contains a u, v-path. In each case, the edges of the u, v-path we construct all belong to W.



b) (extremality) Given a u, v-walk W, consider a shortest u, v-walk W' contained in W. If this is not a path, then it has a repeated vertex, and the portion between the instances of one vertex can be removed to obtain a shorter u, v-walk in W than W'.

1.2.14. *The union of the edge sets of distinct u, v-paths contains a cycle.*

Proof 1 (extremality). Let P and Q be distinct u, v-paths. Since a path in a simple graph is determined by its set of edges, we may assume (by symmetry) that P has an edge e not belonging to Q. Within the portion of P before P traverses e, let y be the last vertex that belongs to Q. Within the portion of P after P traverses e, let z be the first vertex that belongs to Q. The vertices y, z exist, because $u, v \in V(Q)$. The y, z-subpath of P combines with the y, z- or z, y-subpath of Q to form a cycle, since this subpath of Q contains no vertex of P between y and z.

Proof 2 (induction). We use induction on the sum l of the lengths of the two paths, for all vertex pairs simultaneously. If P and Q are u, v-paths, then $l \geq 2$. If $l = 2$, then we have distinct edges consisting of u and v, and together they form a cycle of length 2. For the induction step, suppose $l > 2$. If P and Q have no common internal vertices, then their union is a cycle. If P and Q have a common internal vertex w, then let P', P'' be the u, w-subpath of P and the w, v-subpath of P. Similarly define Q', Q''. Then P', Q' are u, w-paths with total length less than l. Similarly, P'', Q'' are w, v-paths with total length less than l. Since P, Q are distinct, we must have P', Q' distinct or P'', Q'' distinct. We can apply the induction hypothesis to the pair that is a pair of distinct paths joining the same endpoints. This pair contains the edges of a cycle, by the induction hypothesis, which in turn is contained in the union of P and Q.

The union of distinct u, v-walks need not contain a cycle. Let $G = P_3$, with vertices u, x, v in order. The distinct u, v-walks with vertex lists u, x, u, x, v and u, x, v, x, v do not contain a cycle in their union.

1.2.15. *If W is a nontrivial closed walk that does not contain a cycle, then some edge of W occurs twice in succession (once in each direction).*

Proof 1 (induction on the length l of W). We are given $l \geq 1$. A closed walk of length 1 is a loop, which is a cycle. Thus we may assume $l \geq 2$.

Basis step: $l = 2$. Since it contains no cycle, the walk must take a step and return immediately on the same edge.

Induction step: $l > 2$. If there is no vertex repetition other than first vertex = last vertex, then W traverses a cycle, which is forbidden. Hence there is some other vertex repetition. Let W' be the portion of W between the instances of such a repetition. The walk W' is a shorter closed walk than W and contains no cycle, since W has none. By the induction hypothesis, W' has an edge repeating twice in succession, and this repetition also appears in W.

Proof 2. Let w be the first repetition of a vertex along W, arriving from v on edge e. From the first occurrence of w to the visit to v is a w, v-walk, which is a cycle if $v = w$ or contains a nontrivial w, v-path P. This completes a cycle with e unless in fact P is the path of length 1 with edge e, in which case e repeats immediately in opposite directions.

1.2.16. *If edge e appears an odd number of times in a closed walk W, then W contains the edges of a cycle through e.*

Proof 1 (induction on the length of W, as in Lemma 1.2.7). The shortest closed walk has length 1. Basis step ($l = 1$): The edge e in a closed walk of length 1 is a loop and thus a cycle. Induction step ($l > 1$): If there is no vertex repetition, then W is a cycle. If there is a vertex repetition, choose two appearances of some vertex (other than the beginning and end of the walk). This splits the walk into two closed walks shorter than W. Since each step is in exactly one of these subwalks, one of them uses e an odd number of times. By the induction hypothesis, that subwalk contains the edges of a cycle through e, and this is contained in W.

Proof 2 (parity first, plus Lemma 1.2.6). Let x and y be the endpoints of e. As we traverse the walk, every trip through e is x, e, y or y, e, x. Since the number of trips is odd, the two types cannot alternate. Hence some two successive trips through e have the same direction. By symmetry, we may assume that this is $x, e, y, \ldots, x, e, y$.

The portion of the walk between these two trips through e is a y, x-walk that does not contain e. By Lemma 1.2.6, it contains a y, x-path (that does not contain e. Adding e to this path completes a cycle with e consisting of edges in W.

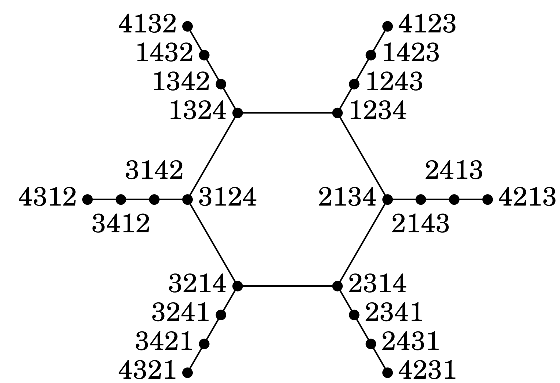
Proof 3 (contrapositive). If edge e in walk W does not lie on a cycle consisting of edges in W, then by our characterization of cut-edges, e is a cut-edge of the subgraph H consisting of the vertices and edges in W. This means that the walk can only return to e at the endpoint from which it most recently left e. This requires the traversals of e to alternate directions along e. Since a closed walk ends where it starts (that is, in the same component of $H - e$), the number of traversals of e by W must be even.

1.2.17. *The "adjacent-transposition graph" G_n on permutations of $[n]$ is connected.* Note that since every permutation of $[n]$ has $n - 1$ adjacent pairs that can be transposed, G_n is $(n-1)$-regular. Therefore, showing that G_n is connected shows that it is Eulerian if and only if n is odd.

Proof 1 (path to fixed vertex). We show that every permutation has a path to the identity permutation $I = 1, \ldots, n$. By the transitivity of the connection relation, this yields for all $u, v \in V(G)$ a u, v-path in G. To create a v, I-path, move element 1 to the front by adjacent interchanges, then move 2 forward to position 2, and so on. This builds a walk to I, which contains a path to I. (Actually, this builds a path.)

Proof 2 (direct u, v-path). Each vertex is a permutation of $[n]$. Let $u = a_1, \ldots, a_n$ and $v = b_1, \ldots, b_n$; we construct at u, v-path. The element b_1 appears in u as some a_i; move it to the front by adjacent transpositions, beginning a walk from u. Next find b_2 among $a_2, \ldots, a_n$ and move it to position 2. Iterating this procedure brings the elements of v toward the front, in order, while following a walk. It reaches v when all positions have been "corrected". (Actually, the walk is a u, v-path.) Note that since we always bring the desired element forward, we never disturb the position of the elements that were already moved to their desired positions.

Proof 3 (induction on n). If $n = 1$, then $G_n \cong K_1$ and G is connected (we can also start with $n = 2$). For $n > 1$, assume that G_{n-1} is connected. In G_n, the subgraph H induced by the vertices having n at the end is isomorphic to G_{n-1}. Every vertex of G is connected to a vertex of H by a path formed by moving element n to the end, one step at a time. For $u, v \in V(G)$, we thus have a path from u to a vertex $u' \in V(H)$, a path from v to a vertex $v' \in V(H)$, and a u', v'-path in H that exists by the induction hypothesis. By the transitivity of the connection relation, there is a u, v-path in G. This completes the proof of the induction step. (The part of G_4 used in the induction step appears below.)



Proof 4 (induction on n). The basis is as in Proof 3. For $n > 1$, note that for each $i \in [n]$, the vertices with i at the end induce a copy H_i of G_{n-1}. By the induction hypothesis, each such subgraph is connected. Also, H_n has vertices with i in position $n - 1$ whenever $1 \le i \le n - 1$. We can interchange the last two positions to obtain a neighbor in H_i. Hence there is an edge from each H_i to H_n, and transitivity of the connection relation again completes the proof.

1.2.18. *For $k \ge 1$, there are two components in the graph G_k whose vertex set is the set of binary k-tuples and whose edge set consists of the pairs that differ in exactly two places.* Changing two coordinates changes the number of 1s in the name of the vertex by zero or by ± 2. Thus the parity of the number of 1s remains the same along every edge. This implies that G_k has at least two components, because there is no edge from an k-tuple with an even number of 1s to an k-tuple with an odd number of 1s.

To show that G_k has at most two components, there are several approaches. In each, we prove that any two vertices with the same parity lie on a path, where "parity" means parity of the number of 1s.

Proof 1. If u and v are vertices with the same parity, then they differ in an even number of places. This is true because each change of a bit in obtaining one label from the other switches the parity. Since they differ in an even number of places, we can change two places at a time to travel from u to v along a path in G_k.

Proof 2. We use induction on k. Basis step ($k = 1$): G_1 has two components, each an isolated vertex. Induction step ($k > 1$): when $k > 1$, G_k consists of two copies of G_{k-1} plus additional edges. The two copies are obtained by appending 0 to all the vertex names in G_{k-1} or appending 1 to them all. Within a copy, the edges don't change, since these vertices all agree in the new place. By the induction hypothesis, each subgraph has two components. The even piece in the 0-copy has $0\cdots000$, which is adjacent to $0\cdots011$ in the odd piece of the 1-copy. The odd piece in the 0-copy has $0\cdots010$, which is adjacent to $0\cdots001$ in the even piece of the 1-copy. Thus the four pieces reduce to (at most) two components in G_k.

1.2.19. *For $n, r, s \in \mathbb{N}$, the simple graph G with vertex set $\mathbb{Z}_n$ and edge set $\{ij\colon |j-i| \in \{r, s\}\}$ has* $\gcd(n, r, s)$ *components.* **Note**: The text gives the vertex set incorrectly. When $r = s = 2$ and n is odd, it is necessary to go up to $n \equiv 0$ to switch from odd to even.

Let $k = \gcd(n, r, s)$. Since k divides n, the congruence classes modulo n fall into congruence classes modulo k in a well-defined way. All neighbors of vertex i differ from i by a multiple of k. Thus all vertices in a component lie in the same congruence class modulo k, which makes at least k components.

To show that there are only k components, we show that all vertices with indices congruent to $i \pmod k$ lie in one component (for each i). It suffices to build a path from i to $i+k$. Let $l = \gcd(r, s)$, and let $a = r/l$ and $b = s/l$. Since there are integers (one positive and one negative) such that $pa + qb = 1$, moving p edges with difference $+r$ and q edges with difference $+s$ achieves a change of $+l$.

We thus have a path from i to $i + l$, for each i. Now, $k = \gcd(l, n)$. As above, there exist integers p', q' such that $p'(l/k) + q'(n/k) = 1$. Rewriting this as $p'l = k - q'n$ means that if we use p' of the paths that add l, then we will have moved from i to $i + k \pmod n$.

1.2.20. *If v is a cut-vertex of a simple graph G, then v is not a cut-vertex of $\overline{G}$.* Let $V_1, \ldots, V_k$ be the vertex sets of the components of $G - v$; note that $k \ge 2$. Then $\overline{G}$ contains the complete multipartite graph with partite sets $V_1, \ldots, V_k$. Since this includes all vertices of $\overline{G} - v$, the graph $\overline{G} - v$ is connected. Hence v is not a cut-vertex of $\overline{G}$.

1.2.21. *A self-complementary graph has a cut-vertex if and only if it has a vertex of degree 1.* If there is a vertex of degree 1, then its neighbor is a cut-vertex (K_2 is not self-complementary).

For the converse, let v be a cut-vertex in a self-complementary graph G. The graph $\overline{G - v}$ has a *spanning biclique*, meaning a complete bipartite subgraph that contains all its vertices. Since G is self-complementary, also G must have a vertex u such that $G - u$ has a spanning biclique.

Since each vertex of $G - v$ is nonadjacent to all vertices in the other components of $G - v$, a vertex other than u must be in the same partite set of the spanning biclique of $G - u$ as the vertices not in the same component as u in $G - v$. Hence only v can be in the other partite set, and v has degree at least $n - 2$. We conclude that v has degree at most 1 in $\overline{G}$, so G has a vertex of degree at most 1. Since a graph and its complement cannot both be disconnected, G has a vertex of degree 1.

1.2.22. *A graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.*

Necessity. Let G be a connected graph. Given a partition of $V(G)$ into nonempty sets S, T, choose $u \in S$ and $v \in T$. Since G is connected, G has a u, v-path P. After its last vertex in S, P has an edge from S to T.

Sufficiency.

Proof 1 (contrapositive). We show that if G is not connected, then for some partition there is no edge across. In particular, if G is disconnected, then let H be a component of G. Since H is a maximal connected subgraph of G and the connection relation is transitive, there cannot be an edges with one endpoint in $V(H)$ and the other endpoint outside. Thus for the partition of $V(G)$ into $V(H)$ and $V(G) - V(H)$ there is no edge with endpoints in both sets.

Proof 2 (algorithmic approach). We grow a set of vertices that lie in the same equivalence class of the connection relation, eventually accumulating all vertices. Start with one vertex in S. While S does not include all vertices, there is an edge with endpoints $x \in S$ and $y \notin S$. Adding y to S produces a larger set within the same equivalence class, using the transitivity of the connection relation. This procedure ends only when there are no more vertices outside S, in which case all of G is in the same equivalence class, so G has only one component.

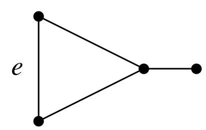
Proof 3 (extremality). Given a vertex $x \in V(G)$, let S be the set of all vertices that can be reached from x via paths. If $S \ne V(G)$, consider the partition into S and $V(G) - S$. By hypothesis, G has an edge with endpoints

$u \in S$ and $v \notin S$. Now there is an x,v-path formed by extending an x,u-path along the edge uv. This contradicts the choice of S, so in fact S is all of $V(G)$. Since there are paths from x to all other vertices, the transitivity of the connection relation implies that G is connected.

1.2.23. *a) If a connected simple graph G is not a complete graph, then every vertex of G belongs to some induced subgraph isomorphic to P_3.* Let v be a vertex of G. If the neighborhood of v is not a clique, then v has a pair x, y of nonadjacent neighbors; $\{x, v, y\}$ induces P_3. If the neighborhood of v is a clique, then since G is not complete there is some vertex y outside the set S consisting of v and its neighbors. Since G is connected, there is some edge between a neighbor w of v and a vertex x that is not a neighbor of v. Now the set $\{v, w, x\}$ induces P_3, since x is not a neighbor of v.

One can also use cases according to whether v is adjacent to all other vertices or not. The two cases are similar to those above.

b) When a connected simple graph G is not a complete graph, G may have edges that belong to no induced subgraph isomorphic to P_3. In the graph below, e lies in no such subgraph.



1.2.24. *If a simple graph with no isolated vertices has no induced subgraph with exactly two edges, then it is a complete graph.* Let G be such a graph. If G is disconnected, then edges from two components yield four vertices that induce a subgraph with two edges. If G is connected and not complete, then G has nonadjacent vertices x and y. Let Q be a shortest x, y-path; it has length at least 2. Any three successive vertices on Q induce P_3, with two edges.

Alternatively, one can use proof by contradiction. If G is not complete, then G has two nonadjacent vertices. Considering several cases (common neighbor or not, etc.) always yields an induced subgraph with two edges.

1.2.25. *Inductive proof that every graph G with no odd cycles is bipartite.*

Proof 1 (induction on $e(G)$). Basis step ($e(G) = 0$): Every graph with no edges is bipartite, using any two sets covering $V(G)$.

Induction step ($e(G) > 0$): Discarding an edge e introduces no odd cycles. Thus the induction hypothesis implies that $G - e$ is bipartite.

If e is a cut-edge, then combining bipartitions of the components of $G - e$ so that the endpoints of e are in opposite sets produces a bipartition of G. If e is not a cut-edge of G, then let u and v be its endpoints, and let X, Y be a bipartition of $G - e$. Adding e completes a cycle with a u, v-path in $G - e$; by hypothesis, this cycle has even length. This forces u and v to be in opposite sets in the bipartition X, Y. Hence the bipartition X, Y of $G - e$ is also a bipartition of G.

Proof 2 (induction on $n(G)$). Basis step ($n(G) = 1$): A graph with one vertex and no odd cycles has no loop and hence no edge and is bipartite.

Induction step ($n(G) > 1$): When we discard a vertex v, we introduce no odd cycles. Thus the induction hypothesis implies that $G - v$ is bipartite. Let $G_1, \ldots, G_k$ be the components of $G - v$; each has a bipartition. If v has neighbors u, w in both parts of the bipartition of G_i, then the edges uv and vw and a shortest u, w-path in G_i form a cycle of odd length. Hence we can specify the bipartition X_i, Y_i of G_i so that X_i contains all neighbors of v in G_i. We now have a bipartition of G by letting $X = \bigcup X_i$ and $Y = \{v\} \cup (\bigcup Y_i)$.

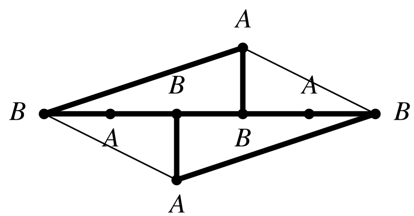
1.2.26. *A graph G is bipartite if and only if for every subgraph H of G, there is an independent set containing at least half of the vertices of H.* Every bipartite graph has a vertex partition into two independent sets, one of which must contain at least half the vertices (though it need not be a maximum independent set). Since every subgraph of a bipartite graph is bipartite, the argument applies to all subgraphs of a bipartite graph, and the condition is necessary.

For the converse, suppose that G is not bipartite. By the characterization of bipartite graphs, G contains an odd cycle H. This subgraph H has no independent set containing at least half its vertices, because every set consisting of at least half the vertices in an odd cycle must have two consecutive vertices on the cycle.

1.2.27. *The "transposition graph" on permutations of $[n]$ is bipartite.* The partite sets are determined by the parity of the number of pairs i, j such that $i < j$ and $a_i > a_j$ (these are called **inversions**). We claim that each transposition changes the parity of the number of inversions, and therefore each edge in the graph joins vertices with opposite parity. Thus the permutations with an even number of inversions form an independent set, as do those with an odd number of inversions. This is a bipartition, and thus the graph is bipartite.

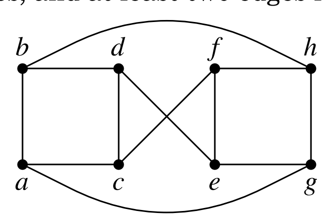
Consider the transposition that interchanges the elements in position r and position s, with $r < s$. No pairs involving elements that are before r or after s have their order changed. If $r < k < s$, then interchanging a_r and a_s changes the order of a_r and a_k, and also it changes the order of a_k and a_s. Thus for each such k the number of inversions changes twice and retains the same parity. This describes all changes in order except for the switch of a_r and a_s itself. Thus the total number of changes is odd, and the parity of the number of inversions changes.

1.2.28. *a) The graph below has a unique largest bipartite subgraph, obtained by deleting the central edge.* Deleting the central edge leaves a bipartite subgraph, since the indicated sets A and B are independent in that subgraph. If deleting one edge makes a graph bipartite, then that edge must belong to all odd cycles in the graph, since a bipartite subgraph has no odd cycles. The two odd cycles in bold have only the central edge in common, so no other edge belongs to all odd cycles.



b) In the graph below, the largest bipartite subgraph has 10 edges, and it is not unique. Deleting edges bh and ag yields an X, Y-bigraph with $X = \{b, c, e, h\}$ and $Y = \{a, d, f, g\}$. Another bipartite subgraph with 10 edges is obtained by deleting edges de and cf; the bipartition is $X = \{b, c, f, g\}$ and $Y = \{a, d, e, h\}$. (Although these two subgraphs are isomorphic, they are two subgraphs, just as the Petersen graph has ten claws, not one.)

It remains to show that we must delete at least two edges to obtain a bipartite subgraph. By the characterization of bipartite graphs, we must delete enough edges to break all odd cycles. We can do this with (at most) one edge if and only if all the odd cycles have a common edge. The 5-cycles (b, a, c, f, h) and (b, d, e, g, h) have only the edge bh in common. Therefore, if there is a single edge lying in all odd cycles, it must be bh. However, (a, c, f, h, g) is another 5-cycle that does not contain this. Therefore no edge lies in all odd cycles, and at least two edges must be deleted.



1.2.29. *A connected simple graph not having P_4 or C_3 as an induced subgraph is a biclique.* Choose a vertex x. Since G has no C_3, $N(x)$ is independent. Let $S = V(G) - N(X) - \{x\}$. Every $v \in S$ has a neighbor in $N(x)$; otherwise, a shortest v, x-path contains an induced P_4. If $v \in S$ is adjacent to w but not z in $N(x)$, then v, w, x, z is an induced P_4. Hence all of S is adjacent to all of $N(x)$. Now $S \cup \{x\}$ is an independent set, since G has no C_3. We have proved that G is a biclique with bipartition $N(x)$, $S \cup \{x\}$.

1.2.30. *Powers of the adjacency matrix.*

a) In a simple graph G, the (i, j)th entry in the kth power of the adjacency matrix $\mathbf{A}$ is the number of (v_i, v_j)-walks of length k in G. We use induction on k. When $k = 1$, $a_{i,j}$ counts the edges (walks of length 1) from i to j. When $k > 1$, every (v_i, v_j)-walk of length k has a unique vertex v_r reached one step before the end at v_j. By the induction hypothesis, the number of (v_i, v_r)-walks of length $k-1$ is entry (i, r) in $\mathbf{A}^{k-1}$, which we write as $a_{i,r}^{(k-1)}$. The number of (v_i, v_j)-paths of length k that arrive via v_r on the last step is $a_{i,r}^{(k-1)} a_{r,j}$, since $a_{r,j}$ is the number of edges from v_r to v_j that can complete the walk. Counting the (v_i, v_j)-walks of length k by which vertex appears one step before v_j yields $\sum_{r=1}^{n} a_{i,r}^{(k-1)} a_{r,j}$. By the definition of matrix multiplication, this is the (i, j)th entry in $\mathbf{A}^k$. (The proof allows loops and multiple edges and applies without change for digraphs. When loops are present, note that there is no choice of "direction" on a loop; a walk is a list of edge traversals).

b) A simple graph G with adjacency matrix A is bipartite if and only if, for each odd integer r, the diagonal entries of the matrix A^r are all 0. By part (a), $A_{i,i}^r$ counts the closed walks of length r beginning at v_i. If this is always 0, then G has no closed walks of odd length through any vertex; in particular, G has no odd cycle and is bipartite. Conversely, if G is bipartite, then G has no odd cycle and hence no closed odd walk, since every closed odd walk contains an odd cycle.

1.2.31. *K_n is the union of k bipartite graphs if and only if $n \le 2^k$ (without using induction).*

a) Construction when $n \le 2^k$. Given $n \le 2^k$, encode the vertices of K_n as distinct binary k-tuples. Let G_i be the complete bipartite subgraph with bipartition X_i, Y_i, where X_i is the set of vertices whose codes have 0 in position i, and Y_i is the set of vertices whose codes have 1 in position i. Since every two vertex codes differ in some position, $G_1 \cup \cdots \cup G_k = K_n$.

b) Upper bound. Given that K_n is a union of bipartite graphs $G_1, \ldots, G_k$, we define a code for each vertex. For $1 \le i \le k$, let X_i, Y_i be a bipartition of G_i. Assign vertex v the code $(a_1, \ldots, a_k)$, where $a_i = 0$ if $v \in X_i$, and $a_i = 1$ if $v \in Y_i$ or $v \notin X_i \cup Y_i$. Since every two vertices are adjacent and the edge joining them must be covered in the union, they lie in opposite partite sets in some G_i. Therefore the codes assigned to the vertices are distinct. Since the codes are binary k-tuples, there are at most 2^k of them, so $n \le 2^k$.

1.2.32. *"Every maximal trail in an even graph is an Eulerian circuit"—FALSE.* When an even graph has more than one component, each component has a maximal trail, and it will not be an Eulerian circuit unless the other components have no edges. The added hypothesis needed is that the graph is connected.

The proof of the corrected statement is essentially that of Theorem 1.2.32. If a maximal trail T is not an Eulerian circuit, then it is incident to a missing edge e, and a maximal trail in the even graph $G - E(T)$ that starts at e can be inserted to enlarge T, which contradicts the hypothesis that T is a maximal trail.

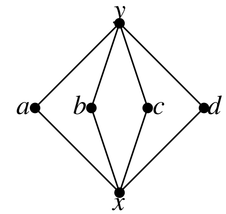
1.2.33. *The edges of a connected graph with $2k$ odd vertices can be partitioned into k trails if $k > 0$.* The assumption of connectedness is necessary, because the conclusion is not true for $G = H_1 + H_2$ when H_1 has some odd vertices and H_2 is Eulerian.

Proof 1 (induction on k). When $k = 1$, we add an edge between the two odd vertices, obtain an Eulerian circuit, and delete the added edge. When $k > 1$, let P be a path connecting two odd vertices. The graph $G' = G - E(P)$ has $2k - 2$ odd vertices, since deleting $E(P)$ changes degree parity only at the ends of P. The induction hypothesis applies to each component of G' that has odd vertices. Any component not having odd vertices has an Eulerian circuit that contains a vertex of P; we splice it into P to avoid having an additional trail. In total, we have used the desired number of trails to partition $E(G)$.

Proof 2 (induction on $e(G)$). If $e(G) = 1$, then $G = K_2$, and we have one trail. If G has an even vertex x adjacent to an odd vertex y, then $G' = G - xy$ has the same number of odd vertices as G. The trail decomposition of G' guaranteed by the induction hypothesis has one trail ending at x and no trail ending at y. Add xy to the trail ending at x to obtain the desired decomposition of G. If G has no even vertex adjacent to an odd vertex, then G is Eulerian or every vertex of G is odd. In this case, deleting an edge xy reduces k, and we can add xy as a trail of length one to the decomposition of $G - xy$ guaranteed by the induction hypothesis.

1.2.34. *The graph below has 6 equivalence classes of Eulerian circuits.* If two Eulerian circuits follow the same circular arrangement of edges, differing only in the starting edges or the direction, then we consider them equivalent. An equivalence class of circuits is characterized by the pairing of edges at each vertex corresponding to visits through that vertex.

A 2-valent vertex has exactly one such pairing; a 4-valent vertex has three possible pairings. The only restriction is that the pairings must yield a single closed trail. Given a pairing at one 4-valent vertex below, there is a forbidden pairing at the other, because it would produce two edge-disjoint 4-cycles instead of a single trail. The other two choices are okay. Thus the answer is $3 \cdot 2 = 6$.



Alternatively, think of making choices while following a circuit. Because each circuit uses each edge, and because the reversal of a circuit C is in the same class as C, we may follow a canonical representative of the class from a along ax. We now count the choices made to determine the circuit. After x we can follow one of 3 choices. This leads us through another neighbor of x to y. Now we cannot use the edge ya or the edge just used, so two choices remain. This determines the rest of the circuit. For each of the three ways to make the initial choice, there was a choice of two later, so there are $3 \cdot 2 = 6$ ways to specify distinct classes of circuits. (Distinct ways of making the choices yields a distinct pairing at some vertex.)

1.2.35. *Algorithm for Eulerian circuits.* Let G be a connected even graph. At each vertex partition the incident edges into pairs (each edge appears in a pair at each endpoint). Start along some edge. At each arrival at a vertex, there is an edge paired with the entering edge; use it to exit. This can end only by arriving at the initial vertex along the edge paired with the initial edge, and it must end since the graph is finite. At the point where the first edge would be repeated, stop; this completes a closed trail. Furthermore, there is no choice in assembling this trail, so every edge appears in exactly one such trail. Therefore, the pairing decomposes G into closed trails.

If there is more than one trail in the decomposition, then there are two trails with a common vertex, since G is connected. (A shortest path connecting vertices in two of the trails first leaves the first trail at some vertex v, and at v we have edges from two different trails.) Given edges from trails A and B at v, change the pairing by taking a pair in A and a pair in B and switching them to make two pairs that pair an edge of A with an edge of B. Now when A is followed from v, the return to A does not end the trail, but rather the trail continues and follows B before returning to the original edge. Thus changing the pairing at v combines these two trails into one trail and leaves the other trails unchanged.

We have shown that if the number of trails in the decomposition exceeds one, then we can obtain a decomposition with fewer trails be changing the pairing. Repeating the argument produces a decomposition using one closed trail. This trail is an Eulerian circuit.

1.2.36. *Alternative characterization of Eulerian graphs.*

a) If G is loopless and Eulerian and $G' = G - uv$, then G' has an odd number of u, v-trails that visit v only at the end.

Proof 1 (exhaustive counting and parity). Every extension of every trail from u in G' eventually reaches v, because a maximal trail ends only at a vertex of odd degree. We maintain a list of trails from u. The number of choices for the first edge is odd. For a trail T that has not yet reached v, there are an odd number of ways to extend T by one edge. We replace T in the list by these extensions. This changes the number of trails in the list by an even number. The process ends when all trails in the list end at v. Since the list always has odd size, the total number of these trails is odd.

Proof 2 (induction and stronger result). We prove that the same conclusion holds whenever u and v are the only vertices of odd degree in a graph H, regardless of whether they are adjacent. This is immediate if H has only the edge uv. For larger graphs, we show that there are an odd number of such trails starting with each edge e incident to u, so the sum is odd. If $e = uv$, then there is one such trail. Otherwise, when $e = uw$ with $w \neq v$, we apply the induction hypothesis to $H - e$, in which w and v are the only vertices of odd degree.

The number of non-paths in this list of trails is even. If T is such a trail that is not a path, then let w be the first instance of a vertex repetition on T. By traversing the edges between the first two occurrences of w in the opposite order, we obtain another trail T' in the list. For T', the first instance of a vertex repetition is again w, and thus $T'' = T$. This defines an involution under which the fixed points are the u, v-paths. The trails we wish to delete thus come in pairs, so there are an even number of them.

b) If v is a vertex of odd degree in a graph G, then some edge incident to v lies in an even number of cycles. Let $c(e)$ denote the number of cycles containing e. Summing $c(e)$ over edges incident to v counts each cycle through v exactly twice, so the sum is even. Since there are an odd number of terms in the sum, $c(e)$ must be even for some e incident to v.

c) A nontrivial connected graph is Eulerian if and only if every edge belongs to an odd number of cycles. Necessity: By part (a), the number of u, v-paths in $G - uv$ is odd. The cycles through uv in G correspond to the u, v-paths in $G - uv$, so the number of these cycles is odd.

Sufficiency: We observe the contrapositive. If G is not Eulerian, then G has a vertex v of odd degree. By part (b), some edge incident to v lies in an even number of cycles.

1.2.37. *The connection relation is transitive.* It suffices to show that if P is a u, v-path and P' is a v, w-path, then P and P' together contain a u, w-path. At least one vertex of P is in P', since both contain v. Let x be the first vertex of P that is in P'. Following P from u to x and then P' from x to w yields a u, w path, since no vertex of P before x belongs to P'.

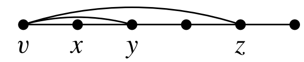
1.2.38. *Every n-vertex graph with at least n edges contains a cycle.*

Proof 1 (induction on n). A graph with one vertex that has an edge has a loop, which is a cycle. For the induction step, suppose that $n > 1$. If our graph G has a vertex v with degree at most 1, then $G - v$ has $n - 1$ vertices and at least $n - 1$ edges. By the induction hypothesis, $G - v$ contains a cycle, and this cycle appears also in G. If G has no vertex of degree at most 1, then every vertex of G has degree at least 2. Now Lemma 1.2.25 guarantees that G contains a cycle.

Proof 2 (use of cut-edges). If G has no cycle, then by Theorem 1.2.14 every edge is a cut-edge, and this remains true as edges are deleted. Deleting all the edges thus produces at least $n + 1$ components, which is impossible.

1.2.39. *If G is a loopless graph and $\delta(G) \geq 3$, then G has a cycle of even length.* An endpoint v of a maximal path P has at least three neighbors on P. Let x, y, z be three such neighbors of v in order on P. Consider three v, y-paths: the edge vy, the edge vx followed by the x, y-path in P, and the edge vz followed by the z, y-path in P.

These paths share only their endpoints, so the union of any two is a cycle. By the pigeonhole principle, two of these paths have lengths with the same parity. The union of these two paths is an even cycle.



1.2.40. *If P and Q are two paths of maximum length in a connected graph G, then P and Q have a common vertex.* Let m be the common length of P and Q. Since G is connected, it has a shortest path R between $V(P)$ and $V(Q)$. Let l be the length of R. Let the endpoints of R be $r \in V(P)$ and $r' \in V(Q)$. The portion P' of P from r to the farther endpoint has length at least $m/2$. The portion Q' of Q from r to the farther endpoint has length at least $m/2$. Since R is a shortest path, R has no internal vertices in P or Q.

If P and Q are disjoint, then P' and Q' are disjoint, and the union of P', Q', and R is a path of length at least $m/2 + m/2 + l = m + l$. Since the maximum path length is m, we have $l = 0$. Thus $r = r'$, and P and Q have a common vertex.

The graph consisting of two edge-disjoint paths of length $2k$ sharing their midpoint is connected and hence shows that P and Q need not have a common edge.

1.2.41. *A connected graph with at least three vertices has two vertices x, y such that 1) $G - \{x, y\}$ is connected and 2) x, y are adjacent or have a common neighbor.* Let x be a endpoint of a longest path P in G, and let v be its neighbor on P. Note that P has at least three vertices. If $G - x - v$ is connected, let $y = v$. Otherwise, a component cut off from $P - x - v$ in $G - x - v$ has at most one vertex; call it w. The vertex w must be adjacent to v, since otherwise we could build a longer path. In this case, let $y = w$.

1.2.42. *A connected simple graph having no 4-vertex induced subgraph that is a path or a cycle has a vertex adjacent to every other vertex.* Consider a vertex x of maximum degree. If x has a nonneighbor y, let x, v, w be the begining of a shortest path to y (w may equal y). Since $d(v) \le d(x)$, some neighbor z of x is not adjacent to v. If $z \leftrightarrow w$, then $\{z, x, v, w\}$ induce C_4; otherwise, $\{z, x, v, w\}$ induce P_4. Thus x must have no nonneighbor.

1.2.43. *The edges of a connected simple graph with $2k$ edges can be partitioned into paths of length 2.* The assumption of connectedness is necessary, since the conclusion does not hold for a graph having components with an odd number of edges.

We use induction on $e(G)$; there is a single such path when $e(G) = 2$. For $e(G) > 2$, let $P = (x, y, z)$ be an arbitrary path of length two in G, and let $G' = G - \{xy, yz\}$. If we can partition $E(G)$ into smaller connected subgraphs of even size, then we can apply the induction hypothesis to each piece and combine the resulting decompositions. One way to do this is to partition $E(G')$ into connected subgraphs of even size and use P.

Hence we are finished unless G' has two components of odd size (G' cannot have more than three components, since an edge deletion increases the number of components by at most one). Each odd component contains at least one of $\{x, y, z\}$. Hence it is possible to add one of xy to one odd component and yz to the other odd component to obtain a partition of G into smaller connected subgraphs.

1.3. VERTEX DEGREES & COUNTING

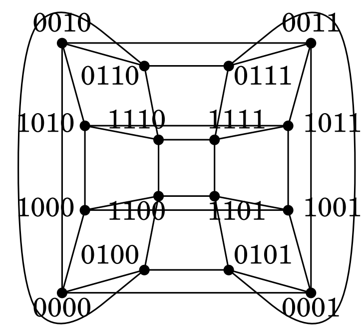
1.3.1. *A graph having exactly two vertices of odd degree must contain a path from one to the other.* The degree of a vertex in a component of G is the same as its degree in G. If the vertices of odd degree are in different components, then those components are graphs with odd degree sum.

1.3.2. *In a class with nine students where each student sends valentine cards to three others, it is not possible that each student sends to and receives cards from the same people.* The sending of a valentine can be represented as a directed edge from the sender to the receiver. If each student sends to and receives cards from the same people, then the graph has $x \to y$ if and

only if $y \to x$. Modeling each opposed pair of edges by a single unoriented edge yields a 3-regular graph with 9 vertices. This is impossible, since every graph has an even number of vertices of odd degree.

1.3.3. *If $d(u) + d(v) = n + k$ for an edge uv in a simple graph on n vertices, then uv belongs to at least k triangles.* This is the same as showing that u and v have at least k common neighbors. Let S be the neighbors of u and T the neighbors of v, and suppose $|S \cap T| = j$. Every vertex of G appears in S or T or none or both. Common neighbors are counted twice, so $n \ge |S| + |T| - j = n + k - j$. Hence $j \ge k$. (Almost every proof of this using induction or contradiction does not need it, and is essentially just this counting argument.)

1.3.4. *The graph below is isomorphic to Q_4.* It suffices to label the vertices with the names of the vertices in Q_4 so that vertices are adjacent if and only if their labels differ in exactly one place.



1.3.5. *The k-dimensional cube Q_k has $\binom{k}{2}2^k$ copies of P_3.*

Proof 1. To specify a particular subgraph isomorphic to P_3, the 3-vertex path, we can specify the middle vertex and its two neighbors. For each vertex of Q_k, there are $\binom{k}{2}$ ways to choose two distinct neighbors, since Q_k is a simple k-regular graph. Thus the total number of P_3's is $\binom{k}{2}2^k$.

Proof 2. We can alternatively choose the starting vertex and the next two. There are 2^k ways to pick the first vertex. For each vertex, there are k ways to pick a neighbor. For each way to pick these vertices, there are $k - 1$ ways to pick a third vertex completing P_3, since Q_k has no multiple edges. The product of these factors counts each P_3 twice, since we build it from each end. Thus the total number of them is $2^k k(k-1)/2$.

Q_k has $\binom{k}{2}2^{k-2}$ copies of C_4.

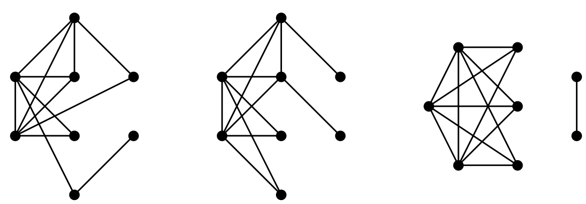
Proof 1 (direct counting). The vertices two apart on a 4-cycle must differ in two coordinates. Their two common neighbors each differ from each in exactly one of these coordinates. Hence the vertices of a 4-cycle must use all 2-tuples in two coordinates while keeping the remaining coordinates fixed. All such choices yield 4-cycles. There are $\binom{k}{2}$ ways to choose the two coordinates that vary and 2^{k-2} ways to set a fixed value in the remaining coordinates.

Proof 2 (prior result). Every 4-cycle contains four copies of P_3, and every P_3 contains two vertices at distance 2 in the cube and hence extends to exactly one 4-cycle. Hence the number of 4-cycles is one-fourth the number of copies of P_3.

1.3.6. *Counting components.* If G has k components and H has l components, then $G + H$ has $k + l$ components. The maximum degree of $G + H$ is $\max\{\Delta(G), \Delta(H)\}$.

1.3.7. *Largest bipartite subgraphs.* P_n is already bipartite. C_n loses one edge if n is odd, none if n is even. The largest bipartite subgraph of K_n is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, which has $\lfloor n^2/4 \rfloor$ edges.

1.3.8. *The lists (5,5,4,3,2,2,2,1), (5,5,4,4,2,2,1,1), and (5,5,5,3,2,2,1,1) are graphic, but (5,5,5,4,2,1,1,1) is not.* The answers can be obtained from the Havel-Hakimi test; a list is graphic if and only if the list obtained by deleting the largest element and deleting that many next-largest elements is graphic. Below are graphs realizing the first three lists, found by the Havel-Hakimi algorithm.



From the last list, we test $(4, 4, 3, 1, 0, 1, 1)$, reordered to $(4, 4, 3, 1, 1, 1, 0)$, then $(3, 2, 0, 0, 1, 0)$. This is not the degree list of a simple graph, since a vertex of degree 3 requires three other vertices with nonzero degree.

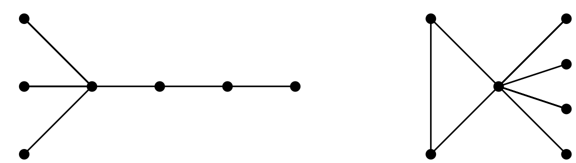
1.3.9. *In a league with two divisions of 13 teams each, no schedule has each team playing exactly nine games against teams in its own division and four games against teams in the other division.* If this were possible, then we could form a graph with the teams as vertices, making two vertices adjacent if those teams play a game in the schedule. We are asking for the subgraph induced by the 13 teams in a single division to be 9-regular. However, there is no regular graph of odd degree with an odd number of vertices, since for every graph the sum of the degrees is even.

1.3.10. *If l, m, n are nonnegative integers with $l + m = n \geq 1$, then there exists a connected simple n-vertex graph with l vertices of even degree and m*

vertices of odd degree if and only if m is even, except for $(l, m, n) = (2, 0, 2)$. Since every graph has an even number of vertices of odd degree, and the only simple connected graph with two vertices has both degrees odd, the condition is necessary.

To prove sufficiency, we construct such a graph G. If $m = 0$, let $G = C_l$ (except $G = K_1$ if $l = 1$). For $m > 0$, we can begin with $K_{1,m-1}$, which has m vertices of odd degree, and then add a path of length l beyond one of the leaves. (Illustration shows $l = 3$, $m = 4$.)

Alternatively, start with a cycle of length l, and add m vertices of degree one with a common neighbor on the cycle. That vertex of the cycle has even degree because m is even. Many other constructions also work. It is also possible to prove sufficiency by induction on n for $n \geq 3$, but this approach is longer and harder to get right than an explicit general construction.



1.3.11. *If C is a closed walk in a simple graph G, then the subgraph consisting of the edges appearing an odd number of times in C is an even graph.* Consider an arbitrary vertex $v \in V(G)$. Let S be the set of edges incident to v, and let $f(e)$ be the number of times an edge e is traversed by C. Each time C passes through v it enters and leaves. Therefore, $\sum_{e \in S} f(e)$ must be even, since it equals twice the number of times that C visits v. Hence there must an even number of odd contributions to the sum, which means there are an even number of edges incident to v that appear an odd number of times in C. Since we can start a closed walk at any of its vertices, this argument holds for every $v \in V(G)$.

1.3.12. *If every vertex of G has even degree, then G has no cut-edge.*

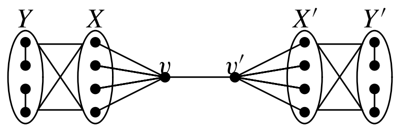
Proof 1 (contradiction). If G has a cut-edge, deleting it leaves two induced subgraphs whose degree sum is odd. This is impossible, since the degree sum in every graph is even.

Proof 2 (construction/extremality). For an edge uv, a maximal trail in $G - uv$ starting at u can only end at v, since whenever we reach a vertex we have use an odd number of edges there. Hence a maximal such trail is a (u, v)-trail. Every (u, v)-trail is a (u, v)-walk and contains a (u, v)-path. Hence there is still a (u, v)-path after deletion of uv, so uv is not a cut-edge.

Proof 3 (prior results). Let G be an even graph. By Proposition 1.2.27, G decomposes into cycles. By the meaning of "decomposition", every edge of G is in a cycle. By Theorem 1.2.14, every edge in a cycle is not a cut-edge. Hence every edge of G is not a cut-edge.

For $k \in \mathbb{N}$, some $(2k+1)$-regular simple graph has a cut-edge.

Construction 1. Let H, H' be copies of $K_{2k,2k}$ with partite sets X, Y for H and X', Y' for H'. Add an isolated edge vv' disjoint from these sets. To $H + H' + vv'$, add edges from v to all of X and from v' to all of X', and add k disjoint edges within Y and k disjoint edges within Y'. The resulting graph G_k is $(2k+1)$-regular with $8k+2$ vertices and has vv' as a cut-edge. Below we sketch G_2; the graph G_1 is the graph in Example 1.3.26.



Construction 2a (inductive). Let G_1 be the graph at the end of Example 1.3.26 (or in Construction 1). This graph is 3-regular with 10 vertices and cut-edge xy; note that $10 = 4 \cdot 1 + 6$. From a $(2k-1)$-regular graph G_{k-1} with $4k+2$ vertices such that $G_{k-1} - xy$ has two components of order $2k+1$, we form G_k. Add two vertices for each component of $G_{k-1} - xy$, adjacent to all the vertices of that component. This adds degree two to each old vertex, gives degree $2k+1$ to each new vertex, and leaves xy as a cut-edge. The result is a $(2k+1)$-regular graph G_k of order $4k+6$ with cut-edge xy.

Construction 2b (explicit). Form H_k from K_{2k+2} by removing k pairwise disjoint edges and adding one vertex that is adjacent to all vertices that lost an incident edge. Now H_k has $2k+2$ vertices of degree $2k+1$ and one of degree $2k$. Form G_k by taking two disjoint copies of H_k and adding an edge joining the vertices of degree $2k$. The graphs produced in Constructions 2a and 2b are identical.

1.3.13. *Meeting on a mountain range.* A **mountain range** is a polygonal curve from $(a,0)$ to $(b,0)$ in the upper half-plane; we start A and B at opposite endpoints. Let P be a highest peak; A and B will meet there. Let the segments from P to $(a,0)$ be $x_1, \ldots, x_r$, and let the segments from P to $(b,0)$ be $y_1, \ldots, y_s$. We define a graph to describe the positions; when A is on x_i and B is on y_j, the corresponding vertex is (i,j). We start at the vertex (r,s) and must reach $(1,1)$. We introduce edges for the possible transitions. We can move from (i,j) to $(i,j+1)$ if the common endpoint of y_j and y_{j+1} has height between the heights of the endpoints of x_i. Similarly, (i,j) is adjacent to $(i+1,j)$ if the common endpoint of x_i and x_{i+1} has height between the heights of the endpoints of x_j. To avoid triviality, we may assume that $r+s>2$.

We prove that (r,s) and $(1,1)$ are the only vertices of odd degree in G. This suffices, because every graph has an even number of vertices of

odd degree, which implies that (r,s) and $(1,1)$ are in the same component, connected by a path.

The possible neighbors of (i,j) are the pairs obtained by changing i or j by 1. Let X and Y be the intervals of heights attained by x_i and y_j, and let $I = X \cap Y$. If the high end of I is the high end of exactly one of X and Y, then exactly one neighboring vertex can be reached by moving past the end of the corresponding segment. If it is the high end of both, then usually one or three neighboring vertices can be reached, the latter when both segments reach "peaks" at their high ends. However, if $(i,j) = (1,1)$, then the high end of both segments is P and there is no neighbor of this type. Similarly, the low end of I generates one or three neighbors, except that when $(i,j) = (r,s)$ there is no neighbor of this type.

No neighbor of (i,j) is generated from both the low end and the high end of I. Since the contributions from the high and low end of I to the degree of (i,j) are both odd, each degree is even, except for (r,s) and $(1,1)$, where exactly one of the contributions is odd.

1.3.14. *Every simple graph with at least two vertices has two vertices of equal degree.* The degree of a vertex in an n-vertex simple graph is in $\{0,\ldots,n-1\}$. These are n distinct values, so if no two are equal then all appear. However, a graph cannot have both an isolated vertex and a vertex adjacent to all others.

This does not hold for graphs allowing loops. In the 2-vertex graph with one loop edge and one non-loop edge, the vertex degrees are 1 and 3.

This does not hold for loopless graphs. In the 3-vertex loopless graph with pairs having multiplicity 0, 1, 2, the vertex degrees are 1, 3, 2.

1.3.15. *Smallest k-regular graphs.* A simple k-regular graph has at least $k+1$ vertices, so K_{k+1} is the smallest. This is the only isomorphism class of k-regular graphs with $k+1$ vertices. With $k+2$ vertices, the complement of a k-regular graph must be 1-regular. There is one such class when k is even ($(k+2)/2$ isolated edges), none when k is odd. (Two graphs are isomorphic if and only if their complements are isomorphic.)

With $k+3$ vertices, the complement is 2-regular. For $k \ge 3$, there are distinct choices for such a graph: a $(k+3)$-cycle or the disjoint union of a 3-cycle and a k-cycle. Since these two 2-regular graphs are nonisomorphic, their complements are nonisomorphic k-regular graphs with $k+3$ vertices.

1.3.16. *For $k \ge 2$ and $g \ge 2$, there exists a k-regular graph with girth g.* We use strong induction on g. For $g = 2$, take the graph consisting of two vertices and k edges joining them.

For the induction step, consider $g > 2$. Here we use induction on k. For $k = 2$, a cycle of length g suffices. For $k > 2$, the induction hypothesis provides a $(k-1)$-regular graph H with girth g. Since $\lceil g/2 \rceil < g$, the global induction hypothesis also provides a graph G with girth $\lceil g/2 \rceil$ that is $n(H)$-regular. Replace each vertex v in G with a copy of H; each vertex in the copy of H is made incident to one of the edges incident to v in G.

Each vertex in the resulting graph inherits $k-1$ incident edges from H and one from G, so the graph is k-regular. It has cycles of length g in copies of H. A cycle C in G is confined to a single copy of H or visits more than one such copy. In the first case, its length is at least g, since H has girth g. In the second case, the copies of H that C visits correspond to a cycle in G, so C visits at least $\lceil g/2 \rceil$ such copies. For each copy, C must enter on one edge and then move to another vertex before leaving, since the copy is entered by only one edge at each vertex. Hence the length of such a cycle is at least $2\lceil g/2 \rceil$.

1.3.17. *Deleting a vertex of maximum degree cannot increase the average degree, but deleting a vertex of minimum degree can reduce the average degree.* Deleting any vertex of a nontrivial regular graph reduces the average degree, which proves the second claim. For the first claim, suppose that G has n vertices and m edges, and let a and a' be the average degrees of G and $G-x$, respectively. Since $G-x$ has $m-d(x)$ edges and degree sum $2m-2d(x)$, we have $a' = \frac{na-2d(x)}{n-1} \le \frac{(n-2)a}{n-1} < a$ if $d(x) \ge a > 0$. Hence deleting a vertex of maximum degree in nontrivial graph reduces the average degree and cannot increase it.

1.3.18. *If $k \ge 2$, then a k-regular bipartite graph has no cut-edge.* Since components of k-regular graphs are k-regular, it suffices to consider a connected k-regular X,Y-bigraph. Let uv be a cut-edge, and let G and H be the components formed by deleting uv. Let $m = |V(G) \cap X|$ and $n = |V(G) \cap Y|$. By symmetry, we may assume that $u \in V(G) \cap Y$ and $v \in V(H) \cap X$.

We count the edges of G. The degree of each vertex of G in X is k, so G has mk edges. The degree of each vertex of G in Y is k except for $d_G(u) = k-1$, so G has $nk-1$ edges. Hence $m_k = nk-1$, which is impossible because one side is divisible by k and the other is not. The proof doesn't work if $k=1$, and the claim is false then.

If vertex degrees k and $k+1$ are allowed, then a cut-edge may exist. Consider the example of $2K_{k,k}$ plus one edge joining the two components.

1.3.19. *A claw-free simple graph with maximum degree at least 5 has a 4-cycle.* Consider five edges incident to a vertex v of maximum degree in such a graph G. Since G has no induced claw, the neighbors of v must induce at least three edges. Since these three edges have six endpoints among the five neighbors of v, two of them must be incident, say xy and yz. Adding the edges xv and zv to these two completes a 4-cycle.

There are arbitrarily large 4-regular claw-free graphs with no 4-cycles.

Consider a vertex v in such a graph G. Since v has degree 4 and is not the center of an induced claw and does not lie on a 4-cycle, the subgraph induced by v and its neighbors consists of two edge-disjoint triangles sharing v (a bowtie). Since this happens at each vertex, G consists of pairwise edge-disjoint triangles, with each vertex lying in two of them. Hence each triangle has three neighboring triangles. Furthermore, two triangles that neighbor a given triangle in this way cannot neighbor each other; that would create a 4-cycle in the graph.

Define a graph H with one vertex for each triangle in G; let vertices be adjacent in H if the corresponding triangles share a vertex in G. Now H is a 3-regular graph with no 3-cycles; a 3-cycle in H would yield a 4-cycle in G using two edges from one of the corresponding triangles. Also H must have no 4-cycles, because a 4-cycle in G could be built using one edge from each of the four triangles corresponding to the vertices of a 4-cycle in H. Note that $e(G) = 2n(G)$ and $n(H) = e(G)/3 = 2n(G)/3$.

On the other hand, given any 3-regular graph H with girth at least 5, reversing the construction yields G with the desired properties and $3n(H)/2$ vertices. Hence it suffices to show that there are arbitrarily large 3-regular graphs with girth at least 5. Disconnected such examples can be formed by taking many copies of the Petersen graph as components. The graph G is connected if and only if H is connected. Connected instances of H can be obtained from multiple copies of the Petersen graph by applying 2-switches (Definition 1.3.32).

Alternatively, arbitrarily large connected examples can be constructed by taking two odd cycles (say length $2m+1$) and joining the ith vertex on the first cycle to the $2i$th vertex (modulo $2m+1$) on the second cycle (this generalizes the Petersen graph). We have constructed a connected 3-regular graph. Since we add disjoint edges between the cycles, there is no triangle. A 4-cycle would have to alternate edges between the two odd cycles with one edge of each, but the neighbors of adjacent vertices on the first cycle are two apart on the second cycle.

1.3.20. *K_n has $(n-1)!/2$ cycles of length n, and $K_{n,n}$ has $n!(n-1)!/2$ cycles of length $2n$.* Each cycle in K_n is a listing of the vertices. These can be listed in $n!$ orders, but we obtain the same subgraph no matter where we start the cycle and no matter which direction we follow, so each cycle is listed $2n$ times. In $K_{n,n}$, we can list the vertices in order on a cycle (alternating between the partite sets), in $2(n!)^2$ ways, but by the same reasoning each cycle appears $(2n)\cdot 2$ times.

1.3.21. *$K_{m,n}$ has $6\binom{m}{3}\binom{n}{3}$ 6-cycles.* To extend an edge in $K_{m,n}$ to a 6-cycle, we choose two more vertices from each side to be visited in order as we follow the cycle. Hence each edge in $K_{n,n}$ appears in $(m-1)(n-1)(m-2)(n-2)$

6-cycles. Since each 6-cycle contains 6 edges, we conclude that $K_{n,n}$ has $mn(m-1)(n-1)(m-2)(n-2)/6$ 6-cycles.

Alternatively, each 6-cycle uses three vertices from each partite set, which we can choose in $\binom{m}{3}\binom{n}{3}$ ways. Each such choice of vertices induces a copy of $K_{3,3}$ with 9 edges. There are $3! = 6$ ways to pick three disjoint edges to be omitted by a 6-cycle, so each $K_{3,3}$ contains 6 6-cycles.

1.3.22. *Odd girth and minimum degree in nonbipartite triangle-free n-vertex graphs.* Let $k = \delta(G)$, and let l be the minimum length of an odd cycle in G. Let C be a cycle of length l in G.

a) Every vertex not in $V(C)$ has at most two neighbors in $V(C)$. It suffices to show that any two neighbors of such a vertex v on C must have distance 2 on C, since having three neighbors would then require $l = 6$.

Since G is triangle-free, v does not have consecutive neighbors on C. If v has neighbors x and y on C separated by distance more than 2 on C, then the detour through v can replace the x, y-path of even length on C to form a shorter odd cycle.

b) $n \geq kl/2$ (and thus $l \leq 2n/k$). Since C is a shortest odd cycle, it has no chords (it is an induced cycle). Since $\delta(G) = k$, each vertex of C thus has at least $k-2$ edges to vertices outside C. However, each vertex outside C has at most two neighbors on C. Letting m be the number of edges from $V(C)$ to $V(G) - V(C)$, we thus have $l(k-2) \leq m \leq 2(n-l)$. Simplifying the inequality yields $n \geq kl/2$.

c) The inequality of part (b) is sharp when k is even. Form G from the cycle C_l by replacing each vertex of C_l with an independent set of size $k/2$ such that two vertices are adjacent if and only if the vertices they replaced were adjacent. Each vertex is now adjacent to the vertices arising from the two neighboring classes, so G is k-regular and has $lk/2$ vertices. Deleting the copies of any one vertex of C_l leaves a bipartite graph, since the partite sets can be labeled alternately around the classes arising from the rest of C_l. Hence every odd cycle uses a copy of each vertex of C_l and has length at least l, and taking one vertex from each class forms such a cycle.

1.3.23. *Equivalent definitions of the k-dimensional cube.* In the direct definition of Q_k, the vertices are the binary k-tuples, with edges consisting of pairs differing in one place. The inductive definition gives the same graph. For $k = 0$ both definitions specify K_1. For the induction step, suppose $k \geq 1$. The inductive definition uses two copies of Q_{k-1}, which by the induction hypothesis is the "1-place difference" graph of the binary $(k-1)$-tuples. If we append 0 to the $(k-1)$-tuples in one copy of Q_{k-1} and 1 to the $(k-1)$-tuples in the other copy, then within each set we still have edges between the labels differing in exactly one place. The inductive construction now adds edges consisting of corresponding vertices in the two copies. This is also what the direction definition does, since k-tuples chosen from the two copies differ in the last position and therefore differ in exactly one position if and only if they are the same in all other positions.

$e(Q_k) = k2^{k-1}$. By the inductive definition, $e(Q_k) = 2e(Q_{k-1}) + 2^{k-1}$ for $k \geq 1$, with $e(Q_0) = 0$. Thus the inductive step for a proof of the formula is $e(Q_k) = 2(k-1)2^{k-2} + 2^{k-1} = k_2^{k-1}$.

1.3.24. *$K_{2,3}$ is the smallest simple bipartite graph that is not a subgraph of the k-dimensional cube for any k.* Suppose the vectors x, y, a, b, c are the vertices of a copy of $K_{2,3}$ in Q_k. Any one of a, b, c differs from x in exactly one coordinate and from y in another (it can't be the same coordinate, because then $x = y$). This implies that x and y differ in two coordinate i, j. Paths from x to y in two steps can be formed by changing i and then j or changing j and then i; these are the only ways. In a cube two vertices have at most two common neighbors. Hence $K_{2,3}$ is forbidden. Any bipartite graph with fewer vertices or edges is contained in $K_{2,3} - e$ or $K_{1,5}$, but $K_{2,3} - e$ is a subgraph of Q_3, and $K_{1,5}$ is a subgraph of Q_5, so $K_{2,3}$ is the smallest forbidden subgraph.

1.3.25. *Every cycle of length $2r$ in a hypercube belongs to a subcube of dimension at most r, uniquely if $r \leq 3$.* Let C be a cycle of length $2r$ in Q_k; $V(C)$ is a collection of binary vectors of length k. Let S be the set of coordinates that change at some step while traversing the vectors in $V(C)$. In order to return to the first vector, each position must flip between 0 and 1 an even number of times. Thus traversing C changes each coordinate in S at least twice, but only one coordinate changes with each edge. Hence $2|S| \leq 2r$, or $|S| \leq r$. Outside the coordinates of S, the vectors of $V(C)$ all agree. Hence $V(C)$ is contained in a $|S|$-dimensional subcube.

As argued above, at most two coordinates vary among the vertices of a 4-cycle; at least two coordinates vary, because otherwise there are not enough vectors available to have four distinct vertices. By the same reasoning, exactly three three coordinates vary among the vertices of any 6-cycle; we cannot find six vertices in a 2-dimensional subcube. Thus the r-dimensional subcube containing a particular cycle is unique when $r \leq 3$.

Some 8-cycles are contained in 3-dimensional subcubes, such as $000x$, $001x$, $011x$, $010x$, $110x$, $111x$, $101x$, $100x$, where x is a fixed vector of length $n-3$. Such an 8-cycle is contained in $n-3$ 4-dimensional subcubes, obtained by letting some position in x vary.

1.3.26. *A 3-dimensional cube contains 16 6-cycles, and the k-dimensional cube Q_k contains $16\binom{k}{3}2^{k-3}$ 6-cycles.* If we show that every 6-cycle appears in exactly one 3-dimensional subcube, then multiplying the number of 3-dimensional subcubes by the number of 6-cycles in each subcube counts each 6-cycle exactly once.

For any set S of vertices not contained in a 3-dimensional subcube, there must be four coordinates in the corresponding k-tuples that are not constant within S. A cycle through S makes changes in four coordinates. Completing the cycle requires returning to the original vertex, so any coordinate that changes must change back. Hence at least eight changes are needed, and each edge changes exactly one coordinate. The cycle has length at least 8; hence 6-cycles are contained in 3-dimensional subcubes.

Furthermore, there are only four vertices possible when $k-2$ coordinates are fixed, so every 6-cycle involves changes in three coordinates. Hence the only 3-dimensional subcube containing the 6-cycle is the one that varies in the same three coordinates as the 6-cycle.

By Example 1.3.8, there are $\binom{k}{3}2^{k-3}$ 3-dimensional subcubes, so it remains only to show that Q_3 has 16 cycles of length 6. We group them by the two omitted vertices. The two omitted vertices may differ in 1, 2, or 3 coordinates. If they differ in one place (they are adjacent), then deleting them leaves a 6-cycle plus one edge joining a pair of opposite vertices. Since Q_3 has 12 edges, there are 12 6-cycles of this type. Deleting two complementary vertices (differing in every coordinate) leaves only a 6-cycle. Since Q_3 has four such pairs, there are four such 6-cycles. The remaining pairs differ in two positions. Deleting such a pair leaves a 4-cycle plus two pendant edges, containing no 6-cycle. This considers all choices for the omitted vertices, so the number of 6-cycles in Q_3 is $12+4$.

1.3.27. *Properties of the "middle-levels" graph.* Let G be the subgraph of Q_{2k+1} induced by vertices in which the numbers of 1s and 0s differs by 1. These are the $(2k+1)$-tuples of weight k and weight $k+1$, where **weight** denotes the number of 1s.

Each vertex of weight k has $k+1$ neighbors of weight $k+1$, and each vertex of weight $k+1$ has $k+1$ neighbors of weight k. There are $\binom{2k+1}{k}$ vertices of each weight. Counting edges by the Degree-Sum Formula,

$$e(G) = (k+1)\frac{n(G)}{2} = (k+1)\binom{2k+1}{k+1} = (2k+1)\binom{2k}{k}.$$

The graph is bipartite and has no odd cycle. The 1s in two vertices of weight k must be covered by the 1s of any common neighbor of weight $k+1$. Since the union of distinct k-sets has size at least $k+1$, there can only be one common neighbor, and hence G has no 4-cycle. On the other hand, G does have a 6-cycle. Given any arbitary fixed vector of weight $k-1$ for the last $2k-2$ positions, we can form a cycle of length six by using 110, 100, 101, 001, 011, 010 successively in the first three positions.

1.3.28. *Alternative description of even-dimensional hypercubes.* The simple graph Q_k' has vertex set $\{0,1\}^k$, with $u \leftrightarrow v$ if and only if u and v *agree*

in exactly one coordinate. Let the *odd vertices* be the vertices whose name has an odd number of 1s; the rest are *even vertices*.

When k is even, $Q_k' \cong Q_k$. To show this, rename all odd vertices by changing 1s into 0s and 0s into 1s. Since k is even, the resulting labels are still odd. Since k is even, every edge in Q_k' joins an even vertex to an odd vertex. Under the new naming, it joins the even vertex to an odd vertex that differs from it in one coordinate. Hence the adjacency relation becomes precisely the adjacency relation of Q_k.

When k is odd, $Q_k' \ncong Q_k$, because Q_k' contains an odd cycle and hence is not bipartite. Starting from one vertex, form a closed walk by successively following k edges where each coordinate is the coordinate of agreement along exactly one of these edges. Hence each coordinate changes exactly $k-1$ times and therefore ends with the value it had at the start. Thus this is a closed walk of odd length and contains an odd cycle.

1.3.29. *Automorphisms of Q_k.*

a) A subgraph H of Q_k is isomorphic to Q_l if and only if it is the subgraph induced by a set of vertices agreeing in some set of $k-l$ coordinates. Let f be an isomorphism from H to Q_l, and let v be the vertex mapped to the vertex $\mathbf{0}$ of Q_l whose coordinates are all 0. Let $u_1,\ldots,u_l$ be the neighbors of v in H mapped to neighbors of $\mathbf{0}$ in Q_l by f. Each u_i differs from v in one coordinate; let S be the set of l coordinates where these vertices differ from v. It suffices to show that vertices of H differ from v only on the coordinates of S. This is immediate for $l \le 1$.

For $l \ge 2$, we prove that each vertex mapped by f to a vertex of Q_l having weight j differs from v in j positions of S, by induction on j. Let x be a vertex mapped to a vertex of weight j in Q_l. For $j \le 1$, we have already argued that x differs from v in j positions of S. For $j \ge 2$, let y and z be two neighbors of x whose images under f have weight $j-1$ in Q_l. By the induction hypothesis, y and z differ from v in j positions of S. Since $f(y)$ and $f(z)$ differ in two places, they have two common neighbors in Q_l, which are x and another vertex w. Since w has weight $j-2$, the induction hypothesis yields that w differs from v in $j-1$ positions of S. Since the images of x,y,z,w induce a 4-cycle in Q_l, also x,y,z,w induce a 4-cycle in H. The only 4-cycle in Q_k that contains all of y,z,w adds the vertex that differs from v in the $j-2$ positions of S where w differs, plus the two positions where y and z differ from w. This completes the proof that x has the desired property.

b) The k-dimensional cube Q_k has exactly $2^k k!$ automorphisms. (Part (a) is unnecessary.) Form automorphisms of Q_k by choosing a subset of the k coordinates in which to complement 0 and 1 and, independently, a permutation of the k coordinates. There are $2^k k!$ such automorphisms.

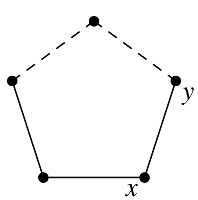
We prove that every automorphism has this form. Let **0** be the all-0 vertex. Let f be the inverse of an automorphism, and let v be the vertex mapped to **0** by f. The neighbors of v must be mapped to the neighbors of **0**. If these choices completely determine f, then f complements the coordinates where v is nonzero, and the correspondence between the neighbors of **0** and the neighbors of v determines the permutation of the coordinates that expresses f as one of the maps listed above.

Suppose that x differs from v in coordinates $r_1, \ldots, r_j$. Let $u_1, \ldots, u_j$ be the neighbors of v differing from v in these coordinates. We prove that $f(x)$ is the k-tuple of weight j having 1 in the coordinates where $f(u_1), \ldots, f(u_j)$ have 1. We use induction on j.

For $j \le 1$, the claim follows by the definition of $u_1, \ldots, u_j$. For $j \ge 2$, let y and z be two neighbors of x that differ from v in $j-1$ coordinates. Let w be the common neighbor of y and z that differs from v in $j-2$ coordinates. By the induction hypothesis, $f(y)$ and $f(z)$ have weight $j-1$ (in the appropriate positions), and $f(w)$ has weight $j-1$. Since $f(x)$ must be the other common neighbor of $f(y)$ and $f(z)$, it has weight j, with 1s in the desired positions.

1.3.30. *The Petersen graph has twelve 5-cycles.* Let G be the Petersen graph. We show first that each edge of G appears in exactly four 5-cycles. For each edge $e = xy$ in G, there are two other edges incident to x and two others incident to y. Since G has no 3-cycles, we can thus extend xy at both ends to form a 4-vertex path in four ways. Since G has no 4-cycle, the endpoints of each such path are nonadjacent. By Proposition 1.1.38, there is exactly one vertex to add to such a path to complete a 5-cycle. Thus e is in exactly four 5-cycles.

When we sum this count over the 15 edges of G, we have counted 60 5-cycles. However, each 5-cycle has been counted five times—once for each of its edges. Thus the total number of 5-cycles in G is $60/5 = 12$.



1.3.31. *Combinatorial proofs with graphs.*

a) For $0 \le k \le n$, $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$. Consider the complete graph K_n, which has $\binom{n}{2}$ edges. If we partition the vertices of K_n into a k-set and an $(n-k)$-set, then we can count the edges as those within one block of the partition and those choosing a vertex from each. Hence the total number of edges is $\binom{k}{2} + \binom{n-k}{2} + k(n-k)$.

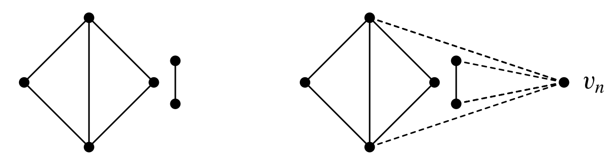
b) If $\sum n_i = n$, *then* $\sum \binom{n_i}{2} \le \binom{n}{2}$. Again consider the edges of K_n, and partition the vertices into sets with n_i being the size of the ith set. The left side of the inequality counts the edges in K_n having both ends in the same S_i, which is at most all of $E(K_n)$.

1.3.32. *For $n \ge 1$, there are $2^{\binom{n-1}{2}}$ simple even graphs with a fixed vertex set of size n.* Let A be the set of simple even graphs with vertex set $v_1, \ldots, v_n$. Since $2^{\binom{n-1}{2}}$ is the size of the set B of simple graphs with vertex set $v_1, \ldots, v_{n-1}$, we establish a bijection from A to B.

Given a graph in A, we obtain a graph in B by deleting v_n. To show that each graph in B arises exactly once, consider a graph $G \in B$. We form a new graph G' by adding a vertex v_n and making it adjacent to each vertex with odd degree in G, as illustrated below.

The vertices with odd degree in G have even degree in G'. Also, v_n itself has even degree because the number of vertices of odd degree in G is even. Thus $G' \in A$. Furthermore, G is the graph obtained from G' by deleting v_n, and every simple even graph in which deleting v_n yields G must have v_n adjacent to the same vertices as in G'.

Since there is a bijection from A to B, the two sets have the same size.



G G'

1.3.33. *Triangle-free graphs in which every two nonadjacent vertices have exactly two common neighbors.*

$n(G) = 1 + \binom{k+1}{2}$, *where k is the degree of a vertex x in G.* For every pair of neighbors of x, there is exactly one nonneighbor of x that they have as a common neighbor. Conversely, every nonneighbor of x has exactly one pair of neighbors of x in its neighborhood, because these are its common neighbors with x. This establishes a bijective correspondence between the pairs in $N(x)$ and the nonneighbors of x. Counting x, $N(x)$, and $\overline{N}(x)$, we have $n(G) = 1 + k + \binom{k}{2} = 1 + \binom{k+1}{2}$. Since this argument holds for every $x \in V(G)$, we conclude that G is k-regular.

Comment: Such graphs exist only for isolated values of k. Unique graphs exist for $k = 1, 2, 5$. Viewing the vertices as x, $N(x) = [k]$, and $\overline{N}(x) = \binom{[k]}{2}$, we have i adjacent to the pair $\{j, k\}$ if and only if $i \in \{j, k\}$. The lack of triangles guarantees that only disjoint pairs in $\binom{[k]}{2}$ can be adjacent, but each pair in $\binom{[k]}{2}$ must have exactly $k-2$ neighbors in $\binom{[k]}{2}$. For $k = 5$, this implies that $\overline{N}(x)$ induces the 3-regular disjointness graph of $\binom{[5]}{2}$, which is the Petersen graph. Since the Petersen graph has girth 5 and diameter 2, each intersecting pair has exactly one common neighbor in $\overline{N}(x)$ in addition to its one common neighbor in $N(x)$, so this graph has the desired properties.

Numerical conditions eliminate $k \equiv 3 \pmod 4$, because G would be regular of odd degree with an odd number of vertices. There are stronger necessary conditions. After $k = 5$, the next possibility is $k = 10$, then 26, 37, 82, etc. A realization for $k = 10$ is known to exist, but in general the set of realizable values is not known.

1.3.34. *If G is a kite-free simple n-vertex graph such that every pair of nonadjacent vertices has exactly two common neighbors, then G is regular.* Since nonadjacent vertices have common neighbors, G is connected. Hence it suffices to prove that adjacent vertices x and y have the same degree. To prove this, we establish a bijection from A to B, where $A = N(x) - N(y)$ and $B = N(y) - N(x)$.

Consider $u \in A$. Since $u \nleftrightarrow y$, there exists $v \in N(u) \cap N(y)$ with $v \neq x$. Since G is kite-free, $v \nleftrightarrow x$, so $v \in B$. Since x and v have common neighbors y and u, the vertex v cannot be generated in this way from another vertex of A. Hence we have defined an injection from A to B. Interchanging the roles of y and x yields an injection from B to A. Since these sets are finite, the injections are bijections, and $d(x) = d(y)$.

1.3.35. *If every induced k-vertex subgraph of a simple n-vertex graph G has the same number of edges, where $1 < k < n-1$, then G is a complete graph or an empty graph.*

a) If $l \geq k$ and G' is a graph on l vertices in which every induced k-vertex subgraph has m edges, then $e(G') = m\binom{l}{k}/\binom{l-2}{k-2}$. Counting the edges in all the k-vertex subgraphs of G' yields $m\binom{l}{k}$, but each edge appears in $\binom{l-2}{k-2}$ of these subgraphs, once for each k-set of vertices containing it. (Both sides of $\binom{l-2}{k-2}e(G') = m\binom{l}{k}$ count the ways to pick an edge of G' and a k-set of vertices in G' containing that edge. On the right, we pick the set first; on the left, we pick the edge first.)

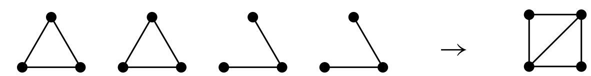
b) Under the stated conditions, $G = K_n$ or $G = \overline{K}_n$. Given vertices u and v, let A and B be the sets of edges incident to u and v, respectively. The set of edges with endpoints u and v is $A \cap B$. We compute

$$|A \cap B| = e(G) - \left|\overline{A \cap B}\right| = e(G) - \left|\overline{A} \cup \overline{B}\right| = e(G) - \left|\overline{A}\right| - \left|\overline{B}\right| + \left|\overline{A} \cap \overline{B}\right|.$$

In this formula, $\overline{A}$ and $\overline{B}$ are the edge sets of induced subgraphs of order $n-1$, and $\overline{A} \cap \overline{B}$ is the edge set of an induced subgraph of order $n-2$. By part (a), the sizes of these sets do not depend on the choice of u and v.

1.3.36. *The unique reconstruction of the graph with vertex-deleted subgraphs below is the kite.*

Proof 1. A vertex added to the first triangle may be joined to 0,1,2, or 3 of its vertices. We eliminate 0 and 1 because no vertex-deleted subgraph has an isolated vertex. We eliminate 3 because every vertex-deleted subgraph of K_4 is a triangle. Joining it to 2 yields the kite.



Proof 2. The graph G must have four vertices, and by Proposition 1.3.11 it has five edges. The only such simple graph is the kite.

1.3.37. *Retrieving a regular graph.* Suppose that H is a graph formed by deleting a vertex from a regular graph G. We have H, so we know $n(G) = n(H) + 1$, but we don't know the vertex degrees in G. If G is d-regular, then G has $dn(G)/2$ edges, and H has $dn(G)/2 - d$ edges. Thus $d = 2e(H)/(n(G)-2)$. Having determined d, we add one vertex w to H and add $d - d_H(v)$ edges from w to v for each $v \in V(H)$.

1.3.38. *A graph with at least 3 vertices is connected if and only if at least two of the subgraphs obtained by deleting one vertex are connected.* The endpoints of a maximal path are not cut-vertices. If G is connected, then the subgraphs obtained by deleted such vertices are connected, and there are at least of these.

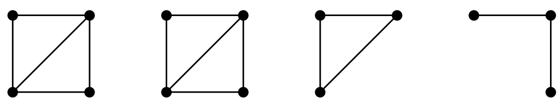
Conversely, suppose that at least two vertex-deleted subgraphs are connected. If $G - v$ is connected, then G is connected unless v is an isolated vertex. If v is an isolated vertex, then all the other subgraphs obtained by deleting one vertex are disconnected. Hence v cannot be isolated, and G is connected.

1.3.39. *Disconnected graphs are reconstructible.* First we show that G is connected if and only if it has at least two connected vertex-deleted subgraphs. Necessity holds, because the endpoints of a maximal path cannot be cut-vertices. If G is disconnected, then $G - v$ is disconnected unless v is an isolated vertex (degree 0) in G and $G - v$ is connected. This happens for at most one vertex in G.

After determining that G is disconnected, we obtain which disconnected graph it is from its vertex-deleted subgraphs. We aim to identify a connected graph M that is a component of G and a vds in the deck that arises by deleting a specified vertex u of M. Replacing $M - u$ by M in that subgraph will reconstruct G.

Among all components of all graphs in the deck, let M be one with maximum order. Since every component H of a potential reconstruction G appears as a component of some $G - v$, M cannot belong to any larger component of G. Hence M is a component of G. Let L be a fixed connected subgraph of M obtained by deleting a leaf u of some spanning tree of M. Then L is a component of $G - u$. We want to reconstruct G by substituting M for L in $G - u$; we must identify $G - u$. There may be several isomorphic copies of $G - u$.

As in the disconnected graph G shown above, M may appear as a component of every vds $G - v$. However, since M cannot be created by a vertex deletion, a vds with the fewest copies of M must arise by deleting a vertex of M. Among these, we seek a subgraph with the most copies of L as components, because in addition to occurrences of L as a component of G, we obtain an additional copy if and only if the deleted vertex of M can play the role of u. This identifies $G - u$, and we obtain G by replacing one of its components isomorphic to L with a component isomorphic to M.



1.3.40. *Largest graphs of specified types.*

a) Largest n-vertex simple graph with an independent set of size a.

Proof 1. Since there are no edges within the independent set, such a graph has at most $\binom{n}{2} - \binom{a}{2}$ edges, which equals $\binom{n-a}{2} + (n-a)a$. This bound is achieved by the graph consisting of a copy H of K_{n-a}, an independent set S of size a, and edges joining each vertex of H to each vertex of S.

Proof 2. Each vertex of an independent set of size a has degree at most $n-a$. Each other vertex has degree at most $n-1$. Thus $\sum d(v) \le a(n-a) + (n-a)(n-1)$. By the Degree-Sum Formula, $e(G) \le (n-a)(n-1+a)/2$. This formula equals those above and is achieved by the same graph, since this graph achieves the bound for each vertex degree.

b) The maximum size of an n-vertex simple graph with k components is $\binom{n-k+1}{2}$. The graph consisting of K_{n-k+1} plus $k-1$ isolated vertices has k components and $\binom{n-k+1}{2}$ edges. We prove that other n-vertex graphs with k components don't have maximum size. Let G be such a graph.

If G has a component that is not complete, then adding edges to make it complete does not change the number of components. Hence we may assume that every component is complete.

If G has components with r and s vertices, where $r \ge s > 1$, then we move one vertex from the s-clique to the r-clique. This deletes $s-1$ edges and creates r edges, all incident to the moved vertex. The other edges remain the same, so we gain $r - s + 1$ edges, which is positive.

Thus the number of edges is maximized only when every component is a complete graph and only one component has more than one vertex.

c) The maximum number of edges in a disconnected simple n-vertex graph is $\binom{n-1}{2}$, with equality only for $K_1 + K_{n-1}$.

Proof 1 (using part (b)). The maximum over graphs with k components is $\binom{n-k+1}{2}$, which decreases as k increases. For disconnected graphs, $k \ge 2$. We maximize the number of edges when $k = 2$, obtaining $\binom{n-1}{2}$.

Proof 2 (direct argument). Given a disconnected simple graph G, let S be the vertex set of one component of G, and let $t = |S|$. Since no edges join S and $\overline{S}$, $e(G) \le \binom{n}{2} - t(n-t)$. This bound is weakest when $t(n-t)$ is smallest, which for $1 \le t \le n-1$ happens when $t \in \{1, n-1\}$. Thus always $e(G) \le \binom{n}{2} - 1(n-1) = \binom{n-1}{2}$, and equality holds when $G = K_1 + K_{n-1}$.

Proof 3 (induction on n). When $n = 2$, the only simple graph with $e(G) > \binom{1}{2} = 1$ is K_2, which is connected. For $n > 2$, suppose $e(G) > \binom{n-1}{2}$. If $\Delta(G) = n-1$, then G is connected. Otherwise, we may select v with $d(v) \le n-2$. Then $e(G - v) > \binom{n-1}{2} - n + 2 = \binom{n-2}{2}$. By the induction hypothesis, $G - v$ is connected. Since $e(G) > \binom{n-1}{2}$ and G is simple, we have $d(v) > 0$, so there is an edge from v to $G - v$, and G is also connected.

Proof 4 (complementation). If G is disconnected, then $\overline{G}$ is connected, so $e(\overline{G}) \ge n-1$ and $e(G) \le \binom{n}{2} - (n-1) = \binom{n-1}{2}$. In fact, $\overline{G}$ must contain a spanning complete bipartite subgraph, which is as small as $n-1$ edges only when $\overline{G} = K_{1,n-1}$ and $G = K_1 + K_{n-1}$.

1.3.41. *Every n-vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$ is connected.* Let x be a vertex of maximum degree. It suffices to show that every vertex not adjacent to x has a common neighbor with x. Choose $y \notin N(x)$. We have $|N(x)| = \lceil n/2 \rceil$ and $|N(y)| \ge \lfloor n/2 \rfloor - 1$. Since $y \not\leftrightarrow x$, we have $N(x), N(y) \subseteq V(G) - \{x, y\}$. Thus

$$|N(x) \cap N(y)| = |N(x)| + |N(y)| - |N(x) \cup N(y)| \ge \lceil n/2 \rceil + \lfloor n/2 \rfloor - 1 - (n-2) = 1.$$

1.3.42. *Strongly independent sets.* If S is an independent set with no common neighbors in a graph G, then the vertices of S have pairwise-disjoint closed neighborhoods of size at least $\delta(G) + 1$. Thus there are at most $\lfloor n(G)/(\delta(G)+1) \rfloor$ of them. Equality is achievable for the 3-dimensional cube using $S = \{000, 111\}$.

Equality is not achievable when $G = Q_4$, since with 16 vertices and minimum degree 4 it requires three parwise-disjoint closed neighborhoods of size 5. If $v \in S$, then no vertex differing from v in at most two places is in S. Also, at most one vertex differing from v in at least three places is in

S, since such vertices differ from each other in at most two places. Thus only two disjoint closed neighborhoods can be found in Q_4.

1.3.43. *Every simple graph has a vertex whose neighbors have average degree as large as the overall average degree.* Let $t(w)$ be the average degree of the neighbors of w. In the sum $\sum_{w\in V(G)} t(w) = \sum_{w\in V(G)} \sum_{y\in N(w)} d(y)/d(w)$, we have the terms $d(u)/d(v)$ and $d(v)/d(u)$ for each edge uv. Since $x/y + y/x \ge 2$ whenever x, y are positive real numbers (this is equivalent to $(x-y)^2 \ge 0$), each such contribution is at least 2. Hence $\sum t(w) \ge \sum_{uv\in E(G)} \frac{d(u)}{d(v)} + \frac{d(v)}{d(u)} \ge 2e(G)$. Hence the average of the neighborhood average degrees is at least the average degree, and the pigeonhole principle yields the desired vertex.

It is possible that every average neighborhood degree exceeds the average degree. Let G be the graph with $2n$ vertices formed by adding a matching between a complete graph and an independent set. Since G has $\binom{n}{2} + n$ edges and $2n$ vertices, G has average degree $(n+1)/2$. For each vertex of the n-clique, the neighborhood average degree is $n - 1 + 1/n$. For each leaf, the neighborhood average degree is n.

1.3.44. *Subgraphs with large minimum degree.* Let G be a loopless graph with average degree a.

a) If $x \in V(G)$, then $G' = G - x$ has average degree at least a if and only if $d(x) \le a/2$. Let a' be the average degree of G', and let n be the order of G. Deleting x reduces the degree sum by $2d(x)$, so $(n-1)a' = na - 2d(x)$. Hence $(n-1)(a'-a) = a - 2d(x)$. For $n > 1$, this implies that $a' \ge a$ if and only if $d(x) \le a/2$.

Alternative presentation. The average degree of G is $2e(G)/n(G)$. Since G' has $e(G) - d(x)$ edges, the average degree is at least a if and only if $\frac{2[e(G)-d(x)]}{n(G)-1} \ge a$. Since $e(G) = n(G)a/2$, we can rewrite this as $n(G)a - 2d(x) = 2e(G) - 2d(x) \ge an(G) - a$. By canceling $n(G)a$, we find that the original inequality is equivalent to $d(x) \le a/2$.

b) If $a > 0$, then G has a subgraph with minimum degree greater than $a/2$. Iteratively delete vertices with degree at most half the current average degree, until no such vertex exists. By part (a), the average degree never decreases. Since G is finite, the procedure must terminate. It ends only by finding a subgraph where every vertex has degree greater than $a/2$.

c) The result of part (b) is best possible. To prove that no fraction of a larger than $\frac{1}{2}a$ can be guaranteed, let G_n be an n-vertex tree. We have $a(G_n) = 2(n-1)/n = 2 - 2/n$, but subgraphs of G_n have minimum degree at most 1. Given $\beta > \frac{1}{2}$, we can choose n large enough so that $1 \le \beta a(G_n)$.

1.3.45. *Bipartite subgraphs of the Petersen graph.*

a) Every edge of the Petersen graph is in four 5-cycles. In every 5-cycle through an edge e, the edge e is the middle edge of a 4-vertex path. Such

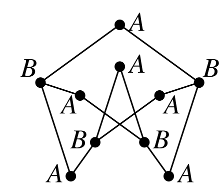
a path can be obtained in four ways, since each edge extends two ways at each endpoint. The neighbors at each endpoint of e are distinct and nonadjacent, since the girth is 5.

Since the endpoints of each such P_4 are nonadjacent, they have exactly one common neighbor. Thus each P_4 yields one 5-cycle, and each 5-cycle through e arises from such a P_4, so there are exactly four 5-cycles containing each edge.

b) The Petersen graph has twelve 5-cycles. Since there are 15 edges, summing the number of 5-cycles through each edge yields 60. Since each 5-cycle is counted five times in this total, the number of 5-cycles is 12.

c) The largest bipartite subgraph has twelve edges.

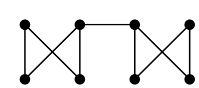
Proof 1 (breaking odd cycles). Each edge is in four 5-cycles, so we must delete at least 12/4 edges to break all 5-cycles. Hence we must delete at least three edges to have a bipartite subgraph. The illustration shows that deleting three is enough; the Petersen graph has a bipartite subgraph with 12 edges (see also the cover of the text).



Proof 2 (study of bipartite subgraphs). The Petersen graph G has an independent set of size 4, consisting of the vertices $\{ab, ac, ad, ae\}$ in the structural description. The 12 edges from these four vertices go to the other six vertices, so this is a bipartite subgraph with 12 edges.

Let X and Y be the partite sets of a bipartite subgraph H. If $|X| \le 4$, then $e(H) \le 12$, with equality only when X is an independent 4-set in G. Hence we need only consider the case $|X| = |Y| = 5$. To obtain $e(G) > 10$, some vertex $x \in X$ must have three neighbors in Y. The two nonneighbors of x in Y have common neighbors with x, and these must lie in $N(x)$, which is contained in Y. Hence $e(G[Y]) \ge 2$. Interchanging X and Y in the argument shows that also $e(G[X]) \ge 2$. Hence $e(H) \le 11$.

1.3.46. *When the algorithm of Theorem 1.4.2 is applied to a bipartite graph, it need not find the bipartite subgraph with the most edges.* For the bipartite graph below, the algorithm may reach the partition between the upper vertices and lower vertices.



This bipartite subgraph with eight edges has more than half of the edges at each vertex, and no further changes are made. However, the bipartite subgraph with the most edges is the full graph.

1.3.47. *Every nontrivial loopless graph G has a bipartite subgraph containing more than half its edges.* We use induction on $n(G)$. If $n(G) = 2$, then G consists of copies of a single edge and is bipartite. For $n(G) > 2$, choose $v \in V(G)$ that is not incident to all of $E(G)$ (at most two vertices can be incident to all of $E(G)$). Thus $e(G - v) > 0$. By the induction hypothesis, $G - v$ has a bipartite subgraph H containing more than $e(G)/2$ edges.

Let X, Y be a bipartition of H. If X contains at least half of $N_G(v)$, then add v to Y; otherwise add v to X. The augmented partition captures a bipartite subgraph of G having more than half of $E(G - v)$ and at least half of the remaining edges, so it has more than half of $E(G)$.

Comment. The statement can also be proved without induction. By Theorem 1.3.19, G has a bipartite subgraph H with at least $e(G)/2$ edges. By the proof of Theorem 1.3.19, equality holds only if $d_H(v) = d_G(v)/2$ for every $v \in V(G)$. Given an edge uv, each of u and v has exactly half its neighbors in its own partite set. Switching both to the opposite set will capture those edges while retaining the edge uv, so the new bipartite subgraph has more edges.

1.3.48. *No fraction of the edges larger than $1/2$ can be guaranteed for the largest bipartite subgraph.* If G_n is the complete graph K_{2n}, then $e(G_n) = \binom{2n}{2} = n(2n-1)$, and the largest bipartite subgraph is $K_{n,n}$, which has n^2 edges. Hence $\lim_{n\to\infty} f(G_n)/e(G_n) = \lim_{n\to\infty} \frac{n^2}{2n^2-n} = \frac{1}{2}$. For large enough n, the fraction of the edges in the largest bipartite subgraph is arbitrarily close to $1/2$. (In fact, in every graph the largest bipartite subgraph has **more** than half the edges.)

1.3.49. *Every loopless graph G has a spanning k-partite subgraph H such that $e(H) \ge (1 - 1/k)e(G)$.*

Proof 1 (local change). Begin with an arbitrary partition of $V(G)$ into k parts $V_1, \ldots, V_k$, and consider the k-partite subgraph H containing all edges of G consisting of two vertices from distinct parts. Given a partition of $V(G)$, let $V(x)$ denote the part containing x. If in G some vertex x has more neighbors in V_j than in some other part, then shifting x to the other part increases the number of edges captured by the k-partite subgraph.

Since G has finitely many edges, this shifting process must terminate. It terminates when for each $x \in V(G)$ the number $|N(x) \cap V_i|$ is minimized by $V_i = V(x)$. Then $d_G(x) = \sum_i |N_G(x) \cap V_i| \ge k|N_G(x) \cap V(x)|$. We conclude that $|N_G(x) \cap V(x)| \le (1/k)d_G(x)$, and hence $d_H(x) \ge (1 - 1/k)d_G(x)$ for all $x \in V(G)$. By the degree-sum formula, $e(H) \ge (1 - 1/k)e(G)$.

Proof 2 (induction on n). We prove that when G is nontrivial, some such H has more than $(1 - 1/k)e(G)$ edges. This is true when $n = 2$. We procede by induction for $n > 2$. Choose a vertex $v \in V(G)$. By the induction hypothesis, $G - v$ has a spanning k-partite subgraph with more than $(1 - 1/k)e(G - v)$ edges. This subgraph partitions $V(G - v)$ into k partite sets. One of these sets contains at most $1/k$ neighbors of v. Add v to that set to obtain the desired k-partite subgraph H. Now $e(H) > (1 - 1/k)e(G - v) + (1 - 1/k)d_G(v) = (1 - 1/k)e(G)$.

1.3.50. For $n \ge 3$, the minimum number of edges in a connected n-vertex graph in which every edge belongs to a triangle is $\lceil 3(n-1)/2 \rceil$. To achieve the minimum, we need only consider simple graphs. Say that connected graphs with each edge in a triangle are *good* graphs. For $n = 3$, the only such graph is K_3, with three edges.

When n is odd, a construction with the claimed size consists of $(n-1)/2$ triangles sharing a common vertex. When n is even, add one vertex to the construction for $n - 1$ and make it adjacent to both endpoints of one edge.

For the lower bound, let G be a smallest n-vertex good graph. Since G has fewer than $3n/2$ edges (by the construction), G has a vertex v of degree 2. Let x and y be its neighbors. Since each edge belongs to a triangle, $x \leftrightarrow y$. If $n > 3$, then we form G' by deleting v and, if xy have no other neighbor, contracting xy. Every edge of G' belongs to a triangle that contained it in G. The change reduces the number of vertices by 1 or 2 and reduces the number of edges by at least $3/2$ times the reduction in the number of vertices. By the induction hypothesis, $e(G') \ge \lceil 3(n(G') - 1)/2 \rceil$, and hence the desired bound holds for G.

1.3.51. Let G be a simple n-vertex graph.

a) $e(G) = \frac{\sum_{v\in V(G)} e(G-v)}{n-2}$. If we count up all the edges in all the subgraphs obtained by deleting one vertex, then each edge of G is counted exactly $n-2$ times, because it shows up in the $n - 2$ subgraphs obtained by deleting a vertex other than its endpoints.

b) If $n \ge 4$ and G has more than $n^2/4$ edges, then G has a vertex whose deletion leaves a graph with more than $(n-1)^2/4$ edges. Since G has more than $n^2/4$ edges and $e(G)$ is an integer, we have $e(G) \ge (n^2 + 4)/4$ when n is even and $e(G) \ge (n^2 + 3)/4$ when n is odd (since $(2k + 1)^2 = 4k^2 + 4k + 1$, every square of an odd number is one more than a multiple of 4). Thus always we have $e(G) \ge (n^2 + 3)/4$.

By part (a), we have $\sum_{v\in V(G)} \frac{e(G-v)}{n-2} \ge (n^2 + 3)/4$. In the sum we have n terms. Since the largest number in a set is at least the average, there is a vertex v such that $\frac{e(G-v)}{n-2} \ge \frac{1}{n}\frac{n^2+3}{4}$. We rewrite this as

$$e(G - v) \ge \frac{(n^2 + 3)(n - 2)}{4n} = \frac{n^3 - 2n^2 + 3n - 6}{4n} = \frac{n^2 - 2n + 1}{4} + \frac{2n - 6}{4n}$$

When $n \geq 4$, the last term is positive, and we obtain the strict inequality $e(G - v) > (n-1)^2/4$.

c) Inductive proof that G contains a triangle if $e(G) > n^2/4$. We use induction on n. When $n \leq 3$, they only simple graph with more than $n^2/4$ edges is when $n = 3$ and $G = K_3$, which indeed contains a triangle. For the induction step, consider $n \geq 4$, and let G be an n-vertex simple graph with more than $n^2/4$ vertices. By part (b), G has a subgraph $G - v$ with $n-1$ vertices and more than $(n-1)^2/4$ edges. By the induction hypothesis, $G - v$ therefore contains a triangle. This triangle appears also in G.

1.3.52. *$K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is the only n-vertex triangle-free graph of maximum size.* As in the proof of Mantel's result, let x be a vertex of maximum degree. Since $N(x)$ is an independent set, x and its non-neighbors capture all the edges, and we have $e(G) \leq (n - \Delta(G))\Delta(G)$. If equality holds, then summing the degrees in $V(G) - N(x)$ counts each edge exactly once. This requires that $V(G) - N(x)$ also is an independent set, and hence G is bipartite. If G is bipartite and has $(n - \Delta(G))\Delta(G)$ edges, then $G = K_{(n-\Delta(G)), \Delta(G)}$. Hence $e(G)$ is maximized only by $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

1.3.53. *The bridge club with 14 members (no game can be played if two of the four people table have previously been partners): If each member has played with four others and then six additional games have been played, then the arrival of a new member allows a game to be played.* We show that the new player yields a set of four people among which no two have been partners. This is true if and only if the previous games must leave three people (in the original 14) among which no two have been partners.

The graph of pairs who have NOT been partners initially is K_{14}. For each game played, two edges are lost from this graph. At the breakpoint in the session, each vertex has lost four incident edges, so 28 edges have been deleted. In the remaining six games, 12 more edges are deleted. Hence 40 edges have been deleted. Since $e(K_{14}) = 91$, there remain 51 edges for pairs that have not yet been partners.

By Mantel's Theorem (Theorem 1.3.23), the maximum number of edges in a simple 14-vertex graph with no triangle is $\lfloor 14^2/4 \rfloor$. Since $51 > 49$, the graph of remaining edges has a triangle. Thus, when the 15th person arrives, there will be four people of whom none have partnered each other.

1.3.54. *The minimum number of triangles $t(G)$ in an n-vertex graph G and its complement.*

a) $t(G) = \binom{n}{3} - (n-2)e + \sum_{v \in V(G)} \binom{d(v)}{2}$. Let $d_1, \ldots, d_n$ denote the vertex degrees. We prove that the right side of the formula assigns weight 1 to the vertex triples that induce a triangle in G or $\overline{G}$ and weight 0 to all other triples. Among these terms, $\binom{n}{3}$ counts all triples, $(n-2)e$ counts those determined by an edge of G and a vertex off that edge, and $\sum \binom{d_i}{2}$ counts 1 for each pair of incident edges. In the table below, we group these contributions by how many edges the corresponding triple induces in G.

$t(G)$	in G	$\binom{n}{3}$	$-(n-2)e$	$\sum \binom{d_i}{2}$
1	3 edges	1	-3	3
0	2 edges	1	-2	1
0	1 edge	1	-1	0
1	0 edges	1	-0	0

b) $t(G) \geq n(n-1)(n-5)/24$. Begin with the formula for $k_3(G) + k_3(\overline{G})$ from part (a). Using the convexity of quadratic functions, we get a lower bound for the sum on the right by replacing the vertex degrees by the average degree $2e/n$. The bound is $\binom{n}{3} - (n-2)e + n\binom{2e/n}{2}$, which reduces to $\binom{n}{3} - 2e(\binom{n}{2} - e)/n$. As a function of e, this is minimized when $e = \frac{1}{2}\binom{n}{2}$. This substitution and algebraic simplification produce $t(G) \geq n(n-1)(n-5)/24$.

Comment. The proof of part (b) uses two minimizations. These imply that equality can hold only for a regular graph ($d_i = 2e/n$ for all i) with $e = \frac{1}{2}\binom{n}{2}$. There is such a regular graph if and only if n is odd and $(n-1)/2$ is even. Thus we need $n = 4k+1$ and G is $2k$-regular.

1.3.55. *Maximum size with no induced P_4. a) If G is a simple connected graph and $\overline{G}$ is disconnected, then $e(G) \leq \Delta(G)^2$, with equality only for $K_{\Delta(G),\Delta(G)}$.* Since $\overline{G}$ is disconnected, $\Delta(G) \geq n(G)/2$, with equality only if $G = K_{\Delta(G),\Delta(G)}$. Thus $e(G) = \sum d_i/2 \leq n(G)\Delta(G)/2 \leq \Delta(G)^2$. As observed, equality when $\overline{G}$ is disconnected requires $G = K_{\Delta(G),\Delta(G)}$.

b) If G is a simple connected graph with maximum degree D and no induced subgraph isomorphic to P_4, then $e(G) \leq D^2$. It suffices by part (a) to prove that $\overline{G}$ is disconnected when G is connected and P_4-free. We use induction on $n(G)$ for $n(G) \geq 2$; it is immediate when $n(G) = 2$. For the induction step, let v be a non-cut-vertex of G. The graph $G' = G - v$ is also P_4-free, so its complement is disconnected, by the induction hypothesis. Thus $V(G) - v$ has a vertex partition X, Y such that all of X is adjacent to all of Y in G. Since G is connected, v has a neighbor $z \in X \cup Y$; we may assume be symmetry that $z \in Y$. If $\overline{G}$ is connected, then $\overline{G}$ has a v, z-path. Let y be the vertex before z on this path; note that $y \in Y$. Also $\overline{G}$ connected requires $x \in X$ such that $vx \in E(\overline{G})$. Now $\{v, z, x, y\}$ induces P_4 in G.

1.3.56. *Inductive proof that for $\sum d_i$ even there is a multigraph with vertex degrees $d_1, \ldots, d_n$.*

Proof 1 (induction on $\sum d_i$). If $\sum d_i = 0$, then all d_i are 0, and the n-vertex graph with no edges has degree list d. For the induction step, suppose $\sum d_i > 0$. If only one d_i is nonzero, then it must be even, and the graph consisting of $n-1$ isolated vertices plus $d_i/2$ loops at one vertex has degree list d (multigraphs allow loops).

Otherwise, d has at least two nonzero entries, d_i and d_j. Replacing these with $d_i - 1$ and $d_j - 1$ yeilds a list d' with smaller even sum. By the induction hypothesis, some graph G' with degree list d'. Form G by adding an edge with endpoints u and v to G', where $d_{G'}(u) = d_i - 1$ and $d_{G'}(v) = d_j - 1$. Although u and v may already be adjacent in G', the resulting multigraph G has degree list d.

Proof 2 (induction on n). For $n = 1$, put $d_1/2$ loops at v_1. If d_n is even, put $d_n/2$ loops at v_n and apply the induction hypothesis. Otherwise, put an edge from v_n to some other vertex corresponding to positive d_i (which exists since $\sum d_i$ is even) and proceed as before.

1.3.57. *An n-tuple of nonnegative integers with largest entry k is graphic if the sum is even, $k < n$, and every entry is k or $k-1$.* Let $A(n)$ be the set of n-tuples satisfying these conditions. Let $B(n)$ be the set of graphic n-tuples. We prove by induction on n that n-tuples in $A(n)$ are also in $B(n)$. When $n = 1$, the only list in $A(n)$ is (0), and it is graphic.

For the induction step, let d be an n-tuple in $A(n)$, and let k be its largest element. Form d' from d by deleting a copy of k and subtracting 1 from k largest remaining elements. The operation is doable because $k < n$. To apply the induction hypothesis, we need to prove that $d' \in A(n-1)$. Since we delete an instance of k and subtract one from k other values, we reduce the sum by $2k$ to obtain d' from d, so d' does have even sum.

Let q be the number of copies of k in d. If $q > k+1$, then d' has ks and $(k-1)$s. If $q = k+1$, then d' has only $(k-1)$s. If $q < k+1$, then d' has $(k-1)$s and $(k-2)$s. Also, if $k = n-1$, then the first possibility cannot occur. Thus d' has length $n-1$, its largest value is less than $n-1$, and its largest and smallest values differ by at most 1. Thus $d' \in A(n-1)$, and we can apply the induction hypothesis to d'.

The induction hypothesis $(d' \in A(n-1)) \Rightarrow (d' \in B(n-1))$ tells us that d' is graphic. Now the Havel-Hakimi Theorem implies that d is graphic. (Actually, we use only the easy part of the HH Theorem, adding a vertex joined to vertices with desired degrees.)

1.3.58. *If d is a nonincreasing list of nonnegative integers, and d' is obtained by deleting d_k and subtracting 1 from the k largest other elements, then d is graphic if and only if d' is graphic.* The proof is like that of the Havel–Hakimi Theorem. Sufficiency is immediate. For necessity, let w be a vertex of degree d_k in a simple graph with degree sequence d. Alter G by 2-switches to obtain a graph in which w has the d_k highest-degree other vertices as neighbors. The argument to find a 2-switch increasing the number of desired neighbors of w is as in the proof of the Havel–Hakimi Theorem.

1.3.59. *The list $d = (d_1, \ldots, d_{2k})$ with $d_{2i} = d_{2i-1} = i$ for $1 \le i \le k$ is graphic.* This is the degree list for the bipartite graph with vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ defined by $x_r \leftrightarrow y_s$ if and only if $r + s > k$. Since the neighborhood of x_r is $\{y_k, y_{k-1}, \ldots, y_{k-r+1}\}$, the degree of x_r is r. Thus the graph has two vertices of each degree from 1 to k.

1.3.60. *Necessary and sufficient conditions for a list d to be graphic when d consists of k copies of a and $n-k$ copies of b, with $a \ge b \ge 0$.* Since the degree sum must be even, the quantity $ka + (n-k)b$ must be even. In addition, the inequality $ka \le k(k-1) + (n-k)\min\{k, b\}$ must hold, since each vertex with degree b has at most $\min\{k, b\}$ incident edges whose other endpoint has degree a. We construct graphs with the desired degree sequence when these conditions hold. Note that the inequality implies $a \le n-1$.

Case 1: *$b \ge k$ and $a \ge n-k$.* Begin with $K_{k,n-k}$, having partite sets X of size k and Y of size $n-k$. If $k(a-n+k)$ and $(n-k)(b-k)$ are even, then add an $(a-n+k)$-regular graph on X and a $(b-k)$-regular graph on Y. To show that this is possible, note first that $0 \le a-n+k \le k-1$ and $0 \le b-k \le a-k \le n-k-1$. Also, when pq is even, a q-regular graph on p vertices in a circle can be constructed by making each vertex adjacent to the $\lfloor q/2 \rfloor$ nearest vertices in each direction and also to the opposite vertex if q is odd (since then p is even).

Note that $k(a-n+k)$ and $(n-k)(b-k)$ have the same parity, since their difference $ak - (n-k)b$ differs from the given even number $ka + (n-k)b$ by an even amount. If they are both odd, then we delete one edge from $K_{k,n-k}$, and now one vertex in the subgraph on X should have degree $a-n+k+1$ and one in the subgraph on Y should have degree $b-k+1$. When pq is odd, such a graph on vertices $v_0, \ldots, v_{p-1}$ in a circle (q-regular except for one vertex of degree $q+1$) can be constructed by making each vertex adjacent to the $(q-1)/2$ nearest vertices in each direction and then adding the edges $\{v_i v_{i+(p-1)/2}$: $0 \le i \le (p-1)/2$. Note that all vertices are incident to one of the added edges, except that $v_{(p-1)/2}$ is incident to two of them.

Case 2: $k-1 \le a < n-k$. Begin by placing a complete graph on a set S of k vertices. These vertices now have degree $k-1$ and will become the vertices of degree a, which is okay since $a \ge b$. Put a set T of $n-k$ additional vertices in a circle. For each vertex in S, add $a-k+1$ consecutive neighbors in T, starting the next set immediately after the previous set ends. Since $a \le n-1$, each vertex in S is assigned $a-k+1$ distinct neighbors in T. Since $k(a-k+1) \le (n-k)b$ and the edges are distributed nearly equally to vertices of T, there is room to add these edges.

For the subgraph induced by T, we need a graph with $n-k$ vertices and $[(n-k)b - k(a-k+1)]/2$ edges and degrees differing by at most 1. The desired number of edges is integral, since $ka + (n-k)b$ is even, and it is nonnegative, since $k(a-k+1) \le (n-k)b$. The largest degree needed is $\lceil (n-k)b - k(a-k+1) \rceil n-k$. This is at most b, which is less than $n-k$ since $b \le a < n-k$. The desired graph now exists by Exercise 1.3.57.

Case 3: $b < k$ *and* $a \ge n-k$. Put the set S of size k in a circle. For each vertex in the set T of size $n-k$, assign b consecutive neighbors in S; these are distinct since $b < k$. Since $a \ge n-k$, no vertex of S receives too many edges. On S we put an almost-regular graph with k vertices and $[ak - b(n-k)]/2$ edges. Again, this number of edges is integral, and in the case specified it is nonnegative. Existence of such a graph requires $a - b(n-k)/k \le k-1$, which is equivalent to the given inequality $k(a-k+1) \le (n-k)b$. Now again Exercise 1.3.57 provides the needed graph.

Case 4: $b < k$ *and* $a < \min\{k-1, n-k\}$. Since $a < n-k$, also $b < n-k$. Therefore, we can use the idea of Case 1 without the complete bipartite graph. Again take disjoint vertex sets X of size k and Y of size $n-k$. If ka and $(n-k)b$ are even, then we use an a-regular graph on X and a b-regular graph on Y. As observed before, these exist.

Note that ka and $(n-k)b$ have the same parity, since their sum is given to be even. If they are both odd, then we put $\min\{k, n-k\}$ disjoint edges with endpoints in both X and Y. We now complete the graph with a regular graph of even degree on one of these sets and an almost-regular graph guaranteed by Exercise 1.3.57 on the other.

1.3.61. *If G is a self-complementary n-vertex graph and n is odd, then G has a vertex of degree $(n-1)/2$.* Let $d_1, \ldots, d_n$ be the degree list of G in nonincreasing order. The degree list of $\overline{G}$ in nonincreasing order is $n-1-d_n, \ldots, n-1-d_1$. Since $G \cong \overline{G}$, the lists are the same. Since n is odd, the central elements in the list yield $d_{(n+1)/2} = n-1-d_{(n+1)/2}$, so $d_{(n+1)/2} = (n-1)/2$.

1.3.62. *When n is congruent to 0 or 1 modulo 4, there is an n-vertex simple graph G with $\frac{1}{2}\binom{n}{2}$ edges such that $\Delta(G) - \delta(G) \le 1$.* This is satisfied by the construction given in the answer to Exercise 1.1.31.

More generally, let G be any $2k$-regular simple graph with $4k+1$ vertices, where $n = 4k+1$. Such a graph can be constructed by placing $4k+1$ vertices around a circle and joining each vertex to the k closest vertices in each direction. By the Degree-Sum Formula, $e(G) = (4k+1)(2k)/2 = \frac{1}{2}\binom{n}{2}$.

For the case where $n = 4k$, delete one vertex from the graph constructed above to form G'. Now $e(G') = e(G) - 2k = (4k-1)(2k)/2 = \frac{1}{2}\binom{n}{2}$.

1.3.63. *The non-negative integers $d_1 \ge \cdots \ge d_n$ are the vertex degrees of a loopless graph if and only if $\sum d_i$ is even and $d_1 \le d_2 + \cdots + d_n$. Necessity.* If such a graph exists, then $\sum d_i$ counts two endpoints of each edge and must be even. Also, every edge incident to the vertex of largest degree

has its other end counted among the degrees of the other vertices, so the inequality holds.

Sufficiency. Specify vertices $v_1, \ldots, v_n$ and construct a graph so that $d(v_i) = d_i$. Induction on n has problems: It is not enough to make d_n edges join v_1 and v_n degrees and apply the induction hypothesis to $(d_1 - d_n), d_2, \ldots, d_{n-1}$. Although $d_1 - d_n \le d_2 + \cdots + d_{n-1}$ holds, $d_1 - d_n$ may not be the largest of these numbers.

Proof 1 (induction on $\sum d_i$). The basis step is $\sum d_i = 0$, realized by an independent set. Suppose that $\sum d_i > 0$; we consider two cases. If $d_1 = \sum_{i=2}^n d_i$, then the desired graph consists of d_1 edges from v_1 to $v_2, \ldots, v_n$. If $d_1 < \sum_{i=2}^n d_i$, then the difference is at least 2, because the total degree sum is even. Also, at least two of the values after d_1 are nonzero, since d_1 is the largest. Thus we can subtract one from each of the last two nonzero values to obtain a list d' to which we can apply the induction hypothesis (it has even sum, and the largest value is at most the sum of the others. To the resulting G', we add one edge joining the two vertices whose degrees are the reduced values. (This can also be viewed as induction on $(\sum_{i=2}^n d_i) - d_1$.)

Proof 2 (induction on $\sum d_i$). Basis as above. Consider $\sum d_i > 0$. If $d_1 > d_2$, then we can subtract 1 from d_1 and from d_2 to obtain d' with smaller sum. Still $d_1 - 1$ is a largest value in d' and is bounded by the sum of the other values. If $d_1 = d_2$, then we subtract 1 from each of the two smallest values to form d'. If these are d_1 and d_2, then d' has the desired properties, and otherwise $\sum_{i=2}^n d_i$ exceeds d_1 by at least 2, and again d' has the desired properties. In each case, we can apply the induction hypothesis to d' and complete the proof as in Proof 1.

Proof 3 (local change). Every nonnegative integer sequence with even sum is realizable when loops and multiple edges are allowed. Given such a realization with a loop, we change it to reduce the number of loops without changing vertex degrees. Eliminating them all produces the desired realization. If we have loops at distinct vertices u and v, then we replace two loops with two copies of the edge uv. If we have loops only at v and have an edge xy between two vertices other than v, then we replace one loop and one copy of xy by edges vx and vy. Such an edge xy must exist because the sum of the degrees of the other vertices is as large as the degree of v.

1.3.64. *A simple graph with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ is connected if $d_j \ge j$ for all j such that $j \le n-1-d_n$.* Let $V(G) = \{v_1, \ldots, v_n\}$, with $d(v_i) = d_i$, and let H be the component of G containing v_n; note that H has at least $1 + d_n$ vertices. If G is not connected, then G has another component H'. Let j be the number of vertices in H'. Since H has at least $1 + d_n$ vertices, we have $j \le n-1-d_n$. By the hypothesis, $d_j \ge j$. Since H' has j vertices, its maximum degree is at least d_j. Since $d_j \ge j$, there are at least $j+1$ vertices in H', which contradicts the definition of j. Hence G is in fact connected.

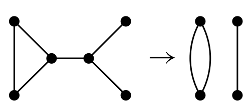
1.3.65. *If $D = \{a_i\}$ is a set of distinct positive integers, with $0 < a_1 < \cdots < a_k$, then there is a simple graph on $a_k + 1$ vertices whose set of vertex degrees (repetition allowed) is D.*

Proof 1 (inductive construction). We use induction on k. For $k = 1$, use K_{a_1+1}. For $k = 2$, use the join $K_{a_1} \vee \overline{K}_{a_2-a_1+1}$. That is, G consists of a clique Q with a_1 vertices, an independent set S with $a_2 - a_1 + 1$ vertices, and all edges from Q to S. The vertices of S have degree a_1, and those of Q have degree a_2.

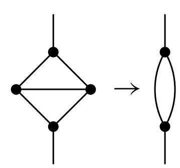
For $k \ge 2$, take a clique Q with a_1 vertices and an independent set S with $a_k - a_{k-1}$ vertices. Each vertex of S has neighborhood Q, and each vertex of Q is adjacent to all other vertices. Other vertices have a_1 neighbors in Q and none in S, so the degree set of $G - Q - S$ should be $\{a_2 - a_1, \ldots, a_{k-1} - a_1\}$. By the induction hypthesis, there is a simple graph H with $a_{k-1} - a_1 + 1$ vertices having this degree set (the degree set is smaller by 2). Using H for $G - Q - S$ completes G as desired.

Proof 2 (induction and complementation). Again use induction on k, using K_{a_1+1} when $k = 1$. For $k > 1$ and $0 < a_1 \cdots < a_k$, the complement of the desired graph with $a_1 + 1$ vertices has degree set $\{a_k - a_1, \ldots, a_k - a_{k-1}, 0\}$. By the induction hypothesis, there is a graph of order $a_k - a_1 + 1$ with degree set $\{a_k - a_1, \cdots, a_k - a_{k-1}\}$. Add a_1 isolated vertices and take the complement to obtain the desired graph G.

1.3.66. *Construction of cubic graphs not obtainable by expansion alone.* A simple cubic graph G that cannot be obtained from a smaller cubic graph by the expansion operation is the same as a cubic graph on which no erasure can be performed, since any erasure yielding a smaller H from G could be inverted by an expansion to obtain G from H. An edge cannot be erased by this operation if and only if one of the subsequent contractions produces a multiple edge. This happens if the other edges incident to the edge being erased belong to a triangle, or in one other case, as indicated below.

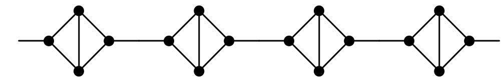


and



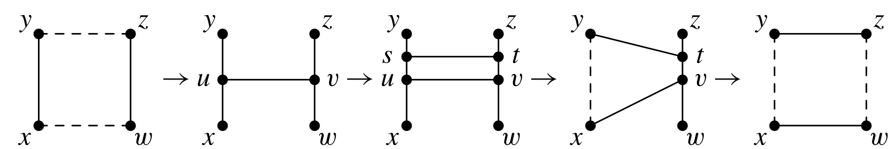
Finally, we need only provide a simple cubic graph with $4k$ vertices where every edge is non-erasable in one of these two ways. To do this place copies of $G_1, \ldots, G_k$ of $K_4 - e$ (the unique 4-vertex graph with 5 edges) around in a ring, and for each consecutive pair G_i, G_{i+1} add an edge joining

a pair of vertices with degree two in the subgraphs, as indicated below, where the wraparound edge has been cut.



1.3.67. *Construction of 3-regular simple graphs*

a) A 2-switch can be performed by performing a sequence of expansions and erasures. We achieve a 2-switch using two expansions and then two erasures as shown below. If the 2-switch deletes xy and zw and introduces xw and yz, then the first expansion places new vertices u and v on xy and zw, the second introduces s and t on the resulting edges ux and vz, the first erasure deletes su and its vertices, and the second erasure deletes tv and its vertices. The resulting vertices are the same as in the original graph, the erasures were legal because they created only edges that were not present originally, and we have deleted xy and zw and introduced xw and yz.



b) Every 3-regular simple graph can be obtained from K_4 by a sequence of expansions and erasures. Erasure is allowed only if no multiple edges result. Suppose H is the desired 3-regular graph. Every 3-regular graph has an even number of vertices, at least four. Any expansion of a 3-regular graph is a 3-regular graph with two more vertices. Hence successive expansions from K_4 produce a 3-regular graph G with $n(H)$ vertices. Since G and H have the same vertex degrees, there is a sequence of 2-switches from G to H. Since every 2-switch can be produced by a sequence of expansions and erasures, we can construct a sequence of expansions and erasures from K_4 to H by going through G.

1.3.68. *If G and H are X,Y-bigraphs, then $d_G(v) = d_H(v)$ for all $v \in X \cup Y$ if and only if there is a sequence of 2-switches that transforms G into H without ever changing the bipartition.* The condition is sufficient, since 2-switches do not change vertex degrees. For necessity, assume that $d_G(v) = d_H(v)$ for all v. We build a sequence of 2-switches transforming G to H.

Proof 1 (induction on $|X|$). If $|X| = 1$, then already $G = H$, so we may assume that $|X| > 1$. Choose $x \in X$ and let $k = d(x)$. Let S be a selection of k vertices of highest degree in Y. If $N(x) \ne S$, choose $y \in S$ and $y' \in Y - S$ so that $x \not\leftrightarrow y$ and $x \leftrightarrow y'$. Since $d(y) \ge d(y')$, there exists $x' \in X$ so that $y \leftrightarrow x'$ and $y' \not\leftrightarrow x'$. Switching $xy', x'y$ for $xy, x'y'$ increases

$|N(x) \cap S|$ with the same bipartition. Iterating this reaches $N(x) = S$; let G' be the resulting graph.

Doing the same in H yields graphs G' from G and H' from H such that $N_{G'}(x) = N_{H'}(x)$. Deleting x and applying the induction hypothesis to the graphs $G^* = G' - x$ and $H^* = H' - x$ completes the construction of the desired sequence of 2-switches.

Proof 2 (induction on number of discrepancies). Let F be the X,Y-bigraph whose edges are those belonging to exactly one of G and H. Let $d = e(F)$. Since G and H have identical vertex degrees, each vertex of F has the same number of incident edges from $E(G)-E(H)$ and $E(H)-E(G)$. When $d > 0$, F therefore has a cycle alternating between $E(G)$ and $E(H)$ (when we enter a vertex on an edge of one type, we can exit on the other type, we can't continue forever, and all cycles have even length).

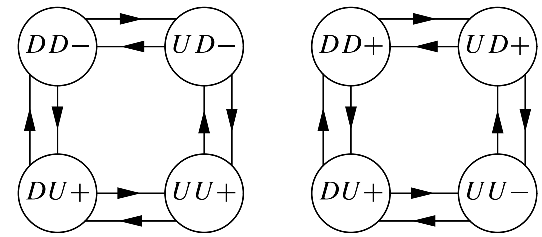
Let C be a shortest alternating cycle in F, with first $xy \in E(G) - E(H)$ and then $yx' \in E(H) - E(G)$ and $x'y' \in E(G) - E(H)$. We consider a 2-switch involving $\{x, y, x', y'\}$. If $y'x \in E(H) - E(G)$, then the 2-switch in G reduces d by 4. If $y'x \in E(G) - E(H)$, then we would have a shorter cycle in F. If $y'x \notin E(G) \cup E(H)$, then we perform the 2-switch in G; if $y'x \in E(G) \cup E(H)$, then we perform the 2-switch in H. Each of these last two cases yields a new pair of graphs with d reduced by 2, and the induction hypothesis applies to this pair to provide the rest of the exchanges.

1.4. DIRECTED GRAPHS

1.4.1. *Digraphs in the real world.* Many digraphs based on temporal order have no cycles. For example, given a set of football games, we can put an edge from game x to game y if game x ends before game y begins. The relation "is a parent of" also works.

Asymmetric digraphs without cycles often arise from tournaments. Each team plays every other team, and there is an edge for each game from the winner to the loser. The result can be without cycles, but usually cycles exist. Another example is the relation "has sent a letter to".

1.4.2. *If the first switch becomes disconnected from the wiring in the lightswitch system of Application 1.4.4, then the digraph for the resulting system is that below.*



1.4.3. *Every u,v-walk in a digraph contains a u,v-path.* The shortest u,v-walk contained in a u,v-walk W is a u,v-path, since the shortest walk has no vertex repetition.

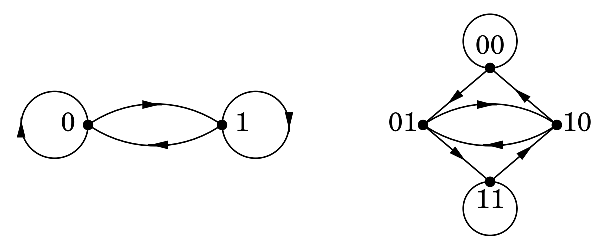
1.4.4. *Every closed walk of odd length in a digraph contains the edges of an odd cycle.* The proof follows that of the corresponding statement for graphs in Lemma 1.2.15, given that the definitions of walk and cycle require the head of each edge to be the tail of the next edge.

We use induction on the length l of a closed odd walk W. Basis step: $l = 1$. A closed walk of length 1 traverses a cycle of length 1.

Induction step: $l > 1$. Assume the claim for closed odd walks shorter than W. If W has no repeated vertex (other than first = last), then W itself forms a cycle of odd length. If vertex v is repeated in W, then we view W as starting at v and break W into two v,v-walks. Since W has odd length, one of these is odd and the other is even. The odd one is shorter than W. By the induction hypothesis, it contains an odd cycle, and this cycle appears in order in W.

1.4.5. *A finite directed graph contains a (directed) cycle if every vertex is the tail of at least one edge (has positive outdegree).* (The same conclusion holds if every vertex is the head of at least one edge.) Let G be such a graph, let P be a maximal (directed) path in G, and let x be the final vertex of P. Since x has at least one edge going out, there is an edge xy. Since P cannot be extended, y must belong to P. Now xy completes a cycle with the y,x-subpath of P.

1.4.6. *The De Bruijn graphs D_2 and D_3.*



1.4.7. *In an orientation of a simple graph with 10 vertices, the vertices can have distinct outdegrees.* Take the orientation of the complete graph with vertices $0, \ldots, 9$ by orienting the edge ij from i to j if $i > j$. In this digraph, the outdegree of vertex i is i.

1.4.8. *There is an n-vertex tournament with $d^+(v) = d^-(v)$ for every vertex v if and only if n is odd.* If n is even, then $d^+(v) + d^-(v) = n - 1$ is odd, so the summands can't be equal integers. For odd n, we construct such a tournament.

Proof 1 (explicit construction). Place the n vertices equally spaced around a circle, and direct the edges from v to the $(n-1)/2$ vertices that follow v in the clockwise direction. After doing this for each vertex, the $(n-1)/2$ nearest vertices in the counterclockwise direction from v have edges directed to v, and each edge has been oriented.

Proof 2 (inductive construction). When $n = 1$, the 1-vertex tournament satisfies the degree condition. For $k > 1$, suppose that T is a tournament with $2k - 1$ vertices that satisfies the condition. Partition $V(T)$ into sets A and B with $|A| = k$ and $|B| = k - 1$. Add two vertices x and y. Add all edges from x to A, from A to y, from y to B, and from B to x. Each vertex in $V(T)$ now has one predecessor and one successor in $\{x, y\}$. We have $d^+(x) = k$, $d^-(x) = k - 1$, $d^+(y) = k - 1$, $d^-(y) = k$. Complete the construction of T' by adding the edge yx. Now T' is a tournament with $2k + 1$ vertices that satisfies the degree condition.

Proof 3 (Eulerian graphs). When n is odd, K_n is a connected even graph and hence is Eulerian. Orienting edges of K_n in the forward direction while following an Eulerian circuit yields the desired tournament.

1.4.9. *For each n, there is an n-vertex digraph in which the vertices have distinct indegrees and distinct outdegrees.* Using vertices $v_1, \ldots, v_n$, let the edges be $\{v_i v_j : 1 \le i < j \le n\}$. Now $d^-(v_i) = i - 1$ and $d^+(v_i) = n - i$. Thus the indegrees are distinct, and the outdegrees are distinct.

1.4.10. *A digraph is strongly connected if and only if for each partition of the vertex set into nonempty sets S and T, there is an edge from S to T.* Given that D is strong, choose $x \in S$ and $y \in T$. Since D has an x, y-path, the path must leave S and enter T and do so along some edge.

Conversely, if there is such an edge for every partition, let S be the set of all vertices reachable from vertex x. If $S \neq V(D)$, then the hypothesis yield an edge leaving S, which adds a vertex to S. Since x was arbitrary, each vertex is reachable from every other, and D is strongly connected.

1.4.11. *In every digraph, some strong component has no entering edges, and some strong component has no exiting edges.*

Proof 1 (using cycles). Given a digraph D, form a digraph D^* with one vertex for each strong component of D. Let the strong components of D be $X_1, \ldots, X_k$, with corresponding vertices $x_1, \ldots, x_k$ in D^*. Put an edge from x_i to x_j in D^* if in D there is an edge from some vertex of X_i to some vertex of X_j. The problem is to show that D^* has a vertex with indegree 0 and a vertex with outdegree 0.

If such vertices do not exist, then D^* has a cycle (by Lemma 1.4.23). If D^* has a cycle, then the union of the strong components of D corresponding to the vertices of the cycle is a strongly connected subgraph of D containing all those components. This is a contradiction, because they were maximal strong subgraphs.

Proof 2 (extremality). For a vertex v in D, let $R(v)$ be the set of vertices reachable from v. Let u be a vertex minimizing $|R(u)|$. If $v \in R(u)$, then $R(v) \subseteq R(u)$, so $R(v) = R(u)$. Since $u \in R(u)$, also u is reachable from v. Thus $R(u)$ induces a strong subdigraph. By the definition of $R(u)$, no edges leave it, so it is a strong component. Applying the same argument to the reverse digraph yields a strong component with no entering edge.

1.4.12. *In a digraph the connection relation is an equivalence relation, and its equivalence classes are the vertex sets of the strong components.* We are defining x to be connected to y if the digraph has both an x, y-path and a y, x-path. The reflexive property holds using paths of length 0. The symmetric property holds by the definition.

For transitivity, consider an x, y-path P_1 and a y, z-path P_2. Let w be the first vertex of P_1 that belongs to P_2. Following P_1 from x to w and P_2 from w to z yields an x, z-path, by the choice of w. Applying this to obtain paths in both directions shows that the connection relation is transitive.

Since a strong component is a strongly connected subdigraph, its pairs of vertices satisfy the connection relation. Hence every strong component is contained in an equivalence class of the connection relation. In order to show that every equivalence class is contained in a strong component, we show that when x is connected to y, there is an x, y-path using only vertices of the equivalence class.

Let P be an x, y-path, and let Q be a y, x-path. The concatenation of Q with P is a closed walk in the digraph; let S be its vertex set. By following the walk, we find a u, v-walk for all $u, v \in S$. Such a walk contains a u, v-path. The same argument yields a v, u-path in the walk. Hence all pairs of vertices on it satisfy the connection relation, and we have found an x, y-path (and y, x-path) witin the equivalence class. Hence the subdigraph induced by the equivalence class is strongly connected.

1.4.13. *Strong components.*

a) Two maximal strongly connected subgraphs of a directed graph share no vertices. If strong components D_1, D_2 of D share a vertex v, then for all

$x \in V(D_1)$ and $y \in V(D_2)$, the union of an x, v-path in D_1 and a v, y-path in D_2 contains an x, y-path in D. Similarly, D has a y, x-path. Thus $D_1 \cup D_2$ is strongly connected.

b) The digraph D^* obtained by contracting the strong components of a digraph D is acyclic (D^* has a vertex v_i for each strong component D_i, with $v_i \to v_j$ if and only if $i \neq j$ and D has an edge from D_i to D_j). If D^* has a cycle with vertices $d_0, \ldots, d_{l-1}$, then D has strong components $D_0, \ldots, D_{l-1}$ such that D has an edge $u_i v_{i+1}$ from D_i to D_{i+1}, for each i (modulo l). If $x \in D_i$ and $y \in D_j$, this means that D contains an x, y-walk consisting of the concatenation of paths with successive endpoints $x, u_i, v_{i+1}, u_{i+1}, v_{i+2}, \ldots, u_{j-1}, v_j, y$. This walk contains an x, y-path. Since x, y were chosen arbitrarily from $D_0 \cup \cdots \cup D_{l-1}$, we conclude that $D_0 \cup \cdots \cup D_{l-1}$ is strongly connected, which contradicts $D_0, \ldots, D_{l-1}$ being maximal strongly connected subgraphs.

1.4.14. *If G is an n-vertex digraph with no cycles, then the vertices of G can be ordered as $v_1, \ldots, v_n$ so that if $v_i v_j \in E(G)$, then $i < j$.* If G has no cycles, then some vertex v has outdegree 0. Put v last in the ordering. Now $G - v$ also has no cycles, and we proceed iteratively. When we choose v_j, it has no successors among $v_1, \ldots, v_{j-1}$, so the desired condition on the edges holds.

1.4.15. *In the simple digraph with vertex set* $\{(i, j) \in \mathbb{Z}^2 \colon 0 \le i \le m$ and $0 \le n\}$ *and an edge from (i, j) to (i', j') if and only if (i', j') is obtained from (i, j) by adding 1 to one coordinate, there are $\binom{m+n}{n}$ paths from $(0, 0)$ to (m, n).* Traversing each edge adds one to each coordinate, so every such path has $m + n$ edges. We can record such a path as a 0, 1-list, recording 0 when we follow an edge that increases the first coordinate, 1 when we follow an edge that increases the second coordinate. Each list with m 0s and n 1s records a unique path. Since there are $\binom{m+n}{n}$ ways to form such a list by choosing positions for the 1s, the bijection implies that the number of paths is $\binom{m+n}{n}$.

1.4.16. *Fermat's Little Theorem.* Let $\mathbb{Z}_n$ denote the set of congruence classes of integers modulo a PRIME NUMBER n (the first printing of the second edition omitted this!). Multiplication by a positive integer a that is not a multiple of n defines a permutation of $\mathbb{Z}_n$, since $ai \equiv aj \pmod n$ yields $a(j - i) \equiv 0 \pmod n$, which requires n to divide $j - i$ when a and n are relatively prime. The functional digraph consists of pairwise disjoint cycles.

a) If G is the functional digraph with vertex set $\mathbb{Z}_n$ for the permutation defined by multiplication by a, then all cycles in G (except the loop on n) have length $l - 1$, where l is the least natural number such that $a^l \equiv a \pmod n$. This is the length of the cycle containing the element 1. Traversing a cycle of length k (not the cycle consisting of n) yields $xa^k \equiv x \pmod n$, or $x(a^k - 1) \equiv 0 \pmod n$, for some x not divisible by n. Since n is prime, this requires $a^k \equiv 1 \pmod n$, and hence $k \ge l - 1$. On the other hand $xa^{l-1} = x$, and hence $k \le l - 1$.

b) $a^{n-1} \equiv 1 \pmod n$. Since all nontrivial cycles have the same length, $l - 1$ divides $n - 1$. Let $m = (n - 1)/(l - 1)$. Now $a^{n-1} = a^{(l-1)m} = (a^{l-1})^m \equiv 1^m \equiv 1 \pmod n$.

1.4.17. *A (directed) odd cycle is a digraph with no kernel.* Let S be a kernel in an odd cycle C. Every vertex must be in S or have a successor in S. Since S is an independent set, exactly one of these two conditions holds at each vertex. Hence we must alternate between vertices in S and vertices not in S as we follow the C. We cannot alternate two conditions as we follow an odd cycle, so there is no kernel.

A digraph having an odd cycle as an induced subgraph and having a kernel. To an odd cycle, add one new vertex as a successor of each vertex on the cycle. The new vertex forms a kernel by itself.

1.4.18. *An acyclic digraph D has a unique kernel.*

Proof 1 (parity of cycles). By Theorem 1.4.16, a digraph with no odd cycles has at least one kernel. We show that a digraph with no even cycles has at most one kernel, by proving the contrapositive. If K and L are distinct kernels (each induces no edges), then every vertex of $K - L$ has a successor in $L - K$, and every vertex of $L - K$ has a successor in $K - L$.

Proof 2 (induction on $n(D)$). In a digraph with one vertex and no cycle, the vertex is a kernel. When $n(D) > 1$, the absence of cycles guarantees a vertex with outdegree 0 (Lemma 1.4.23). Such a vertex lies in every kernel, since it has no successor. Let $S' = \{v \in V(D) \colon d^+(v) = 0\}$. Note that S' induces no edges. Let D' be the subdigraph obtained from D by deleting S' and all vertices having successors in S'. The digraph D' has no cycles; by the induction hypothesis, D' has a unique kernel S''.

Let $S = S' \cup S''$. Since there are no edges from $V(D')$ to S', the set S is a kernel in D. Furthermore, S is the only kernel. We have argued that all of S' is present in every kernel, and independence of the kernel implies that no other vertex outside $V(D')$ is present. The lack of edges from $V(D')$ to S' implies that the remainder of the kernel must be a kernel in D', and there is only one such set.

1.4.19. *A digraph is Eulerian if and only if $d^+(v) = d^-(v)$ for every vertex v and the underlying graph has at most one nontrivial component.*

Necessity. Each passage through a vertex by a circuit uses an entering edge and an exiting edge; this applies also to the "last" and "first" edges of the circuit. Also, two edges can be in the same trail only when they lie in the same component of the underlying graph.

Sufficiency. We use induction on the number of edges, m. Basis step: When $m = 0$, a closed trail consisting of one vertex contains all the edges.

Induction step: Consider $m > 0$. With equal indegree and outdegree, each vertex in the nontrivial component of the underlying graph of our digraph G has outdegree at least 1 in G. By Lemma 1.2.25, G has a cycle C. Let G' be the digraph obtained from G by deleting $E(C)$.

Since C has 1 entering and 1 departing edge at each vertex, G' also has equal indegree and outdegree at each vertex. Each component of the underlying graph H' of G' is the underlying graph of some subgraph of G'. Since G' has fewer than m edges, the induction hypothesis yields an Eulerian circuit of each such subgraph of G'.

To form an Eulerian circuit of G, we traverse C, but when a component of H' is entered for the first time we detour along an Eulerian circuit of the corresponding subgraph of G', ending where the detour began. When we complete the traversal of C, we have an Eulerian circuit of G.

1.4.20. *A digraph is Eulerian if and only if indegree equals outdegree at every vertex and the underlying graph has at most one nontrivial component.* The conditions are necessary, since each passage through a vertex uses one entering edge and one departing edge.

For sufficiency, suppose that G is a digraph satisfying the conditions. We prove first that every non-extendible trail in G is closed. Let T be a non-extendible trail starting at u. Each time T passes through a vertex v other than u, it uses one entering edge and one departing edge. Thus upon each arrival at v, T has used one more edge entering v than departing v. Since $d^+(v) = d^-(v)$, there remains an edge on which T can continue. Hence a non-extendible trail can only end at v and must be closed.

We now show that a trail of maximal length in G must be an Eulerian circuit. Let T be a trail of maximum length; T must also be non-extendible, and hence T is closed. Suppose that T omits some edge e of G. Since the underlying graph of G has only one nontrivial component, it has a shortest path from e to the vertex set of T. Hence some edge e' not in T is incident to some vertex v of T. It may enter or leave v.

Since T is closed, there is a trail T' that starts and ends at v and uses the same edges as T. We now extend T' along e' (forward or backward depending on whether e leaves or enters v) to obtain a longer trail than T. This contradicts the choice of T, and hence T traverses all edges of G.

1.4.21. *A digraph has an Eulerian trail if and only if the underlying graph has only one nontrivial component and $d^-(v) = d^+(v)$ for all vertices or for all but two vertices, in which case in-degree and out-degree differ by one for the other two vertices.* Sufficiency: since the total number of heads equals the total number of tails, the vertices out of balance consist of x with an extra head and y with an extra tail. Add the directed edge xy and apply the characterization above for Eulerian digraphs.

1.4.22. *If D is a digraph with $d^-(v) = d^+(v)$ for every vertex v, except that $d^+(x) - d^-(x) = k = d^-(y) - d^+(y)$, then D contains k pairwise edge-disjoint x, y-paths.* Form a digraph D' by adding k edges from y to x. Since indegree equals outdegree for every vertex of D', the "component" of D' containing x and y is Eulerian. Deleting the added edges from an Eulerian circuit cuts it at k places; the resulting k directed trails are x, y-trails in the digraph D. As proved in Chapter 1, the edge set of every x, y-trail contains an x, y-path; the proof in Chapter 1 applies to both graphs and digraphs.

1.4.23. *Every graph G has an orientation such that $|d^+(v) - d^-(v)| \le 1$ for all v.*

Proof 1 (Eulerian circuits). Add edges to pair up vertices of odd degree (if any exist). Each component of this supergraph G' is Eulerian. Orient G' by following an Eulerian circuit in each component, orienting each edge forward as the circuit is traversed. The circuit leaves each vertex the same number of times as it enters, so the resulting orientation has equal indegree and outdegree at each vertex.

Deleting the edges of $E(G') - E(G)$ now yields the desired orientation of G, because at most one edge was added at each vertex to pair the vertices of odd degree. Deleting at most one incident edge at v produces difference at most one between $d^+(v)$ and $d^-(v)$.

Proof 2 (induction on $e(G)$). If $e(G) = 0$, then the claim holds. For $e(G) > 0$, if G has a cycle H, then orient H consistently, with no imbalance anywhere. If G has no cycle, then find a maximal path H and orient it consistently. This creates imbalance of 1 at the endpoints and 0 elsewhere. The endpoints have degree 1, so no further imbalance occurs there. In both cases, delete $E(H)$ and apply the induction hypothesis to complete the orientation.

1.4.24. *Not every graph has an orientation such that for every vertex subset, the numbers of edges entering and leaving differ by at most one.* Let G be a graph with at least four vertices such that every vertex degree is odd. Let D be an orientation of G. In D, no vertex of G has the same number of vertices entering and leaving. Let $S = \{v \in V : d^+(v) > d^-(v)\}$. Since each edge within S contributes the same amount to $\sum_{v \in S} d^+(v)$ and $\sum_{v \in S} d^-(v)$, there are $\sum_{v \in S} d^+(v) - \sum_{v \in S} d^-(v)$ more edges leaving S than entering. The difference is at least $|S|$. Similarly, for $\overline{S}$ the absolute difference is at least $|\overline{S}|$, so always some set has difference at least $n(G)/2$.

1.4.25. *Orientations and P_3-decomposition. a) Every connected graph has an orientation having at most one vertex with odd outdegree.*

Proof 1 (local change). Given an orientation of G with vertices x and y having odd outdegree, find an x, y-path P in the underlying graph and flip the orientation of every edge on P. This does not change the parity of the outdegree for any internal vertex of P, but it changes the parity of the outdegree for the endpoints, which previously had odd outdegree. Hence this operation reduces the number of vertices of odd outdegree by 2. We can apply this operation whenever at least two vertices have odd outdegree, so we can reduce the number of vertices with odd outdegree to 0 or 1.

Proof 2 (application of Eulerian circuits). Suppose that G has $2k$ vertices of odd degree. Add edges that pair these vertices to form an Eulerian supergraph G'. Follow an Eulerian circuit of G', starting from u along $uv \in E(G)$, producing an orientation of G as follows. Orient uv out from u; now u has odd outdegree and all other vertices have even outdegree. Subsequently, when the circuit traverses an edge $xy \in E(G)$, orient it so that x has even outdegree among the edges oriented so far. At each stage, the only vertex that can have odd outdegree among edges of G is the current vertex. The orientation chosen for the edges not in $E(G)$ is unimportant.

b) A simple connected graph with an even number of edges can be decomposed into paths with two edges. Since the sum of the outdegrees is the number of edges, the parity of the number of vertices with odd outdegree is the same as the parity of the number of edges. Hence part (a) implies that a connected graph with an even number of edges has an orientation in which every vertex has even outdegree. At each vertex, pair up exiting edges arbitrarily. Since G is simple, this decomposes G into copies of P_3.

1.4.26. *De Bruijn cycle for binary words of length 4, avoiding 0101 and 1010.* Make a vertex for each of the 8 sequences of length 3 from the alphabet $S = \{0, 1\}$. Put an edge from sequence a to sequence b, with label $\alpha \in S$, if b is obtained from a by dropping the first letter of a and appending α to the end. Traveling this edge from a corresponds to having α in sequence after a. We want our digraph to have 14 edges corresponding to the desired 14 words, and we want an Eulerian circuit through them to generate the cyclic arrangement of labels. The difference between this digraph and the De Bruijn digraph in Application 1.4.25 is omitting the two edges joining 010 and 101. The resulting digraph still has indegree = outdegree at every vertex, so it is Eulerian. One arrangement of labels generated by an Eulerian circuit is 00001001101111.

1.4.27. *De Bruijn cycle for any alphabet and length.* When A is an alphabet of size k, there exists a cyclic arrangement of k^l characters chosen from A such that the k^l strings of length l in the sequence are all distinct.

Idea: The indegree and outdegree is k at each vertex of the digraph constructed in the matter analogous to that for $k = 2$. Thus the digraph is Eulerian, and recording the edge labels along an Eulerian circuit yields the desired sequence. Below we repeat the details.

Define a digraph $D_{k,l}$ whose vertices are the $(l-1)$-tuples with elements in A. Place an edge from a to b if the last $n-2$ entries of a agree with the first $n-2$ entries of b. Label the edge with the last entry of b. For each vertex a, there are k ways to append a element of A to lengthen its name, and hence there are k edges leaving each vertex.

Similarly, there are k choices for a character deleted from the front of a predecessor's name to obtain name b, so each vertex has indegree k. Also, we can reach $b = (b_1, \dots, b_{n-1})$ from any vertex by successively following the edges labeled $b_1, \dots, b_{n-1}$. Since $D_{k,l}$ is strongly connected and has indegree equal to outdegree at every vertex, the characterization of Eulerian digraphs implies that $D_{k,l}$ is Eulerian.

Let C be an Eulerian circuit of $D_{k,l}$. When we are at the vertex with name $a = (a_1, \dots, a_{n-1})$ while traversing C, the most recent edge had label a_{n-1}, because the label on an edge entering a vertex agrees with the last digit of the sequence at the vertex. Since we delete the front and shift the rest to obtain the rest of the label at the head, the successive earlier labels (looking backward) must have been $a_{n-2}, \dots, a_1$ in order. If C next traverses an edge with label a_n, then the subsequence consisting of the n most recent edge labels at that time is $a_1, \dots, a_n$.

Since the k^{l-1} vertex labels are distinct, and the edges leaving each vertex have distinct labels, and we traverse each edge from each vertex exactly once along C, the k^l strings of length l in the circular arrangement given by the edge labels along C are distinct.

1.4.28. *De Bruijn cycle for length 4 without the constant words.* Make a vertex for each of the m^3 sequences of length 3 from the alphabet S. Put an edge from sequence a to sequence b, with label $\alpha \in S$, if b is obtained from a by dropping the first letter and appending α to the end. Since there are m ways to append a letter, the out-degree of each vertex is m. For each sequence, there are m possible letters that could have been deleted to reach it, so the in-degree of each vertex is m.

Deleting the loops at the m constant vertices (aaa, bbb, etc.) reduces the indegree and outdegree at those vertices by 1, so the resulting digraph has equal indegree and outdegree at every vertex. Also the underlying graph is connected, since vertex abc can be reach from any other vertex by following the edge labeled a, then b, then c.

Thus an Eulerian circuit exists. Recording the edge labels while following an Eulerian circuit yields the desired arrangement. The 4-digit strings obtained are those formed by the 3-digit name of a vertex plus the label on an exiting edge. These $m^4 - m$ strings are distinct and avoid the constant words, since the loops were deleted from the digraph.

Alternative proof. If we know (from Exercise 1.4.27, for example) that there exists a De Bruijn cycle including the constant words, then we can simply delete one letter from each string of four consecutive identical letters, without using graph theory.

1.4.29. *A strong orientation of a graph that has an odd cycle also has an odd (directed) cycle.* Suppose that D is a strong orientation of a graph G that has an odd cycle $v_1, \ldots, v_{2k+1}$. Since D is strongly connected, for each i there is a v_i, v_{i+1}-path in D. If for some i every such path has even length, then the edge between v_i and v_{i+1} points from v_{i+1} to v_i, since the other orientation would be a v_i, v_{i+1}-path of length 1 (odd). In this case, we have an odd cycle through v_i and v_{i+1}. Otherwise, we have a path of odd length from each v_i to v_{i+1}. Combining these gives a closed trail of odd length. In a digraph as well as in a graph (by the same proof), a closed odd trail contains the edges of an odd cycle.

1.4.30. *The maximum length of a shortest spanning closed walk in a strongly-connected n-vertex digraph is $\lfloor (n+1)^2/4 \rfloor$ if $n \ge 3$.* For the lower bound, let G consist of a u, v-path P of $n-l$ vertices, plus l vertices with edges from v and to u. When leaving a vertex not on P, P must be reached and traversed before the next vertex off P. Hence G requires $l(n-l+1)$ steps to walk through every vertex, maximized by setting $l = \lfloor (n+1)/2 \rfloor$. The length of the walk is then $\lfloor (n+1)^2/4 \rfloor$.

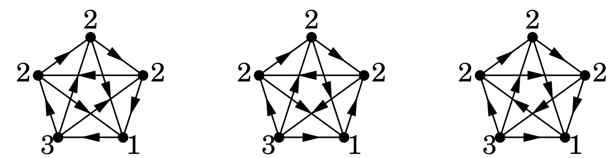
For any strongly-connected n-vertex digraph G, we obtain a spanning closed walk of length at most $\lfloor (n+1)^2/4 \rfloor$. Let m be the maximum length of a path in G; from each vertex to every other, there is a path of length at most m. Begin with a path P of length m; this visits $m+1$ vertices. Next use paths to reach each of the remaining vertices in turn, followed by a path returning to the beginning of P. In this closed walk, $1+(n-m-1)+1$ paths have been followed, each of length at most m. The total length is at most $m(n+1-m)$, which is bounded by $\lfloor (n+1)^2/4 \rfloor$.

1.4.31. *The smallest nonisomorphic pair of tournaments with the same score sequences have five vertices.*

At least five vertices are needed. The score sequence (outdegrees) of an n-vertex tournament can have only one 0 or $n-1$. Nonisomorphic tournaments with such a vertex must continue to be nonisomorphic when that vertex is deleted. Hence a smallest nonisomorphic pair has no vertex with score 0 or $n-1$. The only such score sequences with fewer than 5 vertices are 111 and 2211. The first is realized only by the 3-cycle. For 2211, name the low-degree vertices as v_1 and v_2 such that $v_1 \leftarrow v_2$, and name the high-degree vertices as v_3 and v_4 such that $v_3 \leftarrow v_4$. The only way to complete a tournament with this score sequence is now $N^+(v_1) = \{v_4\}$, $N^+(v_2) = \{v_1\}$, $N^+(v_3) = \{v_1, v_2\}$, and $N^+(v_4) = \{v_2, v_3\}$.

Five vertices suffice, by construction. On five vertices, the sequences to

consider are 33211, 32221, and 22222. There is only one isomorphic class with score sequence 22222, but there are more for the other two sequences. In fact, there are 3 nonisomorphic tournaments with score sequence 32221. They may be characterized as follows: (1) the bottom player beats the top player, and the three middle players induce a cyclic subtournament; (2) the top player beats the bottom player, and the three middle players induce a cyclic subtournament; (3) the top player beats the bottom player, and the three middle players induce a transitive subtournament.



Five vertices suffice, by counting. Each score sequence sums to 10 and has maximum outdegree at most 4; also there is at most one 4 and at most one 0. The possibilities are thus 43210, 43111, 42220, 42211, 33310, 33220, 33211, 32221, 22222. There are 2^{10} tournaments on five vertices; we show that they cannot fit into nine isomorphism classes. The isomorphism class consisting of a 5-cycle plus edges from each vertex to the vertex two later along the cycle occurs 4! times; once for each cyclic ordering of the vertices. Each of the other isomorphism classes occurs at most 5! times. Hence the nine isomorphism classes contain at most $24 + 8 \cdot 120$ of the 2^{10} tournaments. Since $1024 > 984$, there must be at least 10 isomorphism classes among the nine score sequences.

1.4.32. *Characterization of bigraphic sequences.* With $p = p_1, \ldots, p_m$ and $q = q_1, \ldots, q_n$, the pair (p, q) is **bigraphic** if there is a simple bipartite graph in which $p_1, \ldots, p_m$ are the degrees for one partite set and $q_1, \ldots, q_n$ are the degrees for the other.

If p has positive sum, then (p, q) is bigraphic if and only if (p', q') is bigraphic, where (p', q') is obtained from (p, q) by deleting the largest element Δ from p and subtracting 1 from each of the Δ largest elements of q. We follow the method of Theorem 1.3.31. Sufficiency of the condition follows by adding one vertex to a realization of the smaller pair.

For necessity, choose indices in a realization G so that $p_1 \ge \cdots \ge p_m$, $q_1 \ge \cdots \ge q_n$, $d(x_i) = p_i$, and $d(y_j) = q_j$. We produce a realization in which x_1 is adjacent to $y_1, \ldots, y_{p_1}$. If $y_j \not\leftrightarrow x_1$ for some $j \le p_1$, then $y_k \leftrightarrow x_1$ for some $k > p_1$. Since $q_j \ge q_k$, there exists x_i with $i > 1$ such that $x_i \in N(y_j) - N(y_k)$. We perform the 2-switch to replace $\{x_1 y_k, x_i y_j\}$ with $\{x_1 y_j, x_i y_k\}$. This reduces the number of missing neighbors, so we can obtain the desired realization. (Comment: the statement also holds when $m = 1$.)

1.4.33. *Bipartite 2-switch and 0,1-matrices with fixed row and column sums.* With a simple X, Y-bigraph G, we associate a 0,1-matrix $B(G)$ with rows indexed by X and columns indexed by Y. The matrix has a 1 in position i, j if and only if $x_i \leftrightarrow y_j$. Applying a 2-switch to G that exchanges $xy, x'y'$ for $xy', x'y$ (preserving the bipartition) affects $B(G)$ by interchanging the 0's and 1's in the 2 by 2 permutation submatrix induced by rows x, x' and columns y, y'. Hence there is a sequence of 2-switches transforming G to H without changing the bipartition if and only if there is a sequence of switches on 2 by 2 permutation submatrices that transforms $B(G)$ to $B(H)$.

Furthermore, G and H have the same bipartition and same vertex degrees if and only if $B(G)$ and $B(H)$ have the same row sums and the same column sums. Therefore, in the language of bipartite graphs the statement about matrices becomes "all bipartite graphs with the same bipartition and vertex degrees can be reached from each other using 2-switches preserving the bipartition." We prove either statement by induction. We use the phrasing of bipartite graphs.

Proof 1 (induction on m). If $m = 1$, then already $G = H$. For $m > 1$, let G be an X, Y-bigraph. Let x be a vertex of maximum degree in X, with $d(x) = k$. Let S be a set of k vertices of highest degree in Y. Using bipartition-preserving 2-switches, we transform G so that $N(x) = S$. If $N(x) \neq S$, we choose $y \in S$ and $y' \in Y - S$ so that $x \not\leftrightarrow y$ and $x \leftrightarrow y'$. Since $d(y) \geq d(y')$, we have $x' \in X$ so that $y \leftrightarrow x'$ and $y' \not\leftrightarrow x'$. Switching $xy', x'y$ for $xy, x'y'$ increases $|N(x) \cap S|$. Iterating this reaches $N(x) = S$. We can do the same thing in H to reach graphs G' from G and H' from H such that $N_{G'}(x) = N_{H'}(x)$. Now we can delete x and apply the induction hypothesis to the graphs $G^* = G' - x$ and $H^* = H' - x$ to complete the construction of the desired sequence of 2-switches.

Proof 2 (induction on number of discrepancies). Let F be the bipartite graph with the same bipartition as G and H consisting of edges belonging to exactly one of G and H. Let $d = e(F)$. Orient F by directing each edge of $G - E(H)$ from X to Y and each edge of $H - e(G)$ from Y to X. Since G, H have identical vertex degrees, in-degree equals outdegree at each vertex of F. If $d > 0$, this implies that F contains a cycle. There is a 2-switch in G that introduces two edges of $E(G) - E(H)$ and reduces d by 4 if and only if F has a 4-cycle. Otherwise, Let C be a shortest cycle in F, and let x, y, x', y' be consecutive vertices on C. We have $xy \in E(G) - E(H)$, $x'y \in E(H) - E(G)$, and $x'y' \in E(G) - E(H)$. We also have $xy' \notin E(G)$, else we could replace these three edges of C by xy' to obtain a shorter cycle in F. We can now perform the 2-switch in G that replaces $xy, x'y'$ with $xy', x'y$. This reduces d by at least 2.

1.4.34. *If G and H are two tournaments on a vertex set V, then $d_G^+(v) = d_H^+(v)$ for all $v \in V$ if and only if G can be turned into H by a sequence of direction-reversals on cycles of length 3.* Reversal of a 3-cycle changes no outdegree, so the condition is sufficient.

For necessity, let F be the subgraph of G consisting of edges oriented the opposite way in H. Since $d_G^+(v) = d_H^+(v)$ and $d_G^-(v) = d_H^-(v)$ for all v, every vertex has the same indegree and outdegree in F. Let x be a vertex of maximum degree in F, and let $S = N_F^+(x)$ and $T = N_F^-(x)$.

An edge from S to T in G completes a 3-cycle with x whose reversal in G reduces the number of pairs on which G and H disagree. An edge from T to S in H completes a 3-cycle with x whose reversal in H reduces the number of disagreements. If neither of these possibilities occurs, then G orients every edge of $S \times T$ from T to S, and H orients every such edge from S to T. Also F has edges from T to x. This gives every vertex of T higher outdegree than x in F, contradicting the choice of x.

1.4.35. *$p_1 \leq \cdots \leq p_n$ is the sequence of outdegrees of a tournament if and only if $\sum_{i=1}^k p_i \geq \binom{k}{2}$ and $\sum_{i=1}^n p_i = \binom{n}{2}$.* *Necessity*. A tournament has $\binom{n}{2}$ edges in total, and any k vertices have out-degree-sum at least $\binom{k}{2}$ within the subtournament they induce.

Sufficiency. Given a sequence p satisfying the conditions, let $q_k = \sum_{i=1}^k p_k$ and $e_k = q_k - \binom{k}{2}$. We prove sufficiency by induction on $\sum e_k$. The only sequence p with $\sum e_k = 0$ is $0, 1, \ldots, n-1$; this is realized by the *transitive* tournament T_n having $v_k \to v_j$ if and only if $k > j$. If $\sum e_k > 0$, let r be the least k with $e_k > 0$, and let s be the least index above r with $e_k = 0$, which exists since $e_n = 0$. We have $q_{s-1} > \binom{s-1}{2}$, $q_s = \binom{s}{2}$, and $q_{s+1} \geq \binom{s+1}{2}$. This yields $p_{s+1} \geq s$ and $p_s < s-1$, or $p_{s+1} - p_s \geq 2$. Similarly, if $r = 1$ we have $p_1 \geq 1$, and if $r > 1$ we have $p_r - p_{r-1} \geq 2$.

Hence we can subtract one from p_r and add one to p_s to obtain a new sequence p' that is non-decreasing, satisfies the conditions, and reduces $\sum e_k$ by $s - r$. By the induction hypothesis, there is a tournament with score sequence p'. If $v_s \to v_r$ in this tournament, we can reverse this edge to obtain the score sequence p. If not, then the fact that $p'_s \geq p'_r$ implies there is another vertex u such that $v_s \to u$ and $u \to v_r$; obtain the desired tournament by reversing these two edges.

1.4.36. Let T be a tournament having no vertex with indegree 0.

a) If x is a king in T, then T has another king in $N^-(x)$. The subdigraph induced by the vertices of $N^-(x)$ is also a tournament; call it T'. Since every tournament has a king, T' has a king. Let y be a king in T'. Since x is a successor of y and every vertex of $N^+(x)$ is a successor of x, every vertex of $V(T) - V(T')$ is reachable from y by a path in T of length at most T. Hence y is also a king in the original tournament T.

b) T has at least three kings. Since T is a tournament, it has some king, x. By part (a), T has another king y in $N^-(x)$. By part (a) again, T has another king z in $N^-(y)$. Since $y \to x$, we have $x \notin N^-(y)$, and hence $z \neq x$. Thus x, y, z are three distinct kings in T.

c) For $n \geq 3$, an n-vertex tournament T with no source and only three kings. Let $S = \{x, y, z\}$ be a set of three vertices in $V(T)$. Let the subtournament on S be a 3-cycle. For all edges joining S and $V(T) - S$, let the endpoint in S be the tail. Place any tournament on $V(T) - S$. Now x, y, z are kings, but no vertex outside S is a king, because no edge enters S.

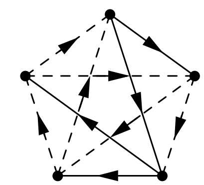
1.4.37. *Algorithm to find a king in a tournament T: Select $x \in V(T)$. If x has indegree 0, call it a king and stop. Otherwise, delete $\{x\} \cup N^+(x)$ from T to form T', and call the output from T' a king in T.* We prove the claims by induction on the number of vertices. The algorithm terminates, because it either stops by selecting a source (indegree 0) or moves to a smaller tournament. By the induction hypothesis, it terminates on the smaller tournament. Thus in each case it terminates and declares a king.

We prove by induction on the number of vertices that the vertex declared a king is a king. When there is only one vertex, it is a king. Suppose that $n(T) > 1$. If the initial vertex x is declared a king immediately, then it has outdegree $n - 1$ and is a king. Otherwise, the algorithm deletes x and its successors and runs on the tournament T' induced by the set of predecessors (in-neighbors) of x.

By the induction hypothesis, the vertex z that the algorithm selects as king in T' is a king in T', reaching each vertex of T' in at most two steps. It suffices to show that z is also a king in the full tournament. Since T' contains only predecessors of x, $z \to x$. Also, z reaches all successors of x in two steps through x. Thus z also reaches all discarded vertices in at most two steps and is a king in T.

1.4.38. *Tournaments with all players kings. a) If n is odd, then there is an tournament with n vertices such that every player is a king.*

Proof 1 (explicit construction). Place the players around a circle. Let each player defeat the $(n-1)/2$ players closest to it in the clockwise direction, and lose to the $(n-1)/2$ players closest to it in the counterclockwise direction. Since every pair of players is separated by fewer players around one side of the circle than the other, this gives a well-defined orientation to each edge. All players have exactly $(n-1)/2$ wins. Thus every outdegree is the maximum outdegree, and we have proved that every vertex of maximum outdegree in a tournament is a king. It is also easy to construct explicit paths. Each player beats the next $(n-1)/2$ players. The remaining $(n-1)/2$ players all lose to the last of these first $(n-1)/2$ players. The construction is illustrated below for five players.



Proof 2 (induction on n). For $n = 3$, every vertex in the 3-cycle is a king. For $n \geq 3$, given a tournament on vertex set S of size n in which every vertex is a king, we add two new vertices x, y. We orient $S \to x \to y \to S$. Every vertex of S reaches x in one step and y in two; x reaches y in one step and each vertex of S in two. Every vertex is a king. (The resulting tournaments are not regular.) Note: Since there is no such tournament when $n = 4$, one must also give an explicit construction for $n = 6$ to include in the basis. The next proof avoids this necessity.

Proof 3 (induction on n). For $n = 3$, we have the cyclic tournament. For $n = 5$, we have the cyclically symmetric tournament in which each vertex beats the two vertices that follow it on the circle. For $n > 5$, let T be an $(n-1)$-vertex tournament in which every vertex is a king, as guaranteed by the induction hypothesis. Add a new vertex x.

If n is odd, then partition $V(T)$ into pairs. For each pair, let a and b be the tail and head of the edge joining them, and add the edges xa and bx.

If n is even, then among any four vertices of $V(T)$ we can find a triple $\{u, v, w\}$ that induces a non-cyclic tournament. Pick one such triple, and partition the remaining vertices of $V(T)$ into pairs. Treat the edges joining x to these pairs as in the other case. Letting u be the vertex of the special triple with edges to the two other vertices, add edges xu, vx, and wx.

b) There is no tournament with four players in which every player is a king. Suppose G is such a tournament. A player with no wins cannot be a king. If some vertex has no losses, then no **other** vertex can be a king. Hence every player of G has 1 or 2 wins. Since the total wins must equal the total losses, there must be two players with 1 win and two players with 2 wins. Suppose x, y are the players with 1 win; by symmetry, suppose x beats y. Since x has no other win and y has exactly one win, the fourth player is not reached in two steps from x, and x is not a king.

1.4.39. *Every loopless digraph D has a vertex subset S such that $D[S]$ has no edges but every vertex is reachable from S by a path of length at most 2.*

Proof 1 (induction). The claim holds when $n(D) = 1$ and when there is a vertex with edges to all others. Otherwise, consider an arbitrary vertex x, and let $D' = D - x - N^+(x)$. Let S' be the subset of $V(D')$ guaranteed by the induction hypothesis. Observe that $S' \cap N^+(x) = \varnothing$. If $yx \in E(D)$ for some $y \in S'$, then $x \cup N^+(x)$ is reachable from y within two steps, and S' is the desired set S. Otherwise, the set $S = S' \cup \{x\}$ works.

Proof 2 (construction). Index the vertices as $v_1, \ldots, v_n$. Process the list in increasing order; when a vertex v_i is reached that has not been deleted, delete all successors of v_i with higher indices. Next process the list in decreasing order; when a vertex v_i is reached that has not been deleted (in either pass), delete all successors of v_i with lower indices.

The set S of vertices that are not deleted in either pass is independent. Every vertex deleted in the second pass has a predecessor in S. Every vertex deleted in the first pass can be reached from S directly or from a vertex deleted in the second pass, giving it a path of length at most two from S. Hence S has the desired properties.

Proof 3 (kernels). By looking at the reverse digraph, it suffices to show that every loopless digraph D has an independent set S that can be reached by a path of length at most 2 from each vertex outside S. Given a vertex ordering $v_1, \ldots, v_n$, decompose D into two acyclic spanning subgraphs G and H consisting of the edges that are forward and backwards in the ordering, respectively. All subgraphs of G and H are acyclic, and hence by Theorem 1.4.16 they have kernels. Let S be a kernel of the subgraph of G induced by a kernel T of H. Every vertex not in T has a successor in T, and every vertex in $T - S$ has a successor in S, so every vertex not in S has a path of length at most 2 to S. (Comment: The set S produced in this way is the same set produced in the reverse digraph by Proof 2. This proof is attributed to S. Thomasse on p. 163 of J. A. Bondy, Short proofs of classical theorems, *J. Graph Theory* 44 (2003), 159–165.)

1.4.40. *The largest unipathic subgraphs of the transitive tournament have $\lfloor n^2/4 \rfloor$ edges.* If a subgraph of T_n contains all three edges of any 3-vertex induced subtournament, then it contains two paths from the least-indexed of these vertices to the highest. Hence a unipathic subgraph must have as its underlying graph a triangle-free subgraph of K_n. By Mantel's Theorem, the maximum number of edges in such a subgraph is $\lfloor n^2/4 \rfloor$, achieved only by the complete equibipartite graph.

This leaves the problem of finding unipathic orientations of $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ in T_n. Suppose G is such a subgraph, with partite sets X, Y. If there are four vertices, say $i < j < k < l$, that alternate from the two partite sets of G or have i, l in one set and j, k in the other, then the oriented bipartite subgraph induced by X, Y as partite sets has two i,l-paths. Hence when $n \ge 4$ all the vertices of X must precede all the vertices of Y, or vice versa. To obtain $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, we will have all edges ij such that $i \le \lfloor n/2 \rfloor$ and $j > \lfloor n/2 \rfloor$, or all edges such that $i \le \lceil n/2 \rceil$ and $j > \lceil n/2 \rceil$. Hence for $n \ge 4$ there are two extremal subgraphs when n is odd and only one when n is even. (There is only one when $n = 1$, and there are three when $n = 3$.)

1.4.41. *Given any listing of the vertices of a tournament, every sequence of switchings of consecutive vertices that induce a reverse edge leads to a list with no reverse edges in at most $\binom{n}{2}$ steps.* Under this algorithm, each switch changes the order of only one pair. Furthermore, the order of two elements in the list can change only when they are consecutive and induce a reverse edge. Hence each pair is interchanged at most once, and the algorithm terminates after at most $\binom{n}{2}$ steps with a spanning path.

1.4.42. *Every ordering of the vertices of a tournament that minimizes the sum of lengths of the feedback edges puts the vertices in nonincreasing order of outdegree.* For the ordering $v_1, \ldots, v_n$, the sum is the sum of $j - i$ over edges $v_j v_i$ such that $j > i$. Consider the interchange of v_i and v_{i+1}. If some vertex is a successor of both or predecessor of both, then the contribution to the sum from the edges involving it remains unchanged. If $x \in N^+(v_i) - N^+(v_{i+1})$, then the switch increases the contribution from these edges by 1. If $x \in N^+(v_{i+1}) - N^+(v_i)$, then the switch decreases the contribution from these edges by 1. If $v_i \to v_{i+1}$, then the switch increases the cost by 1, otherwise it decreases. Hence the net change in the sum of the lengths of feedback edges is $d^+(v_i) - d^+(v_{i+1})$.

This implies that if the ordering has any vertex followed by a vertex with larger outdegree, then the sum can be decreased. Hence minimizing the sum puts the vertices in nonincreasing order of outdegree. Furthermore, permuting the vertices of a given outdegree among themselves does not change the sum of the lengths of feedback edges, so every ordering in nonincreasing order of outdegree minimizes the sum.

2.TREES AND DISTANCE

2.1. BASIC PROPERTIES

2.1.1. *Trees with at most 6 vertices having specified maximum degree or diameter.* For maximum degree k, we start with the star $K_{1,k}$ and append leaves to obtain the desired number of vertices without creating a vertex of larger degree. For diameter k, we start with the path P_{k+1} and append leaves to obtain the desired number of vertices without creating a longer path. Below we list all the resulting isomorphism classes.

For $k = 0$, the only tree is K_1, and for $k = 1$, the only tree is K_2 (diameter or maximum degree k). For larger k, we list the trees in the tables. Let $T_{i,j}$ denote the tree with $i + j$ vertices obtained by starting with one edge and appending $i - 1$ leaves to one endpoint and $j - 1$ leaves at the other endpoint (note that $T_{1,k} = K_{1,k}$ for $k \geq 1$). Let Q be the 6-vertex tree with diameter 4 obtained by growing a leaf from a neighbor of a leaf in P_5. Let n denote the number of vertices.

maximum degree k

k / n	2	3	4	5
3	P_3			
4	P_4	$K_{1,3}$		
5	P_5	$T_{2,3}$	$K_{1,4}$	
6	P_6	$T_{3,3}, Q$	$T_{2,4}$	$K_{1,5}$

diameter k

k / n	2	3	4	5
3	P_3			
4	$K_{1,3}$	P_4		
5	$K_{1,4}$	$T_{2,3}$	P_5	
6	$K_{1,5}$	$T_{2,4}, T_{3,3}$	Q	P_6

2.1.2. *Characterization of trees.*

a) A graph is tree if and only if it is connected and every edge is a cut-edge. An edge e is a cut-edge if and only if e belongs to no cycle, so there are no cycles if and only if every edge is a cut-edge. (To review, edge $e = uv$ is a cut edge if and only if $G - e$ has no u, v-path, which is true if and only if G has no cycle containing e.)

b) A graph is a tree if and only if for all $x, y \in V(G)$, adding a copy of xy as an edge creates exactly one cycle. The number of cycles in $G + uv$ containing the new (copy of) edge uv equals the number of u, v-paths in G, and a graph is a tree if and only if for each pair u, v there is exactly one u, v-path. Note that the specified condition must also hold for addition of extra copies of edges already present; this excludes cliques.

2.1.3. *A graph is a tree if and only if it is loopless and has exactly one spanning tree.* If G is a tree, then G is loopless, since G is acyclic. Also, G is a spanning tree of G. If G contains another spanning tree, then G contains another edge not in G, which is impossible.

Let G be loopless and have exactly one spanning tree T. If G has a edge e not in T, then $T + e$ contains exactly one cycle, because T is a tree. Let f be another edge in this cycle. Then $T + e - f$ contains no cycle. Also $T + e - f$ is connected, because deleting an edge of a cycle cannot disconnect a graph. Hence $T + e - f$ is a tree different from T. Since G contains no such tree, G cannot contain an edge not in T, and G is the tree T.

2.1.4. *Every graph with fewer edges than vertices has a component that is a tree—TRUE.* Since the number of vertices or edges in a graph is the sum of the number in each component, a graph with fewer edges than vertices must have a component with fewer edges than vertices. By the properties of trees, such a component must be a tree.

2.1.5. *A maximal acyclic subgraph of a graph G consists of a spanning tree from each component of G.* We show that if H is a component of G and F is a maximal forest in G, then $F \cap H$ is a spanning tree of H. We may assume that F contains all vertices of G; if not, throw the missing ones in as isolated points to enlarge the forest. Note that $F \cap H$ contains no cycles, since F contains no cycles and $F \cap H$ is a subgraph of F.

We need only show that $F \cap H$ is a connected subgraph of H. If not, then it has more than one component. Since F is spanning and H is connected, H contains an edge between two of these components. Add this edge to F and $F \cap H$. It cannot create a cycle, since F previously did not contain a path between its endpoints. We have made F into a larger forest (more edges), which contradicts the assumption that it was maximal. (Note: the subgraph consisting of all vertices and no edges of G is a spanning subgraph of G; spanning means only that all the vertices appear, and says nothing about connectedness.

2.1.6. *Every tree with average degree a has $2/(2-a)$ vertices.* Let the tree have n vertices and m edges. The average degree is the degree sum divided by n, the degree sum is twice m, and m is $n - 1$. Thus $a = \sum d_i/n = 2(n-1)/n$. Solving for n yields $n = 2/(2-a)$.

2.1.7. *Every n-vertex graph with m edges has at least $m - n + 1$ cycles.* Let k be the number of components in such a graph G. Choosing a spanning tree from each component uses $n-k$ edges. Each of the remaining $m-n+k$ edges completes a cycle with edges in this spanning forest. Each such cycle has one edge not in the forest, so these cycles are distinct. Since $k \ge 1$, we have found at least $m-n+1$ cycles.

2.1.8. *Characterization of simple graphs that are forests.*

a) A simple graph is a forest if and only if every induced subgraph has a vertex of degree at most 1. If G is a forest and H is an induced subgraph of G, then H is also a forest, since cycles cannot be created by deleting edges. Every component of H is a tree, which is an isolated vertex or has a leaf (a vertex of degree 1). If G is not a forest, then G contains a cycle. A shortest cycle in G has no chord, since that would yield a shorter cycle, and hence a shortest cycle is an induced subgraph. This induced subgraph is 2-regular and has no vertex of degree at most 1.

b) A simple graph is a forest if and only if every connected subgraph is an induced subgraph. If G has a connected subgraph H that is not an induced subgraph, then G has an edge xy not in H with endpoints in $V(H)$. Since H contains an x,y-path, $H+xy$ contains a cycle, and G is not a forest. Conversely, if G is not a forest, then G has a cycle C, and every subgraph of G obtained by deleting one edge from C is connected but not induced.

c) The number of components is the number of vertices minus the number of edges. In a forest, each component is a tree and has one less edge than vertex. Hence a forest with n vertices and k components has $n-k$ edges.

Conversely, every component with n_i vertices has at least $n_i - 1$ edges, since it is connected. Hence the number of edges in an n-vertex is n minus the number of components only if every component with n_i vertices has $n_i - 1$ edges. Hence every component is a tree, and the graph is a forest.

2.1.9. *For $2 \le k \le n-1$, the n-vertex graph formed by adding one vertex adjacent to every vertex of P_{n-1} has a spanning tree with diameter k.* Let $v_1, \dots, v_{n-1}$ be the vertices of the path in order, and let x be the vertex adjacent to all of them. The spanning tree consisting of the path $v_1, \dots, v_{k-1}$ and the edges $xv_{k-1}, \dots, xv_{n-1}$ has diameter k.

2.1.10. *If u and v are vertices in a connected n-vertex simple graph, and $d(u,v) > 2$, then $d(u)+d(v) \le n+1-d(u,v)$.* Since $d(u,v) > 2$, we have $N(u) \cap N(v) = \varnothing$, and hence $d(u)+d(v) = |N(u) \cup N(v)|$. Let $k = d(u,v)$. Between u and v on a shortest u,v-path are vertices $x_1, \dots, x_{k-1}$. Since this is a shortest u,v-path, vertices u, v and $x_2, \dots, x_{k-2}$ are forbidden from the neighborhoods of both u and v. Hence $|N(u) \cup N(v)| \le n+1-k$.

The inequality fails when $d(u,v) \le 2$, because in this case u and v can have many common neighbors. When $d(u,v) = 2$, the sum $d(u)+d(v)$ can be as high as $2n-4$.

2.1.11. *If x and y are adjacent vertices in a graph G, then always $|d_G(x,z) - d_G(y,z)| \le 1$.* A z,y-path can be extended (or trimmed) to reach x, and hence $d(z,x) \le d(z,y)+1$. Similarly, $d(z,y) \le d(z,x)+1$. Together, these yield $|d(z,x) - d(z,y)| \le 1$.

2.1.12. *Diameter and radius of $K_{m,n}$.* Every vertex has eccentricity 2 in $K_{m,n}$ if $m,n \ge 2$, which yields radius and diameter 2. For $K_{1,n}$, the radius is 1 and diameter is 2 if $n > 1$. The radius and diameter of $K_{1,1}$ are 1. The radius and diameter of $K_{0,n}$ are infinite if $n > 1$, and both are 0 for $K_{0,1}$.

2.1.13. *Every graph with diameter d has an independent set of size at least $\lceil (1+d)/2 \rceil$.* Let x, y be vertices with $d(x,y) = d$. Vertices that are nonconsecutive on a shortest x,y-path P are nonadjacent. Taking x and every second vertex along P produces an independent set of size $\lceil (1+d)/2 \rceil$.

2.1.14. *Starting a shortest path in the hypercube.* The distance between vertices in a hypercube is the number of positions in which their names differ. From u, a shortest u,v-path starts along any edge to a neighbor whose name differ from u in a coordinate where v also differs from u.

2.1.15. *The complement of a simple graph with diameter at least 4 has diameter at most 2.* The contrapositive of the statement is that if $\overline{G}$ has diameter at least 3, then G has diameter at most 3. Since $G = \overline{\overline{G}}$, this statement has been proved in the text.

2.1.16. *The "square" of a connected graph G has diameter $\lceil \operatorname{diam}(G)/2 \rceil$.* The square is the simple graph G' with $x \leftrightarrow y$ in G' if and only if $d_G(x,y) \le 2$. We prove the stronger result that $d_{G'}(x,y) = \lceil d_G(x,y)/2 \rceil$ for every $x, y \in V(G)$. Given an x,y-path P of length k, we can skip the odd vertices along P to obtain an x,y-path of length $\lceil k/2 \rceil$ in G'.

On the other hand, every x,y-path of length l in G' arises from a path of length at most $2l$ in G. Hence the shortest x,y-path in G' comes from the shortest x,y-path in G by the method described, and $d_{G'}(x,y) = \lceil d_G(x,y)/2 \rceil$. Hence

$$\operatorname{diam}(G') = \min_{x,y} d_{G'}(x,y) = \min_{x,y} \left\lceil \frac{d_G(x,y)}{2} \right\rceil = \left\lceil \min_{x,y} \frac{d_G(x,y)}{2} \right\rceil = \left\lceil \frac{\operatorname{diam}(G)}{2} \right\rceil.$$

2.1.17. *If an n-vertex graph G has $n-1$ edges and no cycles, then it is connected.* Let k be the number of components of G. If $k > 1$, then we adding an edge with endpoints in two components creates no cycles and reduces the number of components by 1. Doing this $k-1$ times creates a graph with $(n-1)+(k-1)$ edges that is connected and has no cycles. Such a graph is a tree and has $n-1$ edges. Therefore, $k=1$, and the original graph G was connected.

2.1.18. *If G is a tree, then G has at least $\Delta(G)$ leaves.* Let $k = \Delta(G)$. Given $n > k \ge 2$, we cannot guarantee more leaves, as shown by growing a path of length $n-k-1$ from a leaf of $K_{1,k}$.

Proof 1a (maximal paths). Deleting a vertex x of degree k produces a forest of k subtrees, and x has one neighbor w_i in the ith subtree G_i. Let P_i be a maximal path starting at x along the edge xw_i. The other end of P_i must be a leaf of G and must belong to G_i, so these k leaves are distinct.

Proof 1b (leaves in subtrees). Deleting a vertex x of degree k produces a forest of k subtrees. Each subtree is a single vertex, in which case the vertex is a leaf of G, or it has at least two leaves, of which at least one is not a neighbor of x. In either case we obtain a leaf of the original tree in each subtree.

Proof 2 (counting two ways). Count the degree sum by edges and by vertices. By edges, it is $2n-2$. Let k be the maximum degree and l the number of leaves. The remaining vertices must have degree at least two each, so the degree sum when counted by vertices is at least $k + 2(n-l-1) + l$. The inequality $2n-2 \ge k + 2(n-l-1) + 1$ simplifies to $l \ge k$. (Note: Similarly, degree $2(n-1) - k$ remains for the vertices other than a vertex of maximum degree. Since all degrees are 1 or at least 2, there must be at least k vertices of degree 1.)

Proof 3: Induction on the number of vertices. For $n \le 3$, this follows by inspecting the unique tree on n vertices. For $n > 3$, delete a leaf u. If $\Delta(T-u) = \Delta(T)$, then by the induction hypothesis $T-u$ has at least k leaves. Replacing u adds a leaf while losing at most one leaf from $T-u$. Otherwise $\Delta(T-u) = \Delta(T) - 1$, which happens only if the neighbor of u is the only vertex of maximum degree in T. Now the induction hypothesis yields at least $k-1$ leaves in $T-u$. Replacing u adds another, since the vertex of maximum degree in T cannot be a leaf in $T-u$ (this is the reason for putting $n = 3$ in the basis step).

2.1.19. *If n_i denotes the number of vertices of degree i in a tree T, then $\sum in_i$ depends only on the number of vertices in T.* Since each vertex of degree i contributes i to the sum, the sum is the degree-sum, which equals twice the number of edges: $2n(T) - 2$.

2.1.20. *Hydrocarbon formulas C_kH_l.* The global method is the simplest one. With cycles forbidden, there are $k + l - 1$ "bonds" - i.e., edges. Twice this must equal the degree sum. Hence $2(k+l-1) = 4k + l$, or $l = 2k + 2$.

Alternatively, (sigh), proof by induction. Basis step ($k = 1$): The formula holds for the only example. Induction step ($k > 1$): In the graph of the molecule, each H has degree 1. Deleting these vertices destroys no cycles, so the subgraph induced by the C-vertices is also a tree. Pick a leaf x in this tree. In the molecule it neighbors one C and three Hs. Replacing x and these three Hs by a single H yields a molecule with one less C that also satisfies the conditions. Applying the induction hypothesis yields $l = [2(k-1)+2] - 1 + 3 = 2k + 2$.

2.1.21. *If a simple n-vertex graph G has a decomposition into k spanning trees, and $\Delta(G) = \delta(G) + 1$, then $2k < n$, and G has $n - 2k$ vertices of degree $2k$ and $2k$ vertices of degree $2k-1$.* Since every spanning tree of G has $n-1$ edges, we have $e(G) = k(n-1)$. Since $e(G) \le n(n-1)/2$ edges, this yields $k \le n/2$. Equality requires $G = K_n$, but $\Delta(K_n) = \delta(K_n)$. Thus $2k < n$.

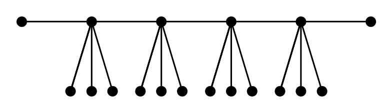
To determine the degree sequence, let l be the number of vertices of degree $\delta(G)$. By the degree-sum formula, $n\Delta(G) - l = 2kn - 2k$. Both sides are between two multiples of n. Since $0 < 2k < n$ and $0 < l < n$, the higher multiple of n is $n\Delta(G) = 2kn$, so $\Delta(G) = 2k$. It then also follows that $l = 2k$. Hence there are $n - 2k$ vertices of degree $2k$ and $2k$ vertices of degree $2k-1$.

2.1.22. *A tree with degree list $k, k-1, \ldots, 2, 1, 1, \ldots, 1$ has $2 + \binom{k}{2}$ vertices.* Since the tree has n vertices and $k-1$ non-leaves, it has $n - k + 1$ leaves. Since $\sum_{i=1}^{k} i = k(k+1)/2$, the degrees of the vertices sum to $k(k+1)/2 + n - k$. The degree-sum is twice the number of edges, and the number of edges is $n-1$. Thus $k(k+1)/2 + n - k = 2n - 2$. Solving for n yields $n = 2 + \binom{k}{2}$.

2.1.23. *For a tree T with vertex degrees in $\{1, k\}$, the possible values of $n(T)$ are the positive integers that are 2 more than a multiple of $k-1$.*

Proof 1 (degree-sum formula). Let m be the number of vertices of degree k. By the degree-sum formula, $mk + (n(T) - m) = 2n(T) - 2$, since T has $n(T) - 1$ edges. The equation simplifies to $n(T) = m(k-1) + 2$. Since m is a nonnegative integer, $n(T)$ must be two more than a multiple of $k-1$.

Whenever $n = m(k-1) + 2$, there is such a tree (not unique for $m \ge 4$). Such a tree is constructed by adjoining $k-2$ leaves to each internal vertex of a path of length $m+1$, as illustrated below for $m = 4$ and $k = 5$.



Proof 2 (induction on m, the number of vertices of degree k). We proof that if T has m vertices of degree k, then $n(T) = m(k-1) + 2$ If $m = 0$, then the tree must have two vertices.

For the induction step, suppose that $m > 0$. For a tree T with m vertices of degree k and the rest of degree 1, let T' be the tree obtained by deleting all the leaves. The tree T' is a tree whose vertices all have degree k in T. Let x be a leaf of T'. In T, x is adjacent to one non-leaf and to $k-1$ leaves. Deleting the leaf neighbors of x leaves a tree T'' with $m-1$ vertices of degree k and the rest of degree 1. By the induction hypothesis,

$n(T'') = (m-1)(k-1)+2$. Since we deleted $k-1$ vertices from T to obtain T'', we obtain $n(T) = m(k-1)+2$. This completes the induction step.

To prove inductively that all such values arise as the number of vertices in such a tree, we start with K_2 and iteratively expand a leaf into a vertex of degree k to add $k-1$ vertices.

2.1.24. *Every nontrivial tree has at least two maximal independent sets, with equality only for stars.* A nontrivial tree has an edge. Each vertex of an edge can be augmented to a maximal independent set, and these must be different, since each contains only one vertex of the edge. A star has exactly two maximal independent sets; the set containing the center cannot be enlarged, and the only maximal independent set not containing the center contains all the other vertices. If a tree is not a star, then it contains a path a, b, c, d. No two of the three independent sets $\{a, c\}$, $\{b, d\}$, $\{a, d\}$ can appear in a single independent set, so maximal independent sets containing these three must be distinct.

2.1.25. *Among trees with n vertices, the star has the most independent sets (and is the only tree with this many).*

Proof 1 (induction on n). For $n = 1$, there is only one tree, the star. For $n > 1$, consider a tree T. Let x be a leaf, and let y be its neighbor. The independent sets in T consist of the independent sets in $T - x$ and all sets formed by adding x to an independent set in $T - x - y$. By the induction hypothesis, the first type is maximized (only) when $T - x$ is a star. The second type contributes at most 2^{n-2} sets, and this is achieved only when $T - x - y$ has no edges, which requires that $T - x$ is a star with center at y. Thus both contributions are maximized when (and only when) T is a star with center y.

Proof 2 (counting). If an n-vertex tree T is not a star, then it contains a copy H of P_4. Of the 16 vertex subsets of $V(H)$, half are independent and half are not. If S is an independent set in T, then $S \cap V(H)$ is also independent. When we group the subsets of $V(T)$ by their intersection with $V(T) - V(H)$, we thus find that at most half the sets in each group are independent. Summing over all groups, we find that at most half of all subsets of $V(T)$, or 2^{n-1}, are independent. However, the star $K_{1,n-1}$ has $2^{n-1}+1$ independent sets.

2.1.26. *For $n \ge 3$, if G is an n-vertex graph such that every graph obtained by deleting one vertex of G is a tree, then $G = C_n$.* Let G_i be the graph obtained by deleting vertex v_i. Since G_i has $n-1$ vertices and is a tree, $e(G_i) = n-2$. Thus $\sum_{i=1}^{n} e(G_i) = n(n-2)$. Since each edge has two endpoints, each edge of G appears in $n-2$ of these graphs and thus is counted $n-2$ times in the sum. Thus $e(G) = n$.

Since G has n vertices and n edges, G must contain a cycle. Since G_i

has no cycle, every cycle in G must contain v_i. Since this is true for all i, every cycle in G must contain every vertex. Thus G has a spanning cycle, and since G has n edges it has no additional edges, so $G = C_n$.

2.1.27. *If $n \ge 2$ and $d_1, \dots, d_n$ are positive integers, then there exists a tree with these as its vertex degrees if and only if $d_n = 1$ and $\sum d_i = 2(n-1)$.* (Some graphs with such degree lists are not trees.) *Necessity:* Every n-vertex tree is connected and has $n-1$ edges, so every vertex has degree at least 1 (when $n \ge 2$) and the total degree sum is $2(n-1)$. *Sufficiency:* We give several proofs.

Proof 1 (induction on n). Basis step ($n = 2$): The only such list is $(1, 1)$, which is the degree list of the only tree on two vertices. Induction step ($n > 2$): Consider $d_1, \dots, d_n$ satisfying the conditions. Since $\sum d_i > n$, some element exceeds 1. Since $\sum d_i < 2n$, some element is at most 1. Let d' be the list obtain by subtracting 1 from the largest element of d and deleting an element that equals 1. The total is now $2(n-2)$, and all elements are positive, so by the induction hypothesis there is a tree on $n-1$ vertices with d' as its vertex degrees. Adding a new vertex and an edge from it to the vertex whose degree is the value that was reduced by 1 yields a tree with the desired vertex degrees.

Proof 2 (explicit construction). Let k be the number of 1s in the list d. Since the total degree is $2n-2$ and all elements are positive, $k \ge 2$. Create a path $x, u_1, \dots, u_{n-k}, y$. For $1 \le i \le n-k$, attach $d_i - 2$ vertices of degree 1 to u_i. The resulting graph is a tree (not the only one with this degree list), and it gives the proper degree to u_i. We need only check that we have the desired number of leaves. Counting x and y and indexing the list so that $d_1, \dots, d_n \ge$, we compute the number of leaves as

$$2 + \sum_{i=1}^{n-k}(d_i - 2) = 2 - 2(n-k) + \sum_{i=1}^{n} d_i - \sum_{i=n-k+1}^{n} d_i = 2 - 2(n-k) + 2(n-1) - k = k.$$

Proof 3 (extremality). Because $\sum d_i = 2(n-1)$, which is even, there is a graph with n vertices and $n-1$ edges that realizes d. Among such graphs, let G (having k components) be one with the fewest components. If $k = 1$, then G is a connected graph with $n-1$ edges and is the desired tree.

If $k > 1$ and G is a forest, then G has $n-k$ edges. Therefore, G has a cycle. Let H be a component of G having a cycle, and let uv be an edge of the cycle. Let H' be another component of G. Because each d_i is positive, H' has an edge, xy. Replace the edges uv and xy by ux and vy (either uv or xy could be a loop.) Because uv was in a cycle, the subgraph induced by $V(H)$ is still connected. The deletion of vy might disconnect H', but each piece is now connected to $V(H)$, so the new graph G' realizes d with fewer components than G, contradicting the choice of G.

2.1.28. *The nonnegative integers $d_1 \ge \cdots \ge d_n$ are the degree sequence of some connected graph if and only if $\sum d_i$ is even, $d_n \ge 1$, and $\sum d_i \ge 2n-2$.* This claim does not hold for simple graphs because the conditions $\sum d_i$ even, $d_n \ge 1$, and $\sum d_i \ge 2n-2$ do not prevent $d_1 \ge n$, which is impossible for a simple graph. Hence we allow loops and multiple edges. Necessity follows because every graph has even degree sum and every connected graph has a spanning tree with $n-1$ edges. For sufficiency, we give several proofs.

Proof 1 (extremality). Since $\sum d_i$ is even, there is a graph with degrees $d_1, \ldots, d_n$. Consider a realization G with the fewest components; since $\sum d_i \ge 2n-2$, G has at least $n-1$ edges. If G has more than one component, then some component as many edges as vertices and thus has a cycle. A 2-switch involving an edge on this cycle and an edge in another component reduces the number of components without changing the degrees. The choice of G thus implies that G has only one component.

Proof 2 (induction on n). For $n=1$, we use loops. For $n=2$, if $d_1 = d_2$, then we use d_1 parallel edges. Otherwise, we have $n>2$ or $d_1 > d_2$. Form a new list $d_1', \ldots, d_{n-1}'$ by deleting d_n and subtracting d_n units from other values. If $n \ge 3$ and $d_n = 1$, we subtract 1 from d_1, noting that $\sum d_i \ge 2n-2$ implies $d_1 > 1$. If $n \ge 3$ and $d_n > 1$, we make the subtractions from any two of the other numbers. In each case, the resulting sequence has even sum and all entries at least 1.

Letting $D = \sum d_i$, we have $\sum d_i' = D - 2d_n$. If $d_n = 1$, then $D - 2d_n \ge 2n-2-2 = 2(n-1)-2$. If $d_n > 1$, then $D \ge nd_n$, and so $D - 2d_n \ge (n-2)d_n \ge 2n-4 = 2(n-1)-2$. Hence the new values satisfy the condition stated for a set of $n-1$ values. By the induction hypothesis, there is a connected graph G' with vertex degrees $d_1', \ldots, d_{n-1}'$.

To obtain the desired graph G, add a vertex v_n with $d_i - d_i'$ edges to the vertex with degree d_i, for $1 \le i \le n-1$. This graph G is connected, because a path from v_n to any other vertex v can be construct by starting from v_n to a neighbor and continuing with a path to v in G'.

Proof 3 (induction on $\sum d_i$ and prior result). If $\sum d_i = 2n-2$, then Exercise 2.1.27 applies. Otherwise, $\sum d_1 \ge 2n$. If $n=1$, then we use loops. If $n>1$, then we can delete 2 from d_1 or delete 1 from d_1 and d_2 without introducing a 0. After applying the induction hypothesis, adding one loop at v_1 or one edge from v_1 to v_2 restores the desired degrees.

2.1.29. *Every tree has a leaf in its larger partite set (in both if they have equal size).* Let X and Y be the partite sets of a tree T, with $|X| \ge |Y|$. If there is no leaf in X, then $e(T) \ge 2|X| = |X| + |X| \ge |X| + |Y| = n(T)$. This contradicts $e(T) < n(T)$.

2.1.30. *If T is a tree in which the neighbor of every leaf has degree at least 3, then some pair of leaves have a common neighbor.*

Proof 1 (extremality). Let P a longest path in T, with endpoint v adjacent to u. Since v is a leaf and u has only one other neighbor on P, u must have a neighbor w off P. If w has a neighbor $z \ne u$, then replacing (u,v) by (u,w,z) yields a longer path. Hence w is a leaf, and v,w are two leaves with a common neighbor.

Proof 2 (contradiction). Suppose all leaves of T have different neighbors. Deleting all leaves (and their incident edges) reduces the degree of each neighbor by 1. Since the neighbors all had degree at least 3, every vertex now has degree at least 2, which is impossible in an acyclic graph.

Proof 3 (counting argument). Suppose all k leaves of T have different neighbors. The $n-2k$ vertices other than leaves and their neighbors have degree at least 2, so the total degree is at least $k + 3k + 2(n-2k) = 2n$, contradicting $\sum d(v) = 2e(T) = 2n-2$.

Proof 4 (induction on $n(T)$). For $n=4$, the only such tree is $K_{1,3}$, which satisfies the claim. For $n>4$, let v be a leaf of T, and let w be its neighbor. If w has no other leaf as neighbor, but has degree at least 3, then $T-v$ is a smaller tree satisfying the hypotheses. By the induction hypothesis, $T-v$ has a pair of leaves with a common neighbor, and these form such a pair in T.

2.1.31. *A simple connected graph G with exactly two non-cut-vertices is a path.* **Proof 1** (properties of trees). Every connected graph has a spanning tree. Every leaf of a spanning tree is not a cut-vertex, since deleting it leaves a tree on the remaining vertices. Hence every spanning tree of G has only two leaves and is a path. Consider a spanning path with vertices $v_1, \ldots, v_n$ in order. If G has an edge v_iv_j with $i < j-1$, then adding v_iv_j to the path creates a cycle, and deleting $v_{j-1}v_j$ from the cycle yields another spanning tree with three leaves. Hence G has no edge off the path.

Proof 2 (properties of paths and distance). Let x and y be the non-cut-vertices, and let P be a shortest x,y-path. If $V(P) \ne V(G)$, then let w be a vertex with maximum distance from $V(P)$. By the choice of w, every vertex of $V(G) - V(P) - \{w\}$ is as close to $V(P)$ as w and hence reaches $V(P)$ by a path that does not use w. Hence w is a non-cut-vertex. Thus $V(P) = V(G)$. Now there is no other edge, because P was a shortest x,y-path.

2.1.32. *Characterization of cut-edges and loops.*

An edge of a connected graph is a cut-edge if and only if it belongs to every spanning tree. If G has a spanning tree T omitting e, then e belongs to a cycle in $T+e$ and hence is not a cut-edge in G. If e is not a cut-edge in G, then $G-e$ is connected and contains a spanning tree T that is also a spanning tree of G; thus some spanning tree omits e.

An edge of a connected graph is a loop if and only if it belongs to no spanning tree. If e is a loop, then e is a cycle and belongs to no spanning tree. If e is not a loop, and T is a spanning tree not containing e, then $T+e$ contains exactly one cycle, which contains another edge f. Now $T+e-f$ is a spanning tree containing e, since it has no cycle, and since deleting an edge from a cycle of the connected graph $T+e$ cannot disconnect it.

2.1.33. *A connected graph with n vertices has exactly one cycle if and only if it has exactly n edges.* Let G be a connected graph with n vertices. If G has exactly one cycle, then deleting an edge of the cycle produces a connected graph with no cycle. Such a graph is a tree and therefore has $n-1$ edges, which means that G has n edges.

For the converse, suppose that G has exactly n edges. Since G is connected, G has a spanning tree, which has $n-1$ edges. Thus G is obtained by adding one edge to a tree, which creates a graph with exactly one cycle.

Alternatively, we can use induction. If G has exactly n edges, then the degree sum is $2n$, and the average degree is 2. When $n=1$, the graph must be a loop, which is a cycle. When $n>2$, if G is 2-regular, then G is a cycle, since G is connected. If G is not 2-regular, then it has a vertex v of degree 1. Let $G'=G-v$. The graph G' is connected and has $n-1$ vertices and $n-1$ edges. By the induction hypothesis, G' has exactly one cycle. Since a vertex of degree 1 belongs to no cycle, G also has exactly one cycle.

2.1.34. *A simple n-vertex graph G with $n>k$ and $e(G)>n(G)(k-1)-\binom{k}{2}$ contains a copy of each tree with k edges.* We use induction on n. For the basis step, let G be a graph with $k+1$ vertices. The minimum allowed number of edges is $(k+1)(k-1)-\binom{k}{2}+1$, which simplifies to $\binom{k}{2}$. Hence $G=K_{k+1}$, and $T\subseteq G$.

For the induction step, consider $n>k+1$. If every vertex has degree at least k, then containment of T follows from Proposition 2.1.8. Otherwise, deleting a vertex of minimum degree (at most $k-1$) yields a subgraph G' on $n-1$ vertices with more than $(n-1)(k-1)-\binom{k}{2}$ edges. By the induction hypothesis, G' contains T, and hence $T\subseteq G$.

2.1.35. *The vertices of a tree T all have odd degree if and only if for all $e\in E(T)$, both components of $T-e$ have odd order.*

Necessity. If all vertices have odd degree, then deleting e creates two of even degree. By the Degree-sum Formula, each component of $T-e$ has an even number of odd-degree vertices. Together with the vertex incident to e, which has even degree in $T-e$, each component of $T-e$ has odd order.

Sufficiency.

Proof 1 (parity). Given that both components of $T-e$ have odd order, $n(T)$ is even. Now consider $v\in V(T)$. Deleting an edge incident to v yields a component containing v and a component not containing v, each of odd order. Together, the components not containing v when we delete the various edges incident to v are $d(v)$ pairwise disjoint subgraphs that together

contain all of $V(T)-\{v\}$. Under the given hypothesis, they all have odd order. Together with v, they produce an even total, $n(T)$. Hence the number of these subgraphs is odd, which means that the number of edges in T incident to v is odd.

Proof 2 (contradiction). Suppose that such a tree T_0 has a vertex v_1 of even degree. Let e_1 be the last edge on a path from a leaf to x. Let T_1 be the component of T_0-e_1 containing v_1. By hypothesis, T_1 has odd order, and v_1 is a vertex of odd degree in T_1. Since the number of odd-degree vertices in T_1 must be even, there is a vertex v_2 of T_1 (different from v_1) having even degree (in both T_1 and T).

Repeating the argument, given v_i of even degree in T_{i-1}, let e_i be the last edge on the v_{i-1},v_i-path in T_{i-1}, and let T_i be the component of $T_{i-1}-e_i$ containing v_i. Also T_i is the component of T_0-e_i that contains v_i, so T_i has odd order. Since v_i has odd degree in T_i, there must be another vertex v_{i+1} with even degree in T_i.

In this way we generate an infinite sequence $v_1,v_2,\ldots$ of distinct vertices in T_0. This contradicts the finiteness of the vertex set, so the assumption that T_0 has a vertex of even degree cannot hold.

2.1.36. *Every tree T of even order has exactly one subgraph in which every vertex has odd degree.*

Proof 1 (Induction). For $n(T)=2$, the only such subgraph is T itself. Suppose $n(T)>2$. Observe that every pendant edge must appear in the subgraph to give the leaves odd degree. Let x be an endpoint of a longest path P, with neighbor u. If u has another leaf neighbor y, add ux and uy to the unique such subgraph found in $T-\{x,y\}$. Otherwise, $d(u)=2$, since P is a longest path. In this case, add the isolated edge ux to the unique such subgraph found in $T-\{u,x\}$.

Proof 2 (Explicit construction). Every edge deletion breaks T into two components. Since the total number of vertices is even, the two components of $T-e$ both have odd order or both have even order. We claim that the desired subgraph G consists of all edges whose deletion leaves two components of odd order.

First, every vertex has odd degree in this subgraph. Consider deleting the edges incident to a vertex u. Since the total number of vertices in T is even, the number of resulting components other than u itself that have odd order must be odd. Hence u has odd order in G.

Furthermore, G is the only such subgraph. If e is a cut-edge of G, then in $G-e$ the two pieces must each have even degree sum. Given that G is a subgraph of T with odd degree at each vertex, parity of the degree sum forces G to e if $T-e$ has components of odd order and omit e if $T-e$ has components of even order.

Comment: Uniqueness also follows easily from symmetric difference. Given two such subgraphs G_1, G_2 , the degree of each vertex in the symmetric difference is even, since its degree is odd in each G_i. This yields a cycle in $G_1 \cup G_2 \subseteq T$, which is impossible.

2.1.37. *If T and T' are two spanning trees of a connected graph G, and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that both $T - e + e'$ and $T' - e' + e$ are spanning trees of G.* Deleting e from T leaves a graph having two components; let U, U' be their vertex sets. Let the endpoints of e be $u \in U$ and $u' \in U'$. Being a tree, T' contains a unique u, u'-path. This path must have an edge from U to U'; choose such an edge to be e', and then $T - e + e'$ is a spanning tree. Since e is the only edge of T between U and U', we have $e' \in E(T') - E(T)$. Furthermore, since e' is on the u, u'-path in T', e' is on the unique cycle formed by adding e to T', and thus $T' - e' + e$ is a spanning tree. Hence e' has all the desired properties.

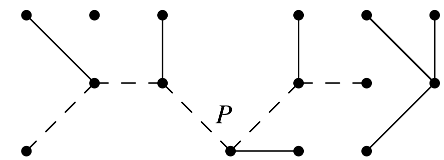
2.1.38. *If T and T' are two trees on the same vertex set such that $d_T(v) = d'_T(v)$ for each vertex v, then T' can be obtained from T' using 2-switches (Definition 1.3.32) with every intermediate graph being a tree.* Using induction on the number n of vertices, it suffices to show when $n \geq 4$ that we can apply (at most) one 2-switch to T to make a given leaf x be adjacent to its neighbor w in T'. We can then delete x from both trees and apply the induction hypothesis. Since the degrees specify the tree when n is at most 3, this argument also shows that at most $n - 3$ 2-switches are needed.

Let y be the neighbor of x in T. Note that w is not a leaf in T, since $d_{T'}(w) = d_T(w)$ and $xw \in E(T)$ and $n \geq 4$. Hence we can choose a vertex z in T that is a neighbor of w not on the x, w-path in T. Cutting xy and wz creates three components: x alone, one containing z, and one containing y, w. Adding the edges zy and xw to complete the 2-switch gives x its desired neighbor and reconnects the graph to form a new tree.

2.1.39. *If G is a nontrivial tree with $2k$ vertices of odd degree, then G decomposes into k paths.*

Proof 1 (induction and stronger result). We prove the claim for every forest G, using induction on k. Basis step ($k = 0$): If $k = 0$, then G has no leaf and hence no edge.

Induction step ($k > 0$): Suppose that each forest with $2k - 2$ vertices of odd degree has a decomposition into $k - 1$ paths. Since $k > 0$, some component of G is a tree with at least two vertices. This component has at least two leaves; let P be a path connecting two leaves. Deleting $E(P)$ changes the parity of the vertex degree only for the endpoints of P; it makes them even. Hence $G - E(P)$ is a forest with $2k - 2$ vertices of odd degree. By the induction hypothesis, $G - E(P)$ is the union of $k - 1$ pairwise edge-disjoint paths; together with P, these paths partition $E(G)$.



Proof 2 (extremality). Since there are $2k$ vertices of odd degree, at least k paths are needed. If two endpoints of paths occur at the same vertex of the tree, then those paths can be combined to reduce the number of paths. Hence a decomposition using the fewest paths has at most one endpoint at each vertex. Under this condition, endpoints occur only at vertices of odd degree. There are $2k$ of these. Hence there are at most $2k$ endpoints of paths and at most k paths.

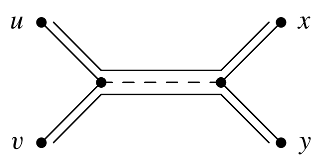
Proof 3 (applying previous result). A nontrivial tree has leaves, so $k > 0$. By Theorem 1.2.33, G decomposes into k trails. Since G has no cycles, all these trails are paths.

2.1.40. *If G is a tree with k leaves, then G is the union of $\lceil k/2 \rceil$ pairwise intersecting paths.* We prove that we can express G in this way using paths that end at leaves. First consider any way of pairing the leaves as ends of $\lceil k/2 \rceil$ paths (one leaf used twice when k is odd). Suppose that two of the paths are disjoint; let these be a u, v-path P and an x, y-path Q. Let R be the path connecting P and Q in G. Replace P and Q by the u, x-path and the v, y-path in G. These paths contain the same edges as P and Q, plus they cover R twice (and intersect). Hence the total length of the new set of paths is larger than before.

Continue this process; whenever two of the paths are disjoint, make a switch between them that increases the total length of the paths. This process cannot continue forever, since the total length of the paths is bounded by the number of paths ($\lceil k/2 \rceil$) times the maximum path length (at most $n - 1$). The process terminates only when the set of paths is pairwise intersecting. (We have not proved that some vertex belongs to all the paths.)

Finally, we show that a pairwise intersecting set of paths containing all the leaves must have union G. If any edge e of G is missing, then $G - e$ has two components H, H', each of which contains a leaf of G. Since e belongs to none of the paths, the paths using leaves in H do not intersect the paths using leaves in H'. This cannot happen, because the paths are pairwise intersecting.

(*Comment*: We can phrase the proof using extremality. The pairing with maximum total length has the desired properties; otherwise, we make a switch as above to increase the total length.)



2.1.41. *For $n \geq 4$, a simple n-vertex graph with at least $2n-3$ edges must have two cycles of equal length.* For such a graph, some component must have size at least twice its order minus 3. Hence we may assume that G is connected. A spanning tree T has $n-1$ edges and diameter at most $n-1$. Each remaining edge completes a cycle with edges of T. The lengths of these cycles belong to $\{3, \ldots, n\}$.

Since there are at least $n-2$ remaining edges, there are two cycles of the same length unless there are exactly $n-2$ remaining cycles and they create cycles of distinct lengths with the edge of T. This forces T to be a path. Now, after adding the edge e between the endpoints of T that produces a cycle of length n, the other remaining edges each produce two additional shorter cycles when added. These $2n-6$ additional cycles fall into the $n-3$ lengths $\{3, \ldots, n-1\}$. Since $2n-6 > n-3$ when $n \geq 4$, the pigeonhole principle yields two cycles of equal length.

2.1.42. *Extendible vertices.* In a nontrivial Eulerian graph G, a vertex is *extendible* if every trail beginning at v extends to an Eulerian circuit.

a) v is extendible if and only if $G - v$ is a forest.

Necessity. We prove the contrapositive. If $G - v$ is not a forest, then $G - v$ has a cycle C. In $G - E(C)$, every vertex has even degree, so the component of $G - E(C)$ containing v has an Eulerian circuit. This circuit starts and ends at v and exhausts all edges of G incident to v, so it cannot be extended to reach C and complete an Eulerian circuit of G.

Sufficiency. If $G-v$ is a forest, then every cycle of G contains v. Given a trail T starting at v, extend it arbitarily at the end until it can be extended no farther. Because every vertex has even degree, the process can end only at v. The resulting closed trail T' must use every edge incident to v, else it could extend farther. Since T' is closed, every vertex in $G - E(T')$ has even degree. If $G - E(T')$ has any edges, then minimum degree at least two in a component of $G - E(T')$ yields a cycle in $G - E(T')$; this cycle avoids v, since T' exhausted the edges incident to v. Since we have assumed that $G - v$ has no cycles, we conclude that $G - E(T')$ has no edges, so T' is an Eulerian circuit that extends T. (Sufficiency can also be proved by contrapositive.)

b) If v is extendible, then $d(v) = \Delta(G)$. An Eulerian graph decomposes into cycles. If this uses m cycles, then each vertex has degree at most $2m$. By part (a) each cycle contains v, and thus $d(v) \geq 2m$. Hence v has maximum degree.

Alternatively, since each cycle contains v, an Eulerian circuit must visit v between any two visits to another vertex u. Hence $d(v) \geq d(u)$.

c) For $n(G) > 2$, all vertices are extendible if and only if G is a cycle. If G is a cycle, then every trail from a vertex extends to become the complete cycle. Conversely, suppose that all vertices are extendible. By part (a), every vertex lies on every cycle. Let C be a cycle in G; it must contain all vertices. If G has any additional edge e, then following the shorter part of C between the endpoints of e completes a cycle with e that does not contain all the vertices. Hence there cannot be an additional edge and $G = C$.

d) If G is not a cycle, then G has at most two extendible vertices. From part (c), we may assume that G is Eulerian but not a cycle. If v is extendible, then $G - v$ is a forest. This forest cannot be a path, since then G is a cycle or has a vertex of odd degree. Since $G - v$ is a forest and not a path, $G - v$ has *more* than $\Delta(G - v)$ leaves unless $G - v$ is a tree with exactly one vertex of degree greater than two. If $G-v$ has more than $\Delta(G-v)$ leaves, all in $N(v)$, then no vertex of $G - v$ has degree as large as v in G, and by part (b) no other vertex is extendible. In the latter case, the one other vertex of degree $d(v)$ may also be extendible, but all vertices except those two have degree 2.

2.1.43. *Given a vertex u in a connected graph G, there is a spanning tree of G that is the union of shortest paths from u to the other vertices.*

Proof 1 (induction on $n(G)$). When $n(G) = 1$, the vertex u is the entire tree. For $n(G) > 1$, let v be a vertex at maximum distance from u. Apply the induction hypothesis to $G - v$ to obtain a tree T in $G - v$. Shortest paths in G from u to vertices other than v do not use v, since v is farthest from u. Therefore, T consists of shortest paths in G from u to the vertices other than v. A shortest u, v-path in G arrives at v from some vertex of T. Adding the final edge of that path to T completes the desired tree in G.

Proof 2 (explicit construction). For each vertex other than u, choose an incident edge that starts a shortest path to u. No cycle is created, since as we follow any path of chosen edges, the distance from u strictly decreases. Also $n(G) - 1$ edges are chosen, and an acyclic subgraph with $n(G) - 1$ edges is a spanning tree. Since distance from u decreases with each step, the v, u-path in the chosen tree is a shortest v, u-path.

Comment: The claim can also be proved using BFS to grow the tree. Proof 1 is a short inductive proof that the BFS algorithm works. Proof 2 is an explicit description of the edge set produced by Proof 1.

2.1.44. *If a simple graph with diameter 2 has a cut-vertex, then its complement has an isolated vertex—TRUE.* Let v be a cut-vertex of a simple graph G with diameter 2. In order to have distance at most 2 to each vertex in the other component(s) of $G - v$, a vertex of $G - v$ must be adjacent to v. Hence v has degree $n(G) - 1$ in G and is isolated in $\overline{G}$.

2.1.45. *If a graph G has spanning trees with diameters 2 and l, then G has spanning trees with all diameters between 2 and l.*

Proof 1 (local change). The only trees with diameter 2 are stars, so G has a vertex v adjacent to all others. Given a spanning tree T with leaf u, replacing the edge incident to u with uv yields another spanning tree T'. For every destroyed path, a path shorter by 1 remains. For every created path, a path shorter by 1 was already present. Hence diam T' differs from diam T by at most 1. Continuing this procedure reaches a spanning tree of diameter 2 without skipping any values along the way, so all the desired values are obtained.

Proof 2 (explicit construction). Since G has a tree with diameter 2, it has a vertex v adjacent to all others. Every path in G that does not contain v extends to v and to an additional vertex if it does not already contain all vertices. Hence for $k < l$ there is a path P of length k in G that contains v as an internal vertex. Adding edges from v to all vertices not in P completes a spanning tree of diameter k.

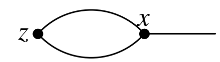
2.1.46. *For $n \ge 2$, the number of isomorphism classes of n-vertex trees with diameter at most 3 is $\lfloor n/2 \rfloor$.* If $n \le 3$, there is only one tree, and its diameter is $n - 1$. If $n \ge 4$, every tree has diameter at least 2. There is one having diameter 2, the star. Every tree with diameter 3 has two centers, x, y, and every non-central vertex is adjacent to exactly one of x, y, so $d(x) + d(y) = n$. By symmetry, we may assume $d(x) \le d(y)$. The unlabeled tree is now completely specified by $d(x)$, which can take any value from 2 through $\lfloor n/2 \rfloor$. Together with the star, the number of trees is $\lfloor n/2 \rfloor$.

2.1.47. *Diameter and radius.*

a) The distance function $d(u, v)$ satisfies the triangle inequality: $d(u, v) + d(v, w) \ge d(u, w)$. A u, v-path of length $d(u, v)$ and a v, w-path of length $d(v, w)$ together form a u, w-walk of length $l = d(u, v) + d(v, w)$. Every u, w-walk contains a u, w-path among its edges, so there is a u, w-path of length at most l. Hence the shortest u, w-path has length at most l.

b) $d \le 2r$, where d is the diameter of G and r is the radius of G. Let u, v be two vertices such that $d(u, v) = d$. Let w be a vertex in the center of G; it has eccentricity r. Thus $d(u, w) \le r$ and $d(w, v) \le r$. By part (a), $d = d(u, v) \le d(u, w) + d(w, v) \le 2r$.

c) Given integers r, d with $0 < r \le d \le 2r$, there is a simple graph with radius r and diameter d. Let $G = C_{2r} \cup H$, where $H \cong P_{d-r+1}$ and the cycle shares with H exactly one vertex x that is an endpoint of H. The distance from the other end of H to the vertex z opposite x on the cycle is d, and this is the maximum distance between vertices. Every vertex of H has distance at least r from z, and every vertex of the cycle has distance r from the vertex opposite it on the cycle. Hence the radius is at least r. The eccentricity of x equals r, so the radius equals r, and x is in the center.



2.1.48. *For $n \ge 4$, the minimum number of edges in an n-vertex graph with diameter 2 and maximum degree $n - 2$ is $2n - 4$.* The graph $K_{2,n-2}$ shows that $2n - 4$ edges are enough. We show that at least $2n - 4$ are needed. Let G be an n-vertex graph with diameter 2 and maximum degree $n - 2$. Let x be a vertex of degree $n - 2$, and let y be the vertex not adjacent to x.

Proof 1. Every path from y through x to another vertex has length at least 3, so diameter 2 requires paths from y to all of $V(G) - \{x, y\}$ in $G - x$. Hence $G - x$ is connected and therefore has at least $n - 2$ edges. With the $n - 2$ edges incident to x, this yields at least $2n - 4$ edges in G.

Proof 2. Let $A = N(y)$. Each vertex of $N(x) - A$ must have an edge to a vertex of A in order to reach y in two steps. These are distinct and distinct from the edges incident to y, so we have at least $|A| + |N(x) - A|$ edges in addition to those incident to x. The total is again at least $2n - 4$.

(Comment: The answer remains the same whenever $(2n - 2)/3 \le \Delta(G) \le n - 5$ but is $2n - 5$ when $n - 4 \le \Delta(G) \le n - 3$.)

2.1.49. *If G is a simple graph with* $\operatorname{rad} G \ge 3$, *then* $\operatorname{rad} \overline{G} \le 2$. The radius is the minimum eccentricity. For $x \in V(G)$, there is a vertex y such that $d_G(x, y) \ge 3$. Let w be the third vertex from x along a shortest x, y-path (possibly $w = y$). For $v \in V(G) - \{x\}$, if $xv \notin E(\overline{G})$, then $xv \in E(G)$. Now $vw \notin E(G)$, since otherwise there is a shorter x, y-path. Thus x, w, v is an x, v-path of length 2 in $\overline{G}$. Hence for all $v \in V(G) - \{x\}$, there is an x, v-path of length at most 2 in $\overline{G}$, and we have $\varepsilon_{\overline{G}}(x) \le 2$ and $\operatorname{rad}(\overline{G}) \le 2$.

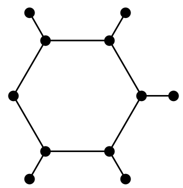
2.1.50. *Radius and eccentricity.*

a) The eccentricities of adjacent vertices differ by at most 1. Suppose that $x \leftrightarrow y$. For each vertex z, $d(x, z)$ and $d(y, z)$ differ by at most 1 (Exercise 2.1.11). Hence

$$\varepsilon(y) = \max_z d(y, z) \le \max_z (d(x, z) + 1) = (\max_z d(x, z)) + 1 = \varepsilon(x) + 1.$$

Similarly, $\varepsilon(x) \le \varepsilon(y) + 1$. The statement can be made more general: $|\varepsilon(x) - \varepsilon(y)| \le d(x, y)$ for all $x, y \in V(G)$.

b) In a graph with radius r, the maximum possible distance from a vertex of eccentricity $r + 1$ to the center of G is r. The distance is at most r, since every vertex is within distance at most r of every vertex in the center, by the definitions of center and radius. The graph consisting of a cycle of length $2r$ plus a pendant edge at all but one vertex of the cycle achieves equality. All vertices of the cycle have eccentricity $r+1$ except the vertex opposite the one with no leaf neighbor, which is the unique vertex with eccentricity r. The leaves have eccentricity $r+2$, except for the one adjacent to the center.



2.1.51. *If x and y are distinct neighbors of a vertex v in a tree G, then* $2\varepsilon(v) \le \varepsilon(x) + \varepsilon(y)$. Let w be a vertex at distance $\varepsilon(v)$ from v. The vertex w cannot be both in the component of $G - xv$ containing x and in the component of $G - yv$ containing y, since this would create a cycle. Hence we may assume that w is in the component of $G - xv$ containing v. Hence $\varepsilon(x) \ge d(x, w) = \varepsilon(v) + 1$. Also $\varepsilon(y) \ge d(y, w) \ge d(v, w) - 1 = \varepsilon(v) - 1$. Summing these inequalities yields $\varepsilon(x) + \varepsilon(y) \ge \varepsilon(v) + \varepsilon(v)$.

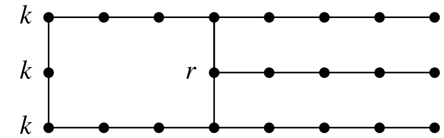
The smallest graph where this inequality can fail is the kite $K_4 - e$. Let v be a vertex of degree 2; it has eccentricity 2. Its neighbors x and y has degree 3 and hence eccentricity 1.

2.1.52. *Eccentricity of vertices outside the center.*

a) If G is a tree, then every vertex x outside the center of G has a neighbor with eccentricity $\varepsilon(x) - 1$. Let y be a vertex in the center, and let w be a vertex with distance at least $\varepsilon(x) - 1$ from x. Let v be the vertex where the unique x, w- and y, w-paths meet; note that v is on the x, y-path in G. Since $d(y, w) \le \varepsilon(y) \le \varepsilon(x) - 1 \le d(x, w)$, we have $d(y, v) \le d(x, v)$. This implies that $v \ne x$. Hence x has a neighbor z on the x, v-path in G.

This argument holds for every such w, and the x, v-path in G is always part of the x, y-path in G. Hence the same neighbor of x is always chosen as z. We have proved that $d(z, w) = d(x, w) - 1$ whenever $d(x, w) \ge \varepsilon(x) - 1$. On the other hand, since z is a neighbor of x, we have $d(z, w) \le d(x, w) + 1 \le \varepsilon(x) - 1$ for every vertex w with $d(x, w) < \varepsilon(x) - 1$. Hence $\varepsilon(z) = \varepsilon(x) - 1$.

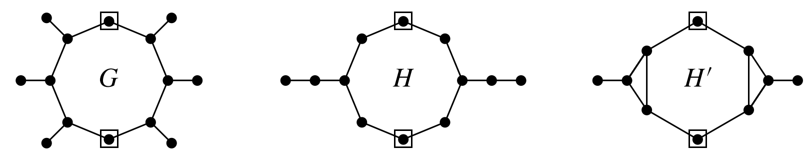
b) For all r and k with $2 \le r \le k < 2r$, there is a graph with radius r in which some vertex and its neighbors all have eccentricity k. Let G consist of a $2r$-cycle C and paths of length $k - r$ appended to three consecutive vertices on C. Below is an example with $r = 5$ and $k = 9$. The desired vertex is the one opposite the middle vertex of degree 3; vertices are labeled with their eccentricities.



2.1.53. *The center of a graph can be disconnected and can have components arbitrarily far apart.* We construct graphs center consists of two (marked) vertices separated by distance k. There are various natural constructions.

The graph G consists of a cycle of length $2k$ plus a pendant edge at all but two opposite vertices. These two are the center; other vertices of the cycle have eccentricity $k + 1$, and the leaves have eccentricity $k + 2$.

For even k, the graph H below consists of a cycle of length $2k$ plus pendant paths of length $k/2$ at two opposite vertices. For odd k, the graph H' consists of a cycle of length $2k$ plus paths of length $\lfloor k/2 \rfloor$ attached at one end to two opposite *pairs* of consecutive vertices.



2.1.54. *Centers in trees.*

a) A tree has exactly one center or has two adjacent centers.

Proof 1 (direct properties of trees). We prove that in a tree T any two centers are adjacent; since T has no triangles, this means it has at most two centers. Suppose u and v are distinct nonadjacent centers, with eccentricity k. There is a unique path R between them containing a vertex $x \notin \{u, v\}$. Given $z \in V(T)$, let P, Q be the unique u, z-path and unique v, z-path, respectively. At least one of P, Q contains x else $P \cup Q$ is a u, v-walk and contains a (u, v)-path other than R. If P passes through x, we have $d(x, z) < d(u, z)$; if Q, we have $d(x, z) < d(v, z)$. Hence $d(x, z) < \max\{d(u, z), d(v, z)\} \le k$. Since z is arbitrary, we conclude that x has smaller eccentricity than u and v. The contradiction implies $u \leftrightarrow v$.

Proof 2 (construction of the center). Let $P = x_1, \ldots, x_2$ be a longest path in T, so that $D = \operatorname{diam} T = d(x_1, x_2)$. Let $r = \lceil D/2 \rceil$. Let $\{u_1, u_2\}$ be the middle of P, with $u_1 = u_2$ if D is even. Label u_1, u_2 along P so that $d(x_i, u_i) = r$. Note that $d(v, u_i) \le r$ for all $v \in T$, else the (v, u_i)-path can be combined with the (u_i, x_i)-path or the (u_i, x_{3-i})-path to form a path longer than P. To show that no vertex outside $\{u_1, u_2\}$ can be a center, it suffices to show that every other vertex v has distance greater than r from x_1 or x_2.

The unique path from v to either x_1 or x_2 meets P at some point w (which may equal v). If w is in the u_1, x_2-portion of P, then $d(v, x_1) > r$. If w is in the u_2, x_1-portion of P, then $d(v, x_2) > r$.

b) A tree has exactly one center if and only if its diameter is twice its radius. Proof 3 above observes that the center or pair of centers is the middle of a longest path. The diameter of a tree is the length of its longest path. The radius is the eccentricity of any center. If the diameter is even, then there is one center, and its eccentricity is half the length of the longest path. If the diameter is odd, say $2k - 1$, then there are two centers, and the eccentricity of each is k, which exceeds $(2k - 1)/2$.

c) Every automorphism of a tree with an odd number of vertices maps at least one vertex to itself. The maximum distance from a vertex must be preserved under any automorphism, so any automorphism of any graph maps the center into itself. A central tree has only one vertex in the center, so it is fixed by any automorphism. A bicentral tree has two such vertices; they are fixed or exchange. If they exchange, then the two subtrees obtained by deleting the edge between the centers are exchanged by the automorphism. However, if the total number of vertices is odd, then the parity of the number of vertices in the two branches is different, so no automorphism can exchange the centers.

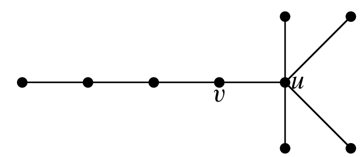
2.1.55. Given $x \in V(G)$, let $s(x) = \sum_{v \in V(G)} d(x, v)$. The *barycenter* of G is the subgraph induced by the set of vertices minimizing $s(x)$.

a) The barycenter of a tree is a single vertex or an edge. Let uv be an edge in a tree G, and let $T(u)$ and $T(v)$ be the components of $G - uv$ containing u and v, respectively. Note that $d(u, x) - d(v, x) = 1$ if $x \in V(T(v))$ and $d(u, x) - d(v, x) = -1$ if $x \in V(T(u))$. Summing the difference over $x \in V(G)$ yields $s(u) - s(v) = n(T(v)) - n(T(u))$.

As a result, $s(u_i) - s(u_{i+1})$ strictly decreases along any path $u_1, u_2, \ldots$; each step leaves more vertices behind. Considering two consecutive steps on a path x, y, z yields $s(x) - s(y) < s(y) - s(z)$, or $2s(y) < s(x) + s(z)$ whenever $x, z \in N(y)$. Thus the minimum of s cannot be achieved at two nonadjacent vertices, because it would be smaller at a vertex between them.

b) The maximum distance between the center and the barycenter in a tree of diameter d is $\lfloor d/2 \rfloor - 1$. By part (a), s is not minimized at a leaf when $n \geq 2$. Since every vertex is distance at most $\lfloor d/2 \rfloor$ from the center, we obtain an upper bound of $\lfloor d/2 \rfloor - 1$.

Part (a) implies that to achieve the bound of $\lfloor d/2 \rfloor - 1$ we need a tree having adjacent vertices u, v such that u is the neighbor of a leaf with eccentricity d, and the number of leaves adjacent to u is at least as large as $n(T(v))$. Since uv lies along a path of length d, we have at least $d - 1$ vertices in $T(v)$. Thus we need at least d vertices in $T(u)$ and at least $2d - 1$ vertices altogether. We obtain the smallest tree achieving the bound by merging an endpoint of P_d with the center of the star $K_{1,d-1}$. In the resulting tree, the barycenter u is the vertex of degree $d - 1$, and the distance between it and the center is $\lfloor d/2 \rfloor - 1$.



2.1.56. *Every tree T has a vertex v such that for all $e \in E(T)$, the component of $T - e$ containing v has at least $\lceil n(T)/2 \rceil$ vertices.*

Proof 1 (orientations). For each edge $xy \in E(T)$, we orient it from x to y if in $T - xy$ the component containing y contains at least $\lceil n(T)/2 \rceil$ vertices (there might be an edge which could be oriented either way). Denote the resulting digraph by $D(T)$.

If $D(T)$ has a vertex x with outdegree at least 2, then $T - x$ has two disjoint subtrees each having at least $\lceil n(T)/2 \rceil$ vertices, which is impossible. Now, since T does not contain a cycle, $D(T)$ does not contain a directed cycle. Hence $D(T)$ has a vertex v with outdegree 0. Since $D(T)$ has no vertex with outdegree at least two, every path in T with endpoint v is an oriented path to v in $D(T)$. Thus every edge xy points towards v, meaning that v is in a component of $T - xy$ with at least $\lceil n(T)/2 \rceil$ vertices.

The only flexibility in the choice of v is that an edge whose deletion leaves two components of equal order can be oriented either way, which yields two adjacent choices for v.

Proof 2 (algorithm). Instead of the existence proof using digraphs, one can march to the desired vertex. For each $v \in V(T)$, let $f(v)$ denote the minimum over $e \in E(T)$ of the order of the component of $T - e$ containing v. Note that $f(v)$ is achieved at some edge e incident to v.

Select a vertex v. If $f(v) < \lceil n(T)/2 \rceil$, then consider an edge e incident to v such that the order of the component of $T - e$ containing v is $f(v)$. Let u be the other endpoint of e. The component of $T - e$ containing u has more than half the vertices. For any other edge e' incident to u, the component of $T - e'$ containing u is strictly larger than the component of $T - e$ containing v. Hence $f(u) > f(v)$.

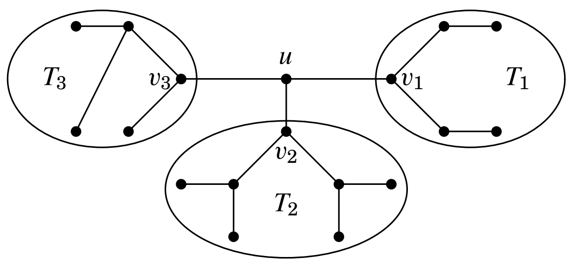
If $f(u) < \lceil n(T)/2 \rceil$, then we repeat the argument. Since f cannot increase indefinitely, we reach a vertex w with $f(v) \geq \lceil n(T)/2 \rceil$.

Uniqueness is as before; if two nonadjacent vertices have this property, then deleting edges on the path joining them yields a contradiction.

2.1.57. *a) If $n_1, \dots, n_k$ are positive integers with sum $n-1$, then $\sum_{i=1}^{k} \binom{n_i}{2} \le \binom{n-1}{2}$.* The graph having pairwise disjoint cliques of sizes $n_1, \dots, n_k$ has $\sum_{i=1}^{k} \binom{n_i}{2}$ edges and is a subgraph of K_{n-1}.

b) $\sum_{v \in V(T)} d(u, v) \le \binom{n}{2}$ when u is a vertex of a tree T. We use induction on n; the result holds trivially for $n = 2$. Consider $n > 2$. The graph $T - u$ is a forest with components $T_1, \dots, T_k$, where $k \ge 1$. Because T is connected, u has a neighbor in each T_i; because T has no cycles, u has exactly one neighbor v_i in each T_i. If $v \in V(T_i)$, then the unique u, v-path in T passes through v_i, and we have $d_T(u, v) = 1 + d_{T_i}(v_i, v)$. Letting $n_i = n(T_i)$, we obtain $\sum_{v \in V(T_i)} d_T(u, v) = n_i + \sum_{v \in V(T_i)} d_{T_i}(v_i, v)$.

By the induction hypothesis, $\sum_{v \in V(T_i)} d_{T_i}(v_i, v) \le \binom{n_i}{2}$. If we sum the formula for distances from u over all the components of $T - u$, we obtain $\sum_{v \in V(T)} d_T(u, v) \le (n-1) + \sum_i \binom{n_i}{2}$. Now observe that $\sum \binom{n_i}{2} \le \binom{m}{2}$ whenever $\sum n_i = m$, because the right side counts the edges in K_m and the left side counts the edges in a subgraph of K_m (a disjoint union of cliques). Hence we have $\sum_{v \in V(T)} d_T(u, v) \le (n-1) + \binom{n-1}{2} = \binom{n}{2}$.



2.1.58. *If S and T are trees with leaf sets $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$, respectively, then $d_S(x_i, x_j) = d_T(y_i, y_j)$ for all $1 \le i \le j \le k$ implies that S and T are isomorphic.* It suffices to show that the numbers $d_S(x_i, x_j)$ determine S uniquely. That is, if S is a tree, then no other tree has the same leaf distances.

Proof 1 (induction on k). If $k = 2$, then S is a path of length $d(x_1, x_2)$. If $k > 2$, then a tree S with leaf distance set D has a shortest path P from x_k to a junction w. Since P has no internal vertices on paths joining other leaves, deleting $V(P) - \{w\}$ leaves a subtree with leaf set $\{x_1, \dots, x_{k-1}\}$ realizing the distances not involving x_k. By the induction hypothesis, this distance set is uniquely realizable; call that tree S'. It remains only to show that the vertex w in $V(S')$ and $d_S(x_k, w)$ are uniquely determined.

Let $t = d_S(x_k, w)$. The vertex w must belong to the path Q joining some leaves x_i and x_j in S'. The paths from x_i and x_j to x_k in S together use the edges of Q, and each uses the path P from w to x_k. Thus $t = (d_S(x_i, x_k) + d_S(x_j, x_k) - d_S(x_i, x_j))/2$.

For arbitrary x_i and x_j, this formula gives the distance in S from x_k to the junction with the x_i, x_j-path. If w is not on the x_i, x_j-path, then the value of the formula exceeds t, since w is the closest vertex of S' to x_k. Hence $t = \min_{i,j<k}(d_S(x_i, x_k) + d_S(x_j, x_k) - d_S(x_i, x_j))/2$. For any i, j that achieves the minimum, $d_{S'}(x_i, w) = d_S(x_i, x_k) - t$, which identifies the vertex w in S'.

Thus there is only one w where the path can be attached and only one length of path that can be put there to form a tree realizing D.

Proof 2 (induction on $n(S)$). When $n(S) = 2$, there is no other tree with adjacent leaves. For $n(S) > 2$, let x_k be a leaf of maximum eccentricity; the eccentricity of a leaf is the maximum among its distances to other leaves.

If some leaf x_j has distance 2 from x_k, then they have a common neighbor. Deleting x_k yields a smaller tree S' with $k-1$ leaves, since the neighbor of x_k is not a leaf in S. The deletion does not change the distances among other leaves. By the induction hypothesis, there is only one way to assemble S' from the distance information, and to form S we must add x_k adjacent to the neighbor of x_j.

If no leaf has distance 2 from x_k, then the neighbor of x_k in S must have degree 2, because having two non-leaf neighbors would contradict the choice of x_k as a leaf of maximum eccentricity. Now $S - x_k$ has the same number of leaves but fewer vertices. The leaf x_k is replaced by x_k', and the distances from the kth leaf to other leaves are all reduced by 1. By the induction hypothesis, there is only one way to assemble $S - x_k$ from the distance information, and to form S we must add x_k adjacent to x_k'.

2.1.59. *If G is a tree with n vertices, k leaves, and maximum degree k, then $2\lceil (n-1)/k \rceil \le \operatorname{diam} G \le n - k + 1$, and the bounds are achievable, except that the lower bound is $2\lceil (n-1)/k \rceil - 1$ when $n \equiv 2 \pmod k$.* Let x be a vertex of degree k. Consider k maximal paths that start at x; these end at distinct leaves. If G has any other edge, it creates a cycle or leads to an additional leaf. Hence G is the union of k edge-disjoint paths with a common endpoint. The diameter of G is the sum of the lengths of two longest such paths.

Upper bound: Since the paths other than the two longest absorb at least $k-2$ edges, at most $n-k+1$ edges remain for the two longest paths; this is achieved by giving one path length $n-k$ and the others length 1.

Lower bound: If the longest and shortest of the k paths differ in length by more than 1, then shortening the longest while lengthening the shortest does not increase the sum of the two longest lengths. Hence the diameter is minimized by the tree G in which the lengths of any pair of the k paths differ by at most 1, meaning they all equal $\lfloor (n-1)/k \rfloor$ or $\lceil (n-1)/k \rceil$. There must be two of length $\lceil (n-1)/k \rceil$ unless $n \equiv 2 \pmod k$.

2.1.60. *If G has diameter d and maximum degree k, then $n(G) \le 1 + [(k-1)^d - 1]k/(k-2)$.* A single vertex x has at most k neighbors. Each of these has at most k other incident edges, and hence there are at most $k(k-1)$ vertices at distance 2 from x. Assuming that new vertices always get generated, the tree of paths from x has at most $k(k-1)^{i-1}$ vertices at distance i from x. Hence $n(G) \le 1 + \sum_{i=1}^{d} k(k-1)^{i-1} = 1 + k\frac{(k-1)^d - 1}{k-1-1}$. (Comment: C_5 and the Petersen graph are among the very few that achieve equality.)

2.1.61. *Every (k, g)-cage has diameter at most g.* (A (k, g)-*cage* is a graph with smallest order among k-regular graphs with girth at least g; Exercise 1.3.16 establishes the existence of such graphs).

Let G be a (k, g)-cage having two vertices x and y such that $d_G(x, y) > g$. We modify G to obtain a k-regular graph with girth at least g that has fewer vertices. This contradicts the choice of G, so there is no such pair of vertices in a cage G.

The modification is to delete x and y and add a matching from $N(x)$ to $N(y)$. Since $d(x, y) > g \ge 3$, the resulting smaller graph G' is simple. Since we have "replaced" edges to deleted vertices, G' is k-regular. It suffices to show that cycles in G' have length at least g. We need only consider cycles using at least one new edge.

Since $d_G(x, y) > g$, every path from $N(x)$ to $N(y)$ has length at least $g-1$. Also every path whose endpoints are within $N(x)$ has length at least $g-2$; otherwise, G has a short cycle through x. Every cycle through a new edge uses one new edge and a path from $N(x)$ to $N(y)$ or at least two new edges and at least two paths of length at least $g-2$. Hence every new cycle has length at least g.

2.1.62. *Connectedness and diameter of the 2-switch graph on spanning trees of G.* Let G be a connected graph with n vertices. The graph G' has one vertex for each spanning tree of G, with vertices adjacent in G' when the corresponding trees have exactly $n(G) - 2$ common edges.

a) G' is connected.

Proof 1 (construction of path). For distinct spanning trees T and T' in G, choose $e \in E(T) - E(T')$. By Proposition 2.1.6, there exists $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G. Let $T_1 = T - e + e'$. The trees T and T_1 are adjacent in G'. The trees T_1 and T' share more edges than T and T' share. Repeating the argument produces a T, T'-path in G' via vertices $T, T_1, T_2, \ldots, T_k, T'$.

Formally, this uses induction on the number m of edges in $E(T) - E(T')$. When $m = 0$, there is a T, T'-path of length 0. When $m > 0$, we generate T_1 as above and apply the induction hypothesis to the pair T_1, T'.

Proof 2 (induction on $e(G)$). If $e(G) = n-1$, then G is a tree, and $G' = K_1$. For the induction step, consider $e(G) > n-1$. A connected n-vertex graph with at least n edges has a cycle C. Choose $e \in E(C)$. The graph $G-e$ is connected, and by the induction hypothesis $(G-e)'$ is connected. Every spanning tree of $G-e$ is a spanning tree of G, so $(G-e)'$ is the induced subgraph of $T(G)$ whose vertices are the spanning trees of G that omit e.

Since $(G-e)'$ is connected, it suffices to show that every spanning tree of G containing e is adjacent in G' to a spanning tree not containing e. If T contains e and T' does not, then there exists $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G omitting e. Thus $T - e + e'$ is the desired tree in $G - e$ adjacent to T in G'.

b) The diameter of G' is at most $n-1$, with equality when G has two spanning trees that share no edges. It suffices to show that $d_{G'}(T, T') = |E(T) - E(T')|$. Each edge on a path from T to T' in G' discards at most one edge of T, so the distance is at least $|E(T) - E(T')|$. Since for each $e \in E(T) - E(T')$ there exists $e' \in E(T') - E(T)$ such that $T - e + e' \in V(G')$, the path built in Proof 1 of part (a) has precisely this length.

Since trees in n-vertex graphs have at most $n-1$ edges, always $|E(T) - E(T')| \le n-1$, so $\operatorname{diam} G' \le n-1$ when G has n vertices. When G has two edge-disjoint spanning trees, the diameter of G' equals $n-1$.

2.1.63. *Every n-vertex graph with $n+1$ edges has a cycle of length at most $\lfloor (2n+2)/3 \rfloor$.* The bound is best possible, as seen by the example of three paths with common endpoints that have total length $n+1$ and nearly-equal lengths. Note that $\lfloor (2n+2)/3 \rfloor = \lceil 2n/3 \rceil$.

Proof 1. Since an n-vertex forest with k components has only $n-k$ edges, an n-vertex graph with $n+1$ edges has at least two cycles. Let C be a shortest cycle. Suppose that $e(C) > \lceil 2n/3 \rceil$. If $G - E(C)$ contains a path connecting two vertices of C, then it forms a cycle with the shorter path on C connecting these two vertices. The length of this cycle is at most

$$\tfrac{1}{2}e(C) + (e(G) - e(C) = e(G) - \tfrac{1}{2}e(C) < n + 1 - n/3 = (2n+3)/3.$$

If the length of this cycle is less than $(2n+3)/3$, then it is at most $(2n+2)/3$, and since it is an integer it is at most $\lfloor (2n+2)/3 \rfloor$.

If there is no such path, then no cycle shares an edge with C. Hence the additional cycle is restricted to a set of fewer than $n + 1 - \lceil 2n/3 \rceil$ edges, and again its length is less than $(2n+3)/3$.

Proof 2. We may assume that the graph is connected, since otherwise we apply the same argument to some component in which the number of edges exceeds the number of vertices by at least two. Consider a spanning tree T, using $n-1$ of the edges. Each of the two remaining edges forms a cycle when added to T. If these cycles share no edges, then the shortest has length at most $(n+1)/2$.

Hence we may assume that the two resulting cycles have at least one common edge; let x, y be the endpoints of their common path in T. Deleting the x, y-path in T from the union of the two cycles yields a third cycle. (The uniqueness of cycles formed when an edge is added to a tree implies that this edge set is in fact a single cycle.) Thus we have three cycles, and each edge in the union of the three cycles appears in exactly two of them. Thus the shortest of the three lengths is at most $2(n+1)/3$.

2.1.64. *If G is a connected graph that is not a tree, then G has a cycle of length at most $2\operatorname{diam} G+1$, and this is best possible.* We use extremality for the upper bound; let C be a shortest cycle in G. If its length exceeds $2\operatorname{diam} G+1$, then there are vertices x, y on C that have no path of length at most $\operatorname{diam} G$ connecting them along C. Following a shortest x, y-path P from its first edge off C until its return to C completes a shorter cycle. This holds because P has length at most k, and we use a portion of P in place of a path along C that has length more than k. We have proved that every shortest cycle in G has length at most $2\operatorname{diam} G+1$.

The odd cycle C_{2k+1} shows that the bound is best possible. It is connected, is not a tree, and has diameter k. Its only cycle has length $2k+1$, so we cannot guarantee girth less than $2k+1$.

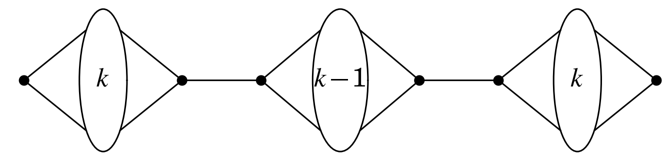
2.1.65. *If G is a connected simple graph of order n and minimum degree k, with $n-3 \ge k \ge 2$, then $\operatorname{diam} G \le 3(n-2)/(k+1)-1$, with equality when $n-2$ is a multiple of $k+1$.* To interpret the desired inequality on $\operatorname{diam} G$, we let $d=\operatorname{diam} G$ and solve for n. Thus it suffices to prove that $n \ge (1+\lfloor d/3\rfloor)(k+1)+j$, where j is the remainder of d upon division by 3. Note that the inequality $n-3 \ge k$ is equivalent to $3(n-2)/(k+1)-1 \ge 2$. Under this constraint, the result is immediate when $d \le 2$, so we may assume that $d \ge 3$.

Let $\langle v_0, \ldots, v_d\rangle$ be a path joining vertices at distance d. For a vertex x, let $N[x]=N(x)\cup\{x\}$. Let $S_i=N[v_{3i}]$ for $0 \le i < \lfloor d/3\rfloor$, and let $S_{\lfloor d/3\rfloor}=N[v_d]$. Since $d \ge 3$, there are $1+\lfloor d/3\rfloor$ such sets, pairwise disjoint (since we have a shortest v_0, v_d-path), and each has at least $k+1$ vertices. Furthermore, v_{d-2} does not appear in any of these sets if $j=1$, and both v_{d-2} and v_{d-3} do not appear if $j=2$. Hence n is as large as claimed.

To obtain an upper bound on d in terms of n, we write $\lfloor d/3\rfloor$ as $(d-j)/3$. Solving for d in terms of n, we find in each case that $d \le 3(n-2)/(k+1)-1-j[1-3/(k+1)]$. Since $k \ge 2$, the bound $d \le 3(n-2)/(k+1)-1$ is valid for every congruence class of d modulo 3.

When $n-2$ is a multiple of $k+1$, the bound is sharp. If $n-2=k+1$, then deleting two edges incident to one vertex of K_n yields a graph with the desired diameter and minimum degree (also $\overline{C}_n$ suffices). For larger multiples, let $m=(n-2)/(k+1)$; note that $m \ge 2$. Begin with cliques $Q_1, \ldots, Q_m$ such that Q_1 and Q_m have order $k+2$ and the others have order $k+1$. For $1 \le i \le m$, choose $x_i, y_i \in Q_i$, and delete the edge $x_i y_i$.

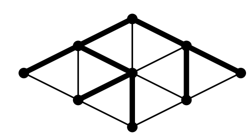
For $1 \le i \le m-1$, add the edge $y_i x_{i+1}$. The resulting graph has minimum degree k and diameter $3m-1$. The figure below illustrates the construction when $m=3$; the ith ellipse represents $Q_m-\{x_i, y_i\}$. (There also exist regular graphs attaining the bound.)



2.1.66. *If $F_1, \ldots, F_m$ are forests whose union is G, then $m \ge \max_{H\subseteq G}\left\lceil \frac{e(H)}{n(H)-1}\right\rceil$.* From a subgraph H, each forest uses at most $n(H)-1$ edges. Thus at least $e(H)/(n(H)-1)$ forests are needed just to cover the edges of H, and the choice of H that gives the largest value of this is a lower bound on m.

2.1.67. *If a graph G has k pairwise edge-disjoint spanning trees in G, then for any partition of $V(G)$ into r parts, there are at least $k(r-1)$ edges of G whose endpoints are in different parts.* Deleting the edges of a spanning tree T that have endpoints in different parts leaves a forest with at least r components and hence at most $n(G)-r$ edges. Since T has $n(G)-1$ edges, T must have at least $r-1$ edges between the parts. The argument holds separately for each spanning tree, yielding $k(r-1)$ distinct edges.

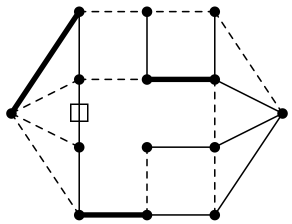
2.1.68. *A decomposition into two isomorphic spanning trees.* One tree turns into the other in the decomposition below upon rotation by 180 degrees.



2.1.69. *An instance of playing Bridg-it.* Indexing the 9 vertical edges as $g_{i,j}$ and the 16 horizontal/slanted edges as $h_{i,j}$, where i is the "row" index and j is the "column" index, we are given these moves:

Player 1:	$h_{1,1}$	$h_{2,3}$	$h_{4,2}$
Player 2:	$g_{2,2}$	$h_{3,2}$	$g_{2,1}$

After the third move of Player 1, the situation is as shown below. The bold edges are those seized by Player 1 and belong to both spanning trees. The two moves by Player 2 have cut the two edges that are missing.



The third move by Player 2 cuts the marked vertical edge. This cuts off three vertices from the rest of the solid tree. Player 1 must respond by choosing a dotted edge that can reconnect it. The choices are $h_{1,2}$, $h_{2,1}$, $h_{2,2}$, $h_{3,1}$, and $h_{4,1}$.

2.1.70. *Bridg-it cannot end in a tie.* That is, when no further moves can be made, one player must have a path connecting his/her goals.

Consider the graph for Player 1 formed in Theorem 2.1.17. At the end of the game, Player 1 has bridges on some of these edges, retaining them as a subgraph H, and the other edges have been cut by Player 2's bridges. Let C be the component of H containing the left goal for Player 1. The edges incident to $V(C)$ that have been cut correspond to a walk built by Player 2 that connects the goals for Player 2. This holds because successive edges around the outside of C are incident to the same "square" in the graph for Player 1, which corresponds to a vertex for Player 2. This can be described more precisely using the language of duality in planar graphs (Chapter 6).

2.1.71. *Player 2 has a winning strategy in Reverse Bridg-it.* A player building a path joining friendly ends is the *loser*, and it is forbidden to stall by building a bridge joining posts on the same end.

We use the same graph as in Theorem 2.1.17, keeping the auxiliary edge so that we start with two edge-disjoint spanning trees T and T'. An edge e that Player 1 can use belongs to only one of the trees, say T. The play by Player 1 will add e to T'. Since $e \in E(T) - E(T')$, Proposition 2.1.7 guarantees an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree. Player 2 makes a bridge to delete the edge e', and the strategy continues with the modified T' sharing the edge e with T. If the only edge of $E(T') - E(T)$ available to break the cycle in $T' + e$ is the auxiliary edge, then Player 1 has already built a path joining the goals and lost the game. The game continues always with two spanning trees available for Player 1, and it can only end with Player 1 completing the required path.

2.1.72. *If $G_1, \dots, G_k$ are pairwise intersecting subtrees of a tree G, then G has a vertex in all of $G_1, \dots, G_k$.* (A special case is the "Helly property" of the real line: pairwise intersecting intervals have a common point.)

Lemma: For vertices u, v, w in a tree G, the u,v-path P, the v,w-path Q, and the u,w-path R in G have a common vertex. Let z be the last vertex shared by P and R. They share all vertices up to z, since distinct paths cannot have the same endpoints. Therefore, the z,v-portion of P and the z,w-portion of R together form a v,w-path. Since G has only one v,w-path, this is Q. Hence z belongs to P, Q, and R.

Main result.

Proof 1 (induction on k). For $k = 2$, the hypothesis is the conclusion. For larger k, apply the inductive hypothesis to both $\{G_1, \dots, G_{k-1}\}$ and $\{G_2, \dots, G_k\}$. This yields a vertex u in all of $\{G_1, \dots, G_{k-1}\}$ and a vertex v in all of $\{G_2, \dots, G_k\}$. Because G is a tree, it has a unique u,v-path. This path belongs to all of $G_2, \dots, G_{k-1}$. Let w be a vertex in $G_1 \cap G_k$. By the Lemma, the paths in G joining pairs in $\{u, v, w\}$ have a common vertex. Since the u,v-path is in $G_2, \dots, G_{k-1}$, the w,u-path is in G_1, and the w,v-path is in G_k, the common vertex of these paths is in $G_1, \dots, G_k$.

Proof 2 (induction on k). For $k = 3$, we let u, v, w be vertices of $G_1 \cap G_2$, $G_2 \cap G_3$, and $G_3 \cap G_1$, respectively. By the Lemma, the three paths joining these vertices have a common vertex, and this vertex belongs to all three subtrees. For $k > 3$, define the $k - 1$ subtrees $G_1 \cap G_k, \dots, G_{k-1} \cap G_k$. By the case $k = 3$, these subtrees are pairwise intersecting. There are $k - 1$ of them, so by the induction hypothesis they have a common vertex. This vertex belongs to all of the original k trees.

2.1.73. *A simple graph G is a forest if and only if pairwise intersecting paths in G always have a common vertex.*

Sufficiency. We prove by contradiction that G is acyclic. If G has a cycle, then choosing any three vertices on the cycle cuts it into three paths that pairwise intersect at their endpoints. However, the three paths do not all have a common vertex. Hence G can have no cycle and is a tree.

Necessity. Let G be a forest. Pairwise intersecting paths lie in a single component of G, so we may assume that G is a tree. We use induction on the number of paths. By definition, two intersecting paths have a common vertex. For $k > 2$, let $P_1, \dots, P_k$ be pairwise intersecting paths. Also $P_1, \dots, P_{k-1}$ are pairwise intersecting, as are $P_2, \dots, P_k$; each consists of $k - 1$ paths. The induction hypothesis guarantees a vertex u belonging to all of $P_1, \dots, P_{k-1}$ and a vertex v belonging to all of $P_2, \dots, P_k$. Since each of $P_2, \dots, P_{k-1}$ contains both u and v and G has exactly one u,v-path Q, this path Q belongs to all of $P_2, \dots, P_{k-1}$.

By hypothesis, P_1 and P_k also have a common vertex z. The unique z,u-path R lies in P_1, and the unique z,v-path S lies in P_k. Starting from z, let w be the last common vertex of R and S. It suffices to show that $w \in V(Q)$. Otherwise, consider the portion of R from w until it first reaches Q, the portion of S from w until it first reaches Q, and the portion of Q between these two points. Together, these form a closed trail and contain a cycle, but this cannot exist in the tree G. The contradiction implies that w belongs to Q and is the desired vertex.

2.1.74. *Every simple n-vertex graph G with $n-2$ edges is a subgraph of its complement.* (We need $e(G) < n-1$, since $K_{1,n-1} \not\subseteq \overline{K_{1,n-1}}$.)

We use induction on n. We will delete two vertices in the induction step, we so we must include $n=2$ and $n=3$ in the basis. When $n=2$, we have $G=\overline{K}_2 \subseteq K_2 = \overline{G}$. When $n=3$, we have $G = K_2 + K_1 \subseteq P_3 = \overline{G}$.

For $n>3$, let G be an n-vertex graph with $n-2$ edges. Suppose first that G has an isolated vertex x. Since $e(G) = n-2$, the Degree-Sum Formula yields a vertex y of degree at least 2. Let $G' = G - \{x,y\}$; this is a graph with $n-2$ vertices and at most $n-4$ edges. By the induction hypotheses, every graph with $n-2$ vertices and $n-4$ edges appears in its complement, so the same holds for smaller graphs (since they are contained in graphs with $n-4$ edges). A copy of G' contained in $\overline{G} - \{x,y\}$ extends to a copy of G in $\overline{G}$ by letting x represent y and letting y represent x.

Hence we may assume that G has no isolated vertices. Every non-tree component of G has at least as many edges as vertices, and trees have one less. Hence at least two components of G are trees. We may therefore choose vertices x and y of degree 1 with distinct neighbors. Let $N(x) = \{x'\}$ and $N(y) = \{y'\}$ with $x' \neq y'$. Let $G' = G - \{x,y\}$; this graph has $n-2$ vertices and $n-4$ edges. By the induction hypothesis, $G' \subseteq \overline{G'} = \overline{G} - x - y$. Let H be a copy of G' in $\overline{G} - x - y$. If x' or y' represents itself in H, then we let x and y switch identities to add their incident edges. Otherwise, we let x and y represent themselves to add their incident edges.

2.1.75. *Every non-star tree is (isomorphic to) a subgraph of its complement.*

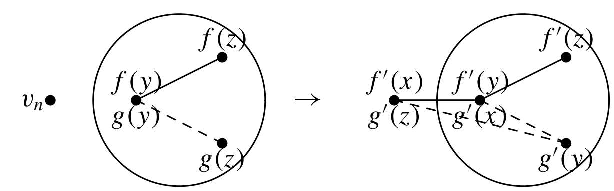
Proof 1 (loaded induction on n). We prove the stronger statement that, given an n-vertex tree T other than $K_{1,n-1}$, the graph K_n with vertex set $\{v_1,\ldots,v_n\}$ contains two edge-disjoint copies of T in which the two copies of each non-leaf vertex of T appear at distinct vertices. The only non-star tree with at most 4 vertices is the path P_4, which is self-complementary via a map that moves each vertex.

Now consider $n>4$. We show first that T has a leaf x such that $T-x$ is not a star. If T is a path, let x be either leaf. Otherwise, T has at least three leaves; let P be a longest path in T, and let x be a leaf other than the endpoints of P. In either case, $T-x$ has a path of length at least 3.

Let $T' = T - x$, and let y be the neighbor of x in T. If y is not a leaf in T', then the induction hypothesis yields embeddings of T' in K_{n-1} in which y occurs at distinct vertices. We can extend both embeddings to K_n by placing x at v_n in each and adding the distinct edges to the images of y.

In this case the non-leaves of T are the same as the non-leaves of T', and the loaded claim holds for T.

If y is a leaf in T, we use the same argument unless $f(y) = g(y)$, where f,g are the mappings from $V(T')$ to $V(K_{n-1})$ for the two embeddings of T' guaranteed by the induction hypothesis. In this case, let z be the other neighbor of y; we have z as a non-leaf of T', and hence $f(z) \neq g(z)$. We cannot have both $g(z) = f(w)$ for some $w \in N(z)$ and $f(z) = g(u)$ for some $u \in N(z)$, because then the edge between $f(z)$ and $g(z)$ is used in both embeddings of T'. By symmetry, we may assume $f(z) \neq g(w)$ for all $w \in N(z)$. For T, we define $f', g' : V(T) \to V(K_n)$ for the edge-disjoint embeddings of T as follows: If $w \notin \{x,y,z\}$, let $f'(w) = f(w)$ and $g'(w) = g(w)$. For the other vertices, let $f'(z) = f(z)$, $f'(y) = f(y)$, $f'(x) = v_n$, $g'(z) = v_n$, $g'(y) = g(z)$, $g'(x) = g(y)$, as illustrated below. By construction the non-leaves of T have pairs of distinct images. The edges not involving x,y,z are mapped as before and hence become edge-disjoint subgraphs of $K_n - \{v_n, f(y), f(z), g(z)\}$. The path x,y,z is explicitly given edge-disjoint images under f', g'. This leaves only the edges involving z. Those under f are the same as under f'. The shift of z from $g(z)$ to $g'(z) = v_n$ does not produce a common edge because $f'(z) = f(z)$ is not the image under g of any neighbor of z.



Proof 2. (induction on $n(T)$ by deleting two leaves—proof due to Fred Galvin). To cover the basis step, we prove first that the claim is true when T has a path P of length at least 3 that includes a endpoint of every edge (see "caterpillars" in Section 2.2). First we embed P in its complement so that every vertex moves. If $n(P)$ is even, say $n(P) = 2k$, then we apply the vertex permutation $\binom{1,2,\ldots,k,k+1,\ldots,2k}{2,4,\ldots,2k,1,\ldots,2k-1}$. When $n(P) = 2k-1$, we use $\binom{1,2,\ldots,k,k+1,\ldots,2k-1}{2k-1,2k-3,\ldots,1,2k,\ldots,2}$. Now, since every vertex on P has moved, we can place the remaining leaves at their original positions and add incident edges from $\overline{T}$ to make them adjacent to their desired neighbors.

All non-star trees with at most six vertices have such a path P. For the induction step, consider a tree T with $n(T) > 6$. Let u and v be endpoints of a longest path in T, so $d(u,v) = \operatorname{diam} T$, and let $T' = T - u - v$. Let x and y be the neighbors of u and v, respectively. If T is not a star and T' is a star, then T is embeddable in its complement using the construction above.

If T' is not a star, then by the induction hypothesis T' embeds in $\overline{T'}$. If the embedding puts x or y at itself, then adding the edges xv and yu yields a copy of T in $\overline{T}$. Otherwise, make u adjacent to the image of x and v adjacent to the image of y to complete the copy of T in $\overline{T}$.

2.1.76. *If $A_1, \ldots, A_n$ are distinct subsets of $[n]$, then there exists $x \in [n]$ such that $A_1 \cup \{x\}, \ldots, A_n \cup \{x\}$ are distinct.* We need to find an element x such that no pair of sets differ by x. Consider the graph G with $V(G) = \{A_1, \ldots, A_n\}$ and $A_i \leftrightarrow A_j$ if only if A_i and A_j differ by the addition or deletion of a single element. Color (label) an edge A_iA_j by the element in which the endpoints differ. Any color that appears in a cycle of G must appear an even number of times in that cycle, because as we traverse the cycle we return to the original set. Hence a subgraph F formed by selecting one edge having each edge-label that appears in G will contain no cycles and must be a forest. Since a forest has at most $n-1$ edges, there must be an element that does not appear on any edge and can serve as x.

2.2. SPANNING TREES & ENUMERATION

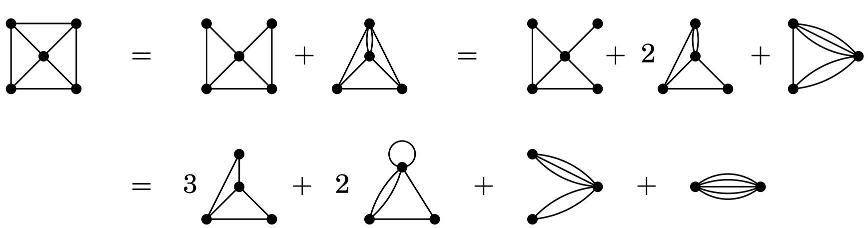
2.2.1. *Description of trees by Prüfer codes.* We use the fact that the degree of a vertex in the tree is one more than the number of times it appears in the corresponding code.

a) The trees with constant Prüfer codes are the stars. The $n-1$ labels that don't appear in the code have degree 1 in the tree; the label that appears $n-2$ times has degree $n-1$.

b) The trees whose codes contain two values are the double-stars. Since $n-2$ labels don't appear in the code, there are $n-2$ leaves in the tree.

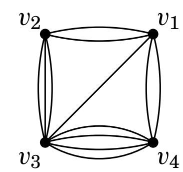
c) The trees whose codes have no repeated entries are the paths. Since $n-2$ labels appear once and two are missing, $n-2$ vertices have degree 2, and two are leaves. All trees with this degree sequence are paths.

2.2.2. *The graph $K_1 \vee C_4$ has 45 spanning trees.* For each graph G in the computation below, we mean $\tau(G)$.



$= \quad 3\cdot 8 + 2\cdot 5 + 3\cdot 2 + 5 \quad = \quad 45$

2.2.3. *Application of the Matrix Tree Theorem.* The matrix $Q = D - A$ for this graph appears on the right below. All rows and columns sum to 0. If we delete any row and column and take the determinant, the result is 106, which is the number of spanning trees. Alternatively, we could apply the recurrence. The number of trees not containing the diagonal edge is $2\cdot 3\cdot 4 + 3\cdot 4\cdot 2 + 4\cdot 2\cdot 2 + 2\cdot 2\cdot 3$, which is 76. The number of trees containing the diagonal edge is $5\cdot 6$, which is 30.

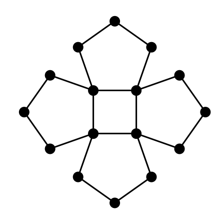


$$\begin{pmatrix} 5 & -2 & -3 & 0 \\ -2 & 5 & -1 & -2 \\ -3 & -1 & 8 & -1 \\ 0 & -2 & -4 & 6 \end{pmatrix}$$

2.2.4. *If a graph G with m edges has a graceful labeling, then K_{2m+1} decomposes into copies of G.* As in the proof of Theorem 2.2.16, view the vertices modulo $2m+1$. Let $a_1, \ldots, a_n$ be the vertex labels on in a graceful labeling of G. By definition, $0 \le a_j \le m$ for each j. For $0 \le i \le 2m$, the ith copy of G uses vertices $i + a_1, \ldots, i + a_n$. Each copy uses one edge from each difference class, and the successive copies use distinct edges from a class, so each edge of K_{2m+1} appears in exactly one of these copies of G.

2.2.5. *The graph below has 2000 spanning trees.* The graph has 16 vertices and 20 edges; we must delete five edges to form a spanning tree. The 5-cycles are pairwise edge-disjoint; we group the deleted edges by the 5-cycles. Each 5-cycle must lose an edge; one 5-cycle will lose two. To avoid disconnecting the graph, one edge lost from the 5-cycle that loses two must be on the 4-cycle, and thus the 4-cycle is also broken.

Every subgraph satisfying these rules is connected with 15 edges, since every vertex has a path to the central 4-cycle, and there is a path from one vertex to the next on the 4-cycle via the 5-cycles that lose just one edge). Hence these are the spanning trees. We can pick the 5-cycle that loses two edges in 4 ways, pick its second lost edge in 4 ways, and pick the edge lost from each remaining 5-cycle in five ways, yielding a total of $4\cdot 4\cdot 5\cdot 5\cdot 5$ spanning trees. The product is 2000.



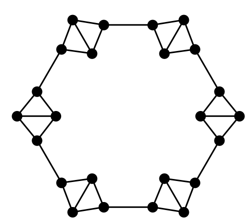
2.2.6. *The 3-regular graph that is a ring of m kites (shown below for $m = 6$) has $2m8^m$ spanning trees.* Call the edges joining kites the "link edges". Deleting two link edges disconnects the graph, so each spanning tree omits at most one link edge.

If a spanning tree uses $m-1$ link edges, then it also contains a spanning tree from each kite. By Example 2.2.6, each kite has eight spanning trees. (Each such spanning tree has three edges; each choice of three edges works except the two forming triangles, and $8 = \binom{5}{2} - 2$.)

To form a spanning tree of this type, we pick one of the m link edges to delete and pick a spanning tree from each kite in 8^k ways. Thus there are $m8^{k-1}$ spanning trees of this sort.

The other possibility is to use all m link edges. Now we must have exactly one kite where the vertices of degree 2 in the kite are not connected by a path within the kite. Since we avoid cycles and spanning trees but must connect the two 3-valent vertices of the kite out to the rest of the graph, we retain exactly two edge from the kite that is cut. Each way of choosing two edges to retain works exept the two that form a path between the 2-valent vertex through one 3-valent vertex: $8 = \binom{5}{2} - 2$.

Since we pick one kite to cut in m ways, pick one of 8 ways to cut it, and pick one of 8 spanning trees in each other kite, there are $m8^m$ spanning trees of this type, for $2m8^m$ spannning trees altogether.



2.2.7. *$K_n - e$ has $(n-2)n^{n-3}$ spanning trees.*

Proof 1 (symmetry and Cayley's Formula—*easiest!*). By Cayley's Formula, there are n^{n-2} spanning trees in K_n. Since each has $n-1$ edges, there are $(n-1)n^{n-2}$ pairs (e, T) such that T is a spanning tree in K_n and $e \in E(T)$. When we group these pairs according to the $\binom{n}{2}$ edges in K_n, we divide by $\binom{n}{2}$ to obtain $2n^{n-3}$ as the number of trees containing any given edge, since by symmetry each edge of K_n appears in the same number of spanning trees.

To count the spanning trees in $K_n - e$, we subtract from the total number of spanning trees in K_n the number that contain the particular edge e. Subtracting $t = 2n^{n-3}$ from n^{n-2} leaves $(n-2)n^{n-3}$ spanning trees in K_n that do not contain e.

Proof 2 (Prüfer correspondence). Given vertex set $[n]$, we count the trees not containing the edge between $n-1$ and n. In the algorithm to generating the Prüfer code of a tree with vertex set $[n]$, we never delete vertex n. Also, we do not delete vertex $n-1$ unless $n-1$ and n are the only leaves, in which case the remaining tree at that stage is a path (because it is a tree with only two leaves).

If the tree contains the edge $(n-1, n)$, then $(n-1, n)$ will be the final edge, and the label last written down is $n-1$ or n. If not, then the path between $n-1$ and n has at least two edges, and we will peel off vertices from one end until only the edge containing n remains. The label n is never recorded during this process, and neither is $n-1$. Thus a Prüfer code corresponds to a tree not containing $(n-1, n)$ if and only if the last term of the list is not $n-1$ or n, and there are $(n-2)n^{n-3}$ such lists.

Proof 3 (Matrix Tree Theorem). For $K_n - e$, the matrix $D - A$ has diagonal $n-1, \dots, n-1, n-2, n-2$, with positions $n-1, n$ and $n, n-1$ equal to 0 and all else -1. Delete the last row and column and take the determinant to obtain the number of spanning trees. To compute the determinant, apply row and column operations as follows: 1) add the $n-2$ other columns to the first so the first column becomes $1, \dots, 1, 0$. 2) subtract the first row from all but the last, so the first row is $1, -1, \dots, -1$, the last is $0, -1, \dots, -1, n-2$, and the others are 0 except for n on the diagonal. The interior rows can then be used to reduce this to a diagonal matrix with entries $1, n, \dots, n, n-2$, whose determinant is $(n-2)n^{n-3}$.

2.2.8. *With vertex set $[n]$, there are $\binom{n}{2}(2^{n-2} - 2)$ trees with $n-2$ leaves and $n!/2$ trees with 2 leaves.* Every tree with two leaves is a path (paths along distinct edges incident to a vertex of degree k leads to k distinct leaves, so having only two leaves in a tree implies maximum degree 2). Every tree with $n-2$ leaves has exactly two non-leaves. Each leaf is adjacent to one of these two vertices, with at least one leaf neighbor for each of the two vertices. These trees are the "double-stars".

To count paths directly, the vertices of a path in order form a permutation of the vertex set. Following the path from the other end produces another permutation. On the other hand, every permutation arises in this way. Hence there are two permutations for every path, and the number of paths is $n!/2$.

To count double-stars directly, we pick the two central vertices in one of $\binom{n}{2}$ ways and then pick the set of leaves adjacent to the lower of the two central vertices. This set is a subset of the $n-2$ remaining vertex labels, and it can be any subset other than the full set and the empty set. The number of ways to do this is the same no matter how the central vertices is chosen, so the number of double-stars is $\binom{n}{2}(2^{n-2} - 2)$.

To solve this using the Prüfer correspondence, we count Prüfer codes for paths and for double-stars. In the Prüfer code corresponding to a tree, the labels of the leaves are the labels that do not appear.

For paths (two leaves), the other $n-2$ labels must each appear in the Prüfer code, so they must appear once each. Having chosen the leaf labels in $\binom{n}{2}$ ways, there are $(n-2)!$ ways to form a Prüfer code in which all the other labels appear. The product is $n!/2$.

For double-stars ($n-2$ leaves), exactly two labels appear in the Prüfer code. We can choose these two labels in $\binom{n}{2}$ ways. To form a Prüfer code (and thus a tree) with these two labels as non-leaves, we choose an arbitrary nonempty proper subset of the positions $1, \ldots, n-2$ for the appearances of the first label. There are $2^{n-2}-2$ ways to do this step. Hence there are $\binom{n}{2}(2^{n-2}-2)$ ways to form the Prüfer code.

2.2.9. *There are $(n!/k!)S(n-2, n-k)$ trees on a fixed vertex set of size n that have exactly k leaves.* Consider the Prüfer sequences of trees. The leaves of a tree are the labels that do not appear in the sequence. We can choose the labels of the leaves in $\binom{n}{k}$ ways. Given a fixed set of leaves, we must count the sequences of length $n-2$ in which the remaining $n-k$ labels all appear. Each label occupies some set of positions in the sequence. We partition the set of positions into $n-k$ nonempty parts, and then we can assign these parts to the labels in $(n-k)!$ ways to complete the sequence. The number of ways to perform the partition, by definition, is $S(n-2, n-k)$. Since these operations are independent, the total number of legal Prüfer sequences is $\binom{n}{k}(n-k)!S(n-2, n-k)$.

2.2.10. *$K_{2,m}$ has $m2^{m-1}$ spanning trees.* Let X, Y be the partite sets, with $|X| = 2$. Each spanning tree has one vertex of Y as a common neighbor of the vertices in X; it can be chosen in m ways. The remaining vertices are leaves; for each, we choose its neighbor in X in one of two ways. Every spanning tree is formed this way, so there are $m2^{m-1}$ trees.

Alternatively, note that $K_{2,m}$ is obtained from the two-vertex multigraph H with m edges by replacing each edge with a path of 2 edges. Since H itself has m spanning trees, Exercise 2.2.12 allows the spanning trees of $K_{2,m}$ to be counted by multiplying m by a factor of $2^{e(H)-n(H)+1} = 2^{m-1}$.

$K_{2,m}$ has $\lfloor (m+1)/2 \rfloor$ isomorphism classes of spanning trees. The vertices in X have one common neighbor, and the isomorphism class is determined by splitting the remaining $m-1$ vertices between them as leaves. We attach k leaves to one neighbor and $m-1-k$ to the other, where $0 \le k \le \lfloor (m-1)/2 \rfloor$. Hence there are $\lfloor (m+1)/2 \rfloor$ isomorphism classes.

2.2.11. $\tau(K_{3,m}) = m^2 3^{m-1}$. Let X, Y be the partite sets, with $|X| = 3$. A spanning tree must have a single vertex in Y adjacent to all of X or two vertices in Y forming P_5 with X. In each case, the remaining vertices of Y are distributed as leaf neighbors arbitrarily to the three vertices of X; each has a choice among the three vertices of X for its neighbor. Hence there are $m3^{m-1}$ spanning trees of the first type and $[3m2(m-1)/2]3^{m-2}$ trees of the second type. and then the remaining vertices in the other

2.2.12. *The effect of graph transformations on the number τ of spanning trees.* Let G be a graph with n vertices and m edges.

a) If H is obtained from G by replacing every edge with k parallel edges, then $\tau(H) = k^{n-1}\tau(G)$.

Proof 1 (direct combinatorial argument). Each spanning tree T of G yields k^{n-1} distinct spanning trees of H by choosing any one of the k copies of each edge in T. This implies $\tau(H) \ge k^{n-1}\tau(G)$. Also, every tree arises in this way. A tree T in H uses at most one edge between each pair of vertices. Since T is connected and acyclic, the edges in G whose copies are used in T form a spanning tree of G that generates T. Hence $\tau(H) \le k^{n-1}\tau(G)$.

Proof 2 (induction on m using the recurrence for τ). If $m = 0$, then $\tau(G) = \tau(H) = 0$, unless $n = 1$, in which case $1 = k^0 \cdot 1$. If $m > 0$, choose $e \in E(G)$. Let H' be the graph obtained from H by contracting all k copies of e. Let H'' be the graph obtained from H by deleting all k copies of e. The spanning trees of H can be grouped by whether they use a copy of e (they cannot use more than one copy). There are $k \times \tau(H')$ of these trees that use a copy of e and $\tau(H'')$ that do not. We can apply the induction hypothesis to H' and H'', since each arises from a graph with fewer than m edges by having k copies of each edge: H' from $G \cdot e$ and H'' from $G - e$. Thus

$$\begin{aligned}\tau(H) = k \times \tau(H') + \tau(H'') &= k \cdot k^{n-2}\tau(G \cdot e) + k^{n-1}\tau(G - e) \\ &= k^{n-1}[\tau(G \cdot e) + \tau(G - e)] = k^{n-1}\tau(G).\end{aligned}$$

Proof 3 (matrix tree theorem). Let Q, Q' be the matrices obtained from G, G', from which we delete one row and column before taking the determinant. By construction, $Q' = kQ$. When we take the determinant of a submatrix of order $n-1$, we thus obtain $\tau(G') = k^{n-1}\tau(G)$.

b) If H is obtained from G by replacing each $e \in E(G)$ with a path $P(e)$ of k edges, then $\tau(H) = k^{m-n+1}\tau(G)$.

Proof 1 (combinatorial argument). A spanning tree T of G yields k^{m-n+1} spanning trees of H as follows. If $e \in E(T)$, include all of $P(e)$. If $e \notin E(T)$, use all but one edge of $P(e)$. Choosing one of the k edges of $P(e)$ to omit for each $e \in E(G) - E(T)$ yields k^{m-n+1} distinct trees (connected and acyclic) in H. Again we must show that all spanning trees have been generated. A tree T' in H omits at most one edge from each path $P(e)$, else some vertex in $P(e)$ would be separated from the remainder of H. Let T be the spanning subgraph of G with $E(T) = \{e \in E(G) \colon P(e) \subseteq T'\}$. If T'

is connected and has no cycles, then the same is true of T, and T' is one of the trees generated from T as described above.

Proof 2 (induction on m). The basis step $m = 0$ is as in (a). For $m > 0$, select an edge $e \in E(G)$. The spanning trees of H use k or $k-1$ edges of $P(e)$. These two types are counted by $\tau(H')$ and $\tau(H'')$, where H' is the graph obtained from H by contracting all edges in $P(e)$, and H'' is the graph obtained from H by deleting $P(e)$ (except for its end-vertices). Since these graphs arise from $G \cdot e$ and $G - e$ (each with $m-1$ edges) by replacing each edge with a path of length k, applying the induction hypothesis yields

$$\begin{aligned}\tau(H) = \tau(H') + k \cdot \tau(H'') &= k^{(m-1)-(n-1)+1}\tau(G \cdot e) + k[k^{(m-1)-n+1}\tau(G-e)] \\ &= k^{m-n+1}[\tau(G \cdot e) + \tau(G-e)] = k^{m-n+1}\tau(G).\end{aligned}$$

2.2.13. *Spanning trees in $K_{n,n}$.* For each spanning tree T of $K_{n,n}$, a list $f(T)$ of pairs of integers (written vertically) is formed as follows: Let u, v be the least-indexed leaves of the remaining subtree that occur in X and Y. Add the pair $\binom{a}{b}$ to the sequence, where a is the index of the neighbor of u and b is the index of the neighbor of v. Delete $\{u, v\}$ and iterate until $n-2$ pairs are generated and one edge remains.

a) Every spanning tree of $K_{n,n}$ has a leaf in each partite set, and hence f is well-defined. If each vertex of one partite set has degree at least 2, then at least $2n$ edges are incident to this partite set, which are too many to have in a spanning tree of a graph with $2n$ vertices.

b) f is a bijection from the set of spanning trees of $K_{n,n}$ to the set of $n-1$-element lists of pairs of elements from $[n]$, and hence $K_{n,n}$ has n^{2n-2} spanning trees. We use an analogue of Prüfer codes. Consider $K_{n,n}$ with partite sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. For each spanning tree T, we form a sequence $f(T)$ of $n-1$ pairs of integers chosen from $[n]$ by recording at each step the ordered pair of subscripts of the neighbors of the least-indexed leaves of T remaining in X and Y, and then deleting these leaves. What remains is a spanning tree in a smaller balanced biclique, so by part (a) the process is well-defined.

Since there n^{2n-2} such lists, it suffices to show that f establishes a bijection from the set of spanning trees of $K_{n,n}$ to the set of lists.

From a list L of $n-1$ pairs of integers chosen from $[n]$, we generate a tree $g(L)$ with vertex set $X \cup Y$. We begin with $X \cup Y$, no edges, and each vertex unmarked. At the ith step, when the ith ordered pair is $\binom{a(i)}{b(i)}$, let u be the least index of an unmarked vertex in Y that does not appear in first coordinates of L at or after position i, and let v be the least index of an unmarked vertex in X that does not appear in second coordinates of L at or after position i. We add the edges $x_{a(i)}y_u$ and $y_{b(i)}x_v$, and then we mark x_v and y_u to eliminate them from further consideration. After $n-1$ pairs, we add one edge joining the two remaining unmarked vertices.

After the ith step, we have $2n - 2i$ components, each containing one unmarked vertex. This follows by induction on i; it holds when $i = 0$. Since indices cannot be marked until after they no longer appear in the list, the two edges created in the ith step join pairs of unmarked vertices. By the induction hypothesis, these come from four different components, and the two added edges combine these into two, each keeping one unmarked vertex. Thus adding the last edge completes the construction of a tree.

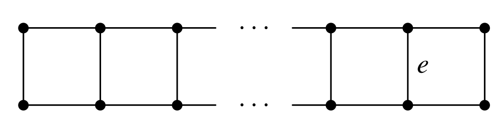
In computing $f(T)$, a label no longer appears in the sequence after it is deleted as a leaf. Hence the vertices marked at the ith step in computing $g(L)$ are precisely the leaves deleted at the ith step in computing $f(g(L))$, which also records $\binom{a(i)}{b(i)}$. Thus $L = f(g(L))$. Similarly, the leaves deleting at the ith step in computing $f(T)$ are the vertices marked at the ith step in computing $g(f(T))$, which yields $T = g(f(T))$. Hence each maps inverts the other, and both are bijections.

2.2.14. *The number of trees with vertices $1, \ldots, r+s$ that have partite sets of sizes r and s is $\binom{r+s}{s}s^{r-1}r^{s-1}$ if $r \neq s$.* It suffices to count the Prüfer codes for such trees. The factor $\binom{r+s}{r}$ counts the assignments of labels to the two partite sets (half that amount if $r = s$). When deleting a vertex in computing the Prüfer code, we record a vertex of the other partite set. Since an edge remains at the end of the construction, the final code has $s-1$ entries from the r-set and $r-1$ entries from the s-set.

It suffices to show that the sublists formed from each partite set determine the full list, because there are $s^{r-1}r^{s-1}$ such pairs of sublists. In reconstructing the code and tree from the pair of lists, the next leaf to be "finished" by receiving its last edge is the least label that is unfinished and doesn't appear in the remainder of the list. The remainder of the list is the remainder of the two sublists. We know which set contains the next leaf to be finished. Its neighbor comes from the other set. This tells us which sublist contributes the next element of the full list. Iterating this merges the two sublists into the full Prüfer code.

When $r = s$, the given formula counts the lists twice.

2.2.15. *For $n \geq 1$, the number of spanning trees in the graph G_n with $2n$ vertices and $3n-2$ edges pictured below satisfies the recurrence $t_n = 4t_{n-1} - t_{n-2}$ for $n \geq 3$, with $t_1 = 1$ and $t_2 = 4$.*

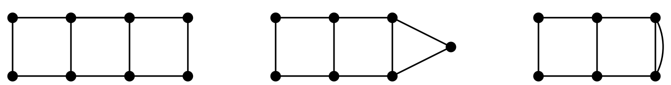


(*Comment:* The solution to the recurrence is $t_n = \frac{1}{2\sqrt{3}}[(2+\sqrt{3})^n - (2-\sqrt{3})^n]$.) Using the recurrence, this follows by induction on n.) We derive the recurrence. Let $t_n = \tau(G_n)$.

Proof 1 (direct argument for recurrence). Each spanning tree in G_n uses two or three of the three rightmost edges. Those with two of the rightmost edges are obtained by adding any two of those edges to any spanning tree of G_{n-1}. Thus there are $3t_{n-1}$ such trees. To prove the recurrence $t_n = 4t_{n-1} - t_{n-2}$, it suffices to show that there are $t_{n-1} - t_{n-2}$ spanning trees that contain the three rightmost edges.

Such trees cannot contain the second-to-last vertical edge e. Therefore, deleting the three rightmost edges and adding e yields a spanning tree of G_{n-1}. Furthermore, each spanning tree of G_{n-1} using e arises exactly once in this way, because we can invert this operation. Hence the number of spanning trees of G_n containing the three rightmost edges equals the number of spanning trees of G_{n-1} containing e. The number of spanning trees of G_{n-1} that don't contain e is t_{n-2}, so the number of spanning trees of G_{n-1} that do contain e is $t_{n-1} - t_{n-2}$.

Proof 2 (deletion/contraction recurrence). Applying the recurrence introduces graphs of other types. Let H_n be the graph obtained by contracting the rightmost edge of G_n, and let F_{n-1} be the graph obtained by contracting one of the rightmost edges of H_n. Below we show G_4, H_4, and F_3.



By using $\tau(G) = \tau(G-e) + \tau(G \cdot e)$ on a rightmost edge e and observing that a pendant edge appears in all spanning trees while a loop appears in none, we obtain

$$\begin{aligned}
\tau(G_n) &= \tau(G_{n-1}) + \tau(H_n) \\
\tau(H_n) &= \tau(G_{n-1}) + \tau(F_{n-1}) \\
\tau(F_n) &= \tau(G_n) + \tau(H_{n-1})
\end{aligned}$$

Substituting in for $\tau(H_n)$ and then for $\tau(F_{n-1})$ and then for $\tau(H_{n-1})$ yields the desired recurrence:

$$\begin{aligned}
\tau(G_n) &= \tau(G_{n-1}) + \tau(G_{n-1}) + \tau(F_{n-1}) = 2\tau(G_{n-1}) + \tau(G_{n-1}) + \tau(H_{n-2}) \\
&= 3\tau(G_{n-1}) + \tau(G_{n-1}) - \tau(G_{n-2}) = 4\tau(G_{n-1}) - \tau(G_{n-2}).
\end{aligned}$$

2.2.16. *Spanning trees in $K_1 \vee P_n$.* The number a_n of spanning trees satisfies $a_n = a_{n-1} + 1 + \sum_{i=1}^{n-1} a_i$ for $n > 1$, with $a_1 = 1$. Let $x_1, \dots, x_n$ be the vertices of the path in order, and let z be the vertex off the path. There are a_{n-1} spanning trees not using the edge zx_n; they combine the edge $x_{n-1}x_n$ with a spanning tree of $K_1 \vee P_{n-1}$. Among trees containing zx_n, let i be the highest index such that all of the path $x_{i+1}, \dots, x_n$ appears in the tree. For each i, there are a_i such trees, since the specified edges are combined with a spanning tree of $K_1 \vee P_i$. The term 1 corresponds to $i = 0$; here the entire tree is $P_n \cup zx_n$. This exhausts all possible spanning trees.

2.2.17. *Cayley's formula from the Matrix Tree Theorem.* The number of labeled n-vertex trees is the number of spanning trees in K_n. Using the Matrix Tree Theorem, we compute this by subtracting the adjacency matrix from the diagonal matrix of degrees, deleting one row and column, and taking the determinant. All degrees are $n-1$, so the initial matrix is $n-1$ on the diagonal and -1 elsewhere. Delete the last row and column. We compute the determinant of the resulting matrix.

Proof 1 (row operations). Add every row to the first row does not change the determinant but makes every entry in the first row 1. Now add the first row to every other row. The determinant remains unchanged, but every row below the first is now 0 everywhere except on the diagonal, where the value is n. The matrix is now upper triangular, so the determinant is the product of the diagonal entries, which are one 1 and $n-2$ copies of n. Hence the determinant is n^{n-2}, as desired.

Proof 2 (eigenvalues). The determinant of a matrix is the product of its eigenvalues. The eigenvalues of a matrix are shifted by λ when λI is added to the matrix. The matrix in question is $nI_{n-1} - J_{n-1}$, where I_{n-1} is the $n-1$-by-$n-1$ identity matrix and J_{n-1} is the $n-1$-by-$n-1$ matrix with every entry 1. The eigenvalues of $-J_{n-1}$ are $-(n-1)$ with multiplicity 1 and 0 with multiplicity $n-2$. Hence the eigenvalues of the desired matrix are 1 with multiplicity 1 and n with multiplicity $n-2$. Hence the determinant is n^{n-2}, as desired.

2.2.18. *Proof that $\tau(K_{r,s}) = s^{r-1}r^{s-1}$ using the Matrix Tree Theorem.* The adjacency matrix of $K_{r,s}$ is $\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$, where $\mathbf{0}$ and $\mathbf{1}$ denote matrices of all 0s and all 1s, and both the row partition and the column partition consist of r in the first block and s in the second block. The diagonal matrix of degrees is $\begin{pmatrix} sI_r & 0 \\ 0 & rI_s \end{pmatrix}$, where I_n is the identity matrix of order n. Hence we may delete the first row and column to obtain $Q^* = \begin{pmatrix} sI_{r-1} & -\mathbf{1} \\ -\mathbf{1} & rI_s \end{pmatrix}$.

We apply row and column operations that do not change the determinant. We subtract column $r-1$ (last of the first block) from the earlier columns and subtract column r (first of the second block) from the later columns. This yields the matrix on the left below, where the values outside the matrix indicate the number of rows or columns in the blocks. Now, we add to row $r-1$ the earlier rows and add to row r the later rows, yielding the matrix on the right below.

$$\begin{array}{c|cccc} & r-2 & 1 & 1 & s-1 \\ \hline r-2 & sI_{r-2} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ 1 & -s\mathbf{1} & s & -1 & \mathbf{0} \\ 1 & \mathbf{0} & -1 & r & -r\mathbf{1} \\ s-1 & \mathbf{0} & \mathbf{-1} & \mathbf{0} & rI_{s-1} \end{array} \qquad \begin{array}{c|cccc} & r-2 & 1 & 1 & s-1 \\ \hline r-2 & sI_{r-2} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ 1 & \mathbf{0} & s & -r+1 & \mathbf{0} \\ 1 & \mathbf{0} & -s & r & \mathbf{0} \\ s-1 & \mathbf{0} & \mathbf{-1} & \mathbf{0} & rI_{s-1} \end{array}$$

Adding row $r-1$ to row r now makes row r all zero except for a single 1 in position r (on the diagonal). Adding row r to the first $r-2$ rows (and $r-1$ times row r to row $r-1$) now leaves the 1 in row r as the only nonzero entry in column r. Also, the s in column $r-1$ of row $r-1$ is now the only nonzero entry in row $r-1$. Hence we can add $1/s$ times row $r-1$ to each of the last $s-1$ rows to eliminate the other nonzero entries in column $r-1$.

The resulting matrix is diagonal, with diagonal entries consisting of $r-1$ copies of s, one copy of 1, and $s-1$ copies of r. Since adding a multiple of a row or column to another does not change the determinant, the determinant of our original matrix equals the determinant of this diagonal matrix. The determinant of a diagonal matrix is the product of its diagonal entries, so the determinant is $s^{r-1}r^{s-1}$.

2.2.19. *The number t_n of labeled trees on n vertices satisfies the recurrence $t_n = \sum_{k=1}^{n-1} k\binom{n-2}{k-1} t_k t_{n-k}$.* For an arbitrary labeled tree on n vertices, delete the edge incident to v_2 on the path from v_2 to v_1. This yields labeled trees on k and $n-k$ vertices for some k, where v_1 belongs to the tree on k vertices and v_2 to the tree on $n-k$ vertices. Each such pair arises from exactly $k\binom{n-2}{k-1}$ labeled trees on n vertices. To see this, reverse the process. First choose the $k-1$ other vertices to be in the subtree containing v_1. Next, choose a tree on k labeled vertices and a tree on $n-k$ labeled vertices (any such choice could arise by deleting the specified edge of a tree on n vertices). Finally, reconnect the tree by adding an edge from v_2 to any one of the k vertices in the tree containing v_1. This counts the trees such that the subtree containing v_1 has k vertices, and summing this over k yields t_n.

2.2.20. *A d-regular graph G has a decomposition into copies of $K_{1,d}$ if and only if G is bipartite.* If G has bipartition X, Y, then for each $x \in X$ we include the copy of $K_{1,d}$ obtained by taking all d edges incident to x. Since every edge has exactly one endpoint in X, and every vertex in X has degree d, this puts every edge of G into exactly one star in our list.

If G has a $K_{1,d}$-decomposition, then we let X be the set of centers of the copies of $K_{1,d}$ in the decomposition. Since G is d-regular, each copy of $K_{1,d}$ uses all edges incident to its center. Since the list is a decomposition, each edge is in exactly one such star, so X is an independent set. Since every edge belongs to some $K_{1,d}$ centered in X, there is no edge with both endpoints outside X. Thus the remaining vertices also form an independent set, and G has bipartition $X, \overline{X}$.

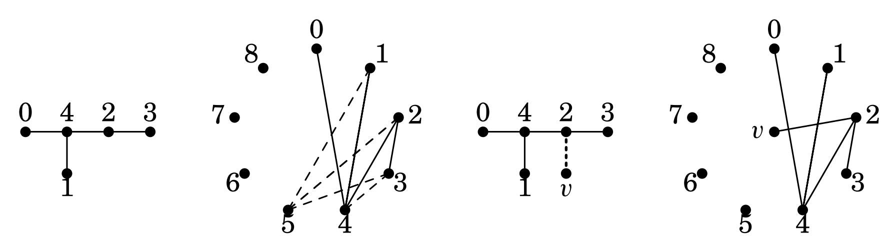
Alternative proof of sufficiency. If G is not bipartite, then G contains an odd cycle. When decomposing a d-regular graph into copies of $K_{1,d}$, each subgraph used consists of all d edges incident to a single vertex. Hence each vertex occurs only as a center or only as a leaf in these subgraphs. Also, every edge joins the center and the leaf in the star containing it. These statements require that centers and leaves alternate along a cycle, but this cannot be done in an odd cycle.

2.2.21. *Decomposition of $K_{2m-1,2m}$ into m spanning paths.* We add a vertex to the smaller partite set and decomposition $K_{2m,2m}$ into m spanning cycles. Deleting the added vertex from each cycle yields pairwise edge-disjoint spanning paths of $K_{2m-1,2m}$.

Let the partite sets of $K_{2m,2m}$ be $x_1, \ldots, x_{2m}$ and $y_1, \ldots, y_{2m}$. Let the kth cycle consist of the edges of the forms $x_i y_{i+2k-1}$ and $x_i y_{i+2k}$, where subscripts above $2m$ are reduced by $2m$. These sets are pairwise disjoint and form spanning cycles.

2.2.22. *If G is an n-vertex simple graph having a decomposition into k spanning trees, and $\Delta(G) = \delta(G) + 1$, then G has $n-2k$ vertices of degree $2k$ and $2k$ vertices of degree $2k-1$.* Each spanning tree has $n-1$ edges, so $e(G) = k(n-1)$. Note that $k < n/2$, since G is simple and is not K_n (since it is not regular). If G has r vertices of minimum degree and $n-r$ of maximum degree, then the Degree-Sum Formula yields $2k(n-1) = n\Delta(G) - r$. Since $1 \le r \le n$, we conclude that $\Delta(G) = 2k = r$.

2.2.23. *If the Graceful Tree Conjecture holds and $e(T) = m$, then K_{2m} decomposes into $2m-1$ copies of T.* Let $T' = T - u$, where u is a leaf of T with neighbor v. Let w be a vertex of K_{2m}. Construct a cyclic T'-decomposition of $K_{2m} - w$ using a graceful labeling of T' as in the proof of Theorem 2.2.16. Each vertex serves as v in exactly one copy of T'. Extend each copy of T' to a copy of T by adding the edge to w from the vertex serving as v. This exhausts the edges to w and completes the T-decomposition of G.

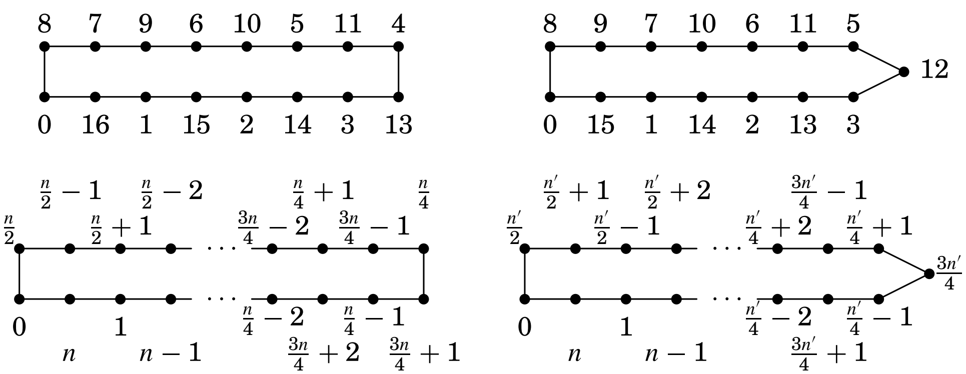


2.2.24. *Of the n^{n-2} trees with vertex set $\{0, \ldots, n-1\}$, how many are gracefully labeled by their vertex names?* This question was incorrectly posed. It should be *of the graphs with vertex set* $\{0,\dots,n-1\}$ *that have* $n-1$ *edges, how many are gracefully labeled by their vertex names?* Such a graph has k choices for the placement of the edge with difference $n-k$, since the lower endpoint can be any of $\{0,\dots,k-1\}$. Hence the number of graphs is $(n-1)!$.

2.2.25. *If a graph G is graceful and Eulerian, then $e(G)$ is congruent to 0 or 3 mod 4.* Let f be a graceful labeling. The parity of the sum of the labels on an edge is the same as the parity of their difference. Hence the sum $\sum_{v\in V(G)} d(v)f(v)$ has the same parity as the sum of the edge differences. The first sum is even, since G is Eulerian. The second has the same parity as the number of odd numbers in the range from 1 to $e(G)$. This is even if and only if $e(G)$ is congruent to 0 or 3 mod 4, which completes the proof.

2.2.26. *The cycle C_n is graceful if and only if 4 divides n or $n+1$.* The necessity of the condition is a special case of Exercise 2.2.25. For sufficiency, we provide a construction for each congruence class. We show an explicit construction ($n=16$ and $n=15$) and a general construction for each class. In the class where $n+1$ is divisible by 4, we let n' denote $n+1$. When n is divisible by 4, let $n'=n$.

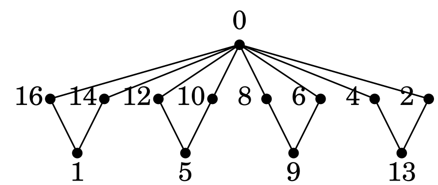
The labeling uses a base edge joining 0 and $n'/2$, plus two paths. The bottom path, starting from 0, alternates labels from the top and bottom to give the large differences: $n, n-1$, and so on down to $n'/2+1$. The top path, starting from $n'/2$, uses labels working from the center to give the small differences: 1, 2, and so on up to $n'/2-1$. The label next to $n'/2$ is $n'/2-1$ when 4 divides n, otherwise $n'/2+1$. When chosen this way, the two paths reach the same label at their other ends to complete the cycle: $n/4$ in the even case, $3n'/4$ in the odd case. Checking this ensures that the intervals of labels used do not overlap. Note that the value $3n/4$ is not used in the even case, and $n'/4$ is not used in the odd case.



2.2.27. *The graph consisting of k copies of C_4 with one common vertex is graceful.* The construction is illustrated below for $k=4$. Let x be the central vertex. Let the neighbors of x be $y_0,\dots,y_{2k-1}$, and let the remaining vertices be $z_0,\dots,z_{k-1}$, such that $N(z_i)=\{y_{2i},y_{2i+1}\}$.

Define a labeling f by $f(x)=0$, $f(y_i)=4k-2i$, and $f(z_i)=4i+1$. The labels on $y_1,\dots,y_{2k}$ are distinct positive even numbers, and those on $z_1,\dots,z_k$ are distinct odd numbers, so f is injective, as desired. The differences on the edges from x are the desired distinct even numbers.

The differences on the remaining edges are odd and less than $2k$; it suffices to show that their values are distinct. Involving z_i, the differences are $4k-1-8i$ and $4k-3-8i$. Starting from z_0 through increasing i, these are $4k-1, 4k-3, 4k-9, 4k-11, \dots$. Starting from z_{k-1} through decreasing i, these are $-4k+5, -4k+7, -4k+13, -4k+15, \dots$. The absolute values are distinct, as needed.



2.2.28. Given positive integers $d_1,\dots,d_n$, there exists a caterpillar with vertex degrees $d_1,\dots,d_n$ if and only if $\sum d_i = 2n-2$. If there is such a caterpillar, it is a tree and has $n-1$ edges, and hence the vertex degrees sum to $n-2$. Hence the condition is necessary. There are various proofs of sufficiency, which construct a caterpillar with these degrees given only the list $d_1,\dots,d_n$ of positive numbers with sum $2n-2$.

Proof 1 (explicit construction). We may assume that $d_1\ge\cdots\ge d_k > 1 = d_{k+1} = \cdots = d_n$. Begin with a path of length $k+1$ with vertices $v_0,\dots,v_{k+1}$. Augment these vertices to their desired degrees by adding d_i-2 edges (and leaf neighbors) at v_i, for $1\le i\le k$. This creates a caterpillar with vertex degrees $d_1,\dots,d_k$ for the nonleaves. We must prove that it has $n-k$ leaves, which is the number of 1s in the list.

Including v_0 and v_{k+1}, the actual number of leaves in the caterpillar we constructed is $2+\sum_{i=1}^{k}(d_i-2)$. This equals $2-2k+(\sum_{i=1}^{n} d_i)-\sum_{i=k+1}^{n} d_i$. Since $\sum_{i=1}^{n} d_i = 2n-2$, the number of leaves is $(2-2k)+(2n-2)-(n-k)=n-k$, as desired. We have created an n-vertex caterpillar with vertex degrees $d_1,\dots,d_n$.

Proof 2 (induction on n). Basis step ($n=2$): the only list is 1, 1, and the one graph realizing this is a caterpillar. Induction step ($n>2$): n positive numbers summing to $2n-2$ must include at least two 1s; otherwise, the sum is at least $2n-1$. If the remaining numbers are all 2s, then P_n is a caterpillar with the desired degrees. Otherwise, some d_i exceeds 2; by symmetry, we may assume that this is d_1. Let d' be the list obtained by reducing d_1 by one and deleting one of the 1s. The list d' has $n-1$ entries, all positive, and it sums to $2n-4 = 2(n-1)-2$. By the induction hypothesis, there is a caterpillar G' with degree list d'.

Let x be a vertex of G' with degree d_1'. Since $d_1 > 2$, we have $d_1' \geq 2$, and hence x is on the spinal path. Growing a leaf at x yields obtain a larger caterpillar G with degree list d. This completes the induction step.

2.2.29. *Every tree transforms to a caterpillar with the same degree list by operations that delete an edge and add another rejoining the two components.* Let P be a longest path in the current tree T. If P is incident to every edge, then T is a caterpillar. Otherwise a path P' of length at least two leaves P at some vertex x. Let uv be an edge of P', with u between x and v, and let y be a neighbor of x on P.

Cut xy and add yu. Now cut uv and add vx. Each operation has the specified type, and together they form a 2-switch preserving the vertex degrees. Also, the new tree has a path whose length is that of P plus $d_T(x,u)$.

Since the length of a path cannot exceed the number of vertices, this process terminates. It can only terminate when the longest path is incident to all edges and the tree is a caterpillar.

2.2.30. *A connected graph is a caterpillar if and only if it can be drawn on a channel without edge crossings.*

Necessity. If G is a caterpillar, let P be the spine of G. Draw P on a channel by alternating between the two sides of the channel. The remaining edges of G consist of a leaf and a vertex of P. If u,v,w are three consecutive vertices on P, then v has an "unobstructed view" of the other side of the channel between the edges vu and vw. Each leaf x adjacent to v can be placed in that portion of the other bank, and the edge vx can then be drawn straight across the channel without crossing another edge.

Sufficiency. Suppose that G is drawn on a channel. The endpoints of an edge e cannot both have neighbors in the same direction along the channel, since that would create a crossing. Hence G has no cycle, since a cycle would leave an edge and return to it via the same direction along the channel. We conclude that G is a tree.

If G contains the 7-vertex tree that is not a caterpillar, then let v be its central vertex. The three neighbors of v occur on the other side of the channel in some order; let u be the middle neighbor. The other edge incident to u must lie in one direction or the other from uv, contradicting the preceding paragraph. Hence G avoids the forbidden subtree and is a caterpillar.

(Alternatively, we can prove this directly by moving along the channel to extract the spine, observing that the remainder of the tree must be leaves attached to the spine.)

2.2.31. *Every caterpillar has an up/down labeling.* Constructive proof. Let $P = v_0,\ldots,v_k$ be a longest path in a caterpillar G with m edges; by the argument above P is the spine of G. We iteratively construct a graceful labeling f for G. Define two parameters l,u that denote the biggest low label and smallest high label used; after each stage the unused labels are $\{l+1,\ldots,u-1\}$. Let r denote the lowest edge difference achieved; after each stage $r,\ldots,m$ have been achieved.

Begin by setting $f(v_0)=0$ and $f(v_1)=m$; hence $l=0$, $u=m$, $r=m$. Before stage i, we will have $\{f(v_i), f(v_{i-1})\} = \{l,u\}$; this is true by construction before stage 1. Suppose this is true before stage i, along with the other claims made for l,u,d. Let $d = d_G(v_i)$. In stage i, label the $d-1$ remaining neighbors of v_i with the $d-1$ numbers nearest $f(v_{i-1})$ that have not been used, ending with v_{i+1}. Since we start with $|f(v_i)-f(v_{i-1})| = u-l = r$, the new differences are $r-1,\ldots,r-d+1$, which have not yet been achieved. To finish stage i, reset r to $r-d+1$; also, if $f(x_{i-1}) = l$ reset l to $l+d-1$, but if $f(x_{i-1}) = u$ reset u to $u-d+1$. Now stage i is complete, and the claims about l,u,r are satisfied as we are ready to start stage $i+1$: $\{f(v_{i+1}), f(v_i)\} = \{l,u\}$, $r = u-l$, and the edge differences so far are $r,\ldots,m$. After stage $k-1$, we have assigned distinct labels in $\{0,\ldots,m\}$ to all $m+1$ vertices, and the differences of labels of adjacent vertices are all distinct, so we have constructed a graceful labeling.

The 7-vertex tree that is not a caterpillar has no up/down labeling. In an up/down labeling of a connected bipartite graph, one partite set must have all labels above the threshold and the other have all labels below the threshold. Also, we can interchange the high side and the low side by subtracting all labels from $n-1$. Hence for this 7-vertex tree we may assume the labels on the vertices of degree 2 are the high labels 4,5,6. Since 0,6 must be adjacent, this leaves two cases: 0 on the center or 0 on the leaf next to 6. In the first case, putting 1 or 2 next to 6 gives a difference already present, but with 3 next to 6 we can no longer obtain a difference of 1 on any edge. In the second case, we can only obtain a difference of 5 by putting 1 on the center, but now putting 2 next to 5 gives two edges with difference 3, while putting 2 next to 4 and 3 next to 5 give two edges with difference 2. Hence there is no way to complete an up/down labeling.

2.2.32. *There are $2^{n-4} + 2^{\lfloor (n-4)/2 \rfloor}$ isomorphism classes of n-vertex caterpillars.* We describe caterpillars by binary lists. Each 1 represents an edge on the spine. Each 0 represents a pendant edge at the spine vertex between the edges corresponding to the nearest 1s on each side. Thus n-vertex caterpillars correspond to binary lists of length $n-1$ with both end bits being 1.

We can generate the lists for caterpillars from either end of the spine; reversing the list yields a caterpillar in the same isomorphism class. Hence we count the lists, add the symmetric lists, and divide by 2. There are 2^{n-3} lists of the specified type. To make a symmetric list, we specify $\lceil (n-3)/2 \rceil$ bits. Thus the result is $(2^{n-3} + 2^{\lceil (n-3)/2 \rceil})/2$.

2.2.33. *If T is an orientation of a tree such that the heads of the edges are all distinct, then T is a union of paths from the root (the one vertex that is not a head), and each each vertex is reached by one path from the root.* We use induction on n, the number of vertices. For $n = 1$, the tree with one vertex satisfies all the conditions. Consider $n > 1$. Since there are $n-1$ edges, some vertex is not a tail. This vertex v is not the root, since the root is the tail of all its incident edges. Since the heads are distinct, v is incident to only one edge and is its head. Hence $T - v$ is an orientation of a smaller tree where the heads of the edges are distinct. By the induction hypothesis, it is a tree of paths from the root (one to each vertex), and replacing the edge to v preserves this desired conclusion for the full tree.

2.2.34. *An explicit de Bruijn cycle of length 2^n is generated by starting with n 0's and subsequently appending a 1 when doing so does not repeat a previous string of length n (otherwise append a 0).* A de Bruijn cycle is formed by recording the successive edge labels along an Eulerian circuit in the de Bruijn digraph. The vertices of the de Bruijn digraph are the 2^{n-1} binary strings of length $n-1$. From each vertex two edges depart, labeled 0 and 1. The edge 0 leaving v goes to the vertex obtained by dropping the first bit of v and appending 0 at the end. The edge 1 leaving v goes to the vertex obtained by dropping the first bit of v and appending 1 at the end.

Let v_0 denote the all-zero vertex, and let e be the loop at v_0 labeled 0. The $2^{n-1} - 1$ edges labeled 0 other than e form a tree of paths in to v_0. (Since a path along these edges never reintroduces a 1, it cannot return to a vertex with a 1 after leaving it.) Starting at v_0 along edge e means starting with n 0's. Algorithm 2.4.7 now tells us to follow the edge labeled 1 at every subsequent step unless it has already been used; that is, unless appending a 1 to the current list creates a previous string of length n. Theorem 2.4.9 guarantees that the result is an Eulerian circuit.

2.2.35. *Tarry's Algorithm (The Robot in the Castle).* The rules of motion are: 1) After entering a corridor, traverse it and enter the room at the other end. 2) After entering a room whose doors are all unmarked, mark I on the door of entry. 3) When in a room having an unmarked door, mark O on some unmarked door and exit through it. 4) When in a room having all doors marked, find one not marked O (if one exists), and exit through it. 5) When in a room having all doors marked O, STOP.

When in a room other than the original room u, the number of entering edges that have been used exceeds the number of exiting edges. Thus an

exiting door has not yet been marked O. This implies that the robot can only terminate in the original room u.

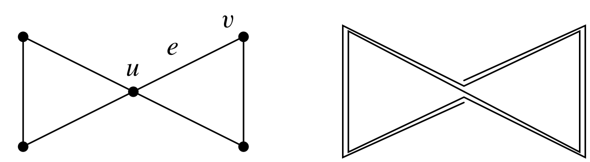
The edges marked I grow from u a tree of paths that can be followed back to u. The rules for motion establish an ordering of the edges leaving each room so that the edge labeled I (for a room other than u) is last.

In order to terminate in u or to leave a room v by the door marked I, every edge entering the room must have been used to enter it, including all edges marked I at the other end. Therefore, for every room actually entered, the robot follows all its incident corridors in both directions.

Thus it suffices to show that every room is reached. Let V be the set of all rooms, and let S be the set of rooms reached in a particular robot tour. If $S \neq V$, then since the castle is connected there is a corridor joining rooms $s \in S$ and $r \notin S$ (the shortest path between S and $\overline{S}$. Since every reached vertex has its incidence corridors followed in both directions, the corridor sr is followed, and r is also reached. The contradiction yields $S = V$.

Comment. Consider a digraph in which each corridor becomes a pair of oppositely-directed edges. Thus indegree equals outdegree at each vertex. The digraph is Eulerian, and the edges marked I form an intree to the initial vertex. The rules for the robot produce an Eulerian circuit by the method in Algorithm 2.4.7.

The portion of the original tour after the initial edge $e = uv$ is not a tour formed according to the rules for a tour in $G - e$, because in the original tour no door of u is ever marked I. If e is not a cut-edge, then tours that follow e, follow $G - e$ from v, and return along e do not include tours that do not start and end with e. There may be such tours, as illustrated below, so such a proof falls into the induction trap.



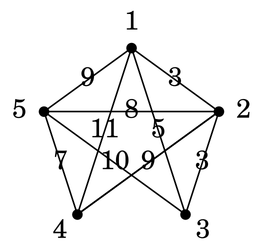
2.3. OPTIMIZATION AND TREES

2.3.1. *In an edge-weighting of K_n, the total weight on every cycle is even if and only if the total weight on every triangle is even.* Necessity is trivial, since triangles are cycles. For sufficiency, suppose that every triangle has even weight. We use induction on the length to prove that every cycle C has even weight. The basis step, length 3, is given by hypothesis. For the induction step, consider a cycle C and a chord e. The chord creates two shorter cycles C_1, C_2 with C. By the induction hypothesis, C_1 and C_2 have even weight. The weight of C is the sum of their weights minus twice the weight of e, so it is still even.

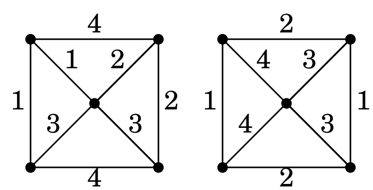
2.3.2. *If T is a minimum-weight spanning tree of a weighted graph G, then the u,v-path in T need not be a minimum-weight u,v-path in G.* If G is a cycle of length of length at least 3 with all edge weights 1, then the cheapest path between the endpoints of the edge omitted by T has cost 1, but the cheapest path between them in T costs $n(G)-1$.

2.3.3. *Computation of minimum spanning tree.* The matrix on the left below corresponds to the weighted graph on the right. Using Kruskal's algorithm, we iteratively select the cheapest edge not creating a cycle. Starting with the two edges of weight 3, the edge of weight 5 is forbidden, but the edge of weight 7 is available. The edge of weight 8 completes the minimum spanning tree, total weight 21. Note that if the edge of weight 8 had weight 10, then either of the edges of weight 9 could be chosen to complete the tree; in this case there would be two spanning trees with the minimum value.

$$\begin{pmatrix} 0 & 3 & 5 & 11 & 9 \\ 3 & 0 & 3 & 9 & 8 \\ 5 & 3 & 0 & \infty & 10 \\ 11 & 9 & \infty & 0 & 7 \\ 9 & 8 & 10 & 7 & 0 \end{pmatrix}$$



2.3.4. *Weighted trees in $K_1 \vee C_4$.* On the left, the spanning tree is unique, using all edges of weights 1 and 2. On the right it can use either edge of weight 2 and either edge of weight 3 plus the edges of weight 1.



2.3.5. *Shortest paths in a digraph.* The direct i to j travel time is the entry $a_{i,j}$ in the first matrix below. The second matrix recordes the least i to j travel time for each pair i,j. These numbers were determined for each i by iteratively updating candidate distances from i and then selecting the closest of the unreached set (Dijkstra's Algorithm). To do this by hand,

make an extra copy of the matrix and use crossouts to update candidate distances in each row, using the original numbers when updating candidate distances. The answer can be presented with more information by drawing the tree of shortest paths that grows from each vertex.

$$\begin{pmatrix} 0 & 10 & 20 & \infty & 17 \\ 7 & 0 & 5 & 22 & 33 \\ 14 & 13 & 0 & 15 & 27 \\ 30 & \infty & 17 & 0 & 10 \\ \infty & 15 & 12 & 8 & 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 0 & 10 & 15 & 25 & 17 \\ 7 & 0 & 5 & 20 & 24 \\ 14 & 13 & 0 & 15 & 25 \\ 30 & 25 & 17 & 0 & 10 \\ 22 & 15 & 12 & 8 & 0 \end{pmatrix}$$

2.3.6. *In an integer weighting of the edges of K_n, the total weight is even on every cycle if and only if the subgraph consisting of the edges with odd weight is a spanning complete bipartite subgraph.*

Sufficiency. Every cycle contains an even number of edges from a spanning complete bipartite subgraph.

Necessity. Suppose that the total weight on every cycle is even. We claim that every component of the spanning subgraph consisting of edges with even weight is a complete graph. Otherwise, it has two vertices x, y at distance 2, which induce P_3 with their common neighbor z. Since xy has odd weight, x, y, z would form a cycle with odd total weight.

If the spanning subgraph of edges with even weight has at least three components, then selecting one vertex from each of three components yields a triangle with odd weight. Hence there are at most two components. This implies that the complement (the graph of edges with odd weight) is a spanning complete bipartite subgraph of G.

2.3.7. *A weighted graph with distinct edge weights has a unique minimum-weight spanning tree (MST).*

Proof 1 (properties of spanning trees). If G has two minimum-weight spanning trees, then let e be the lightest edge of the symmetric difference. Since the edge weights are distinct, this weight appears in only one of the two trees. Let T be this tree, and let T' be the other. Since $e \in E(T)-E(T')$, there exists $e' \in E(T')-E(T)$ such that $T'+e-e'$ is a spanning tree. By the choice of e, $w(e') > w(e)$. Now $w(T'+e-e') < w(T')$, contradicting the assumption that T' is an MST. Hence there cannot be two MSTs.

Proof 2 (induction on $k = e(G)-n(G)+1$). If $k=0$, then G is a tree and has only one spanning tree. If $k>0$, then G is not a tree; let e be the heaviest edge of G that appears in a cycle, and let C be the cycle containing e. We claim that e appears in no MST of G. If T is a spanning tree containing e, then T omits some edge e' of C, and $T-e+e'$ is a cheaper spanning tree than T. Since e appears in no MST of G, every MST of G is an MST of $G-e$. By the induction hypothesis, there is only one such tree.

Proof 3 (Kruskal's Algorithm). In Kruskal's Algorithm, there is no choice if there are no ties between edge weights. Thus the algorithm can produce only one tree. We also need to show that Kruskal's Algorithm can produce every MST. The proof in the text can be modified to show this; if e is the first edge of the algorithm's tree that is not in an MST T', then we obtain an edge e' with the same weight as e such that $e' \in E(T') - E(T)$ and e' is available when e is chosen. The algorithm can choose e' instead. Continuing to modify the choices in this way turns T into T'.

2.3.8. *No matter how ties are broken in choosing the next edge for Kruskal's Algorithm, the list of weights of a minimum spanning tree (in nondecreasing order) is unique.* We consider edges in nondecreasing order of cost. We prove that after considering all edges of a particular cost, the vertex sets of the components of the forest built so far is the same independent of the order of consideration of the edges of that cost. We prove this by induction on the number of different cost values that have been considered. At the start, none have been considered and the forest consists of isolated vertices.

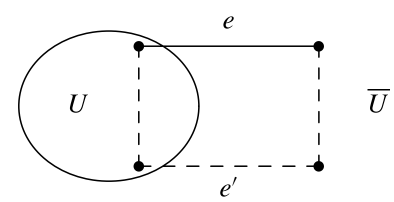
Before considering the edges of cost x, the induction hypothesis tells us that the vertex sets of the components of the forest are fixed. Let H be a graph with a vertex for each such component, and put two vertices adjacent in H if G has an edge of cost x joining the corresponding two components. Suppose that H has k vertices and l components. Independent of the order in which the algorithm consider the edges of cost x, it must select some $k-l$ edges of cost x in G, and it cannot select more, since this would create a cycle among the chosen edges.

2.3.9. *Among the cheapest spanning trees containing a spanning forest F is one containing the cheapest edge joining components of F.* Let T be a cheapest spanning tree containing F. If $e \notin E(T)$, then $T+e$ contains exactly one cycle, since T has exactly one u,v-path. Since u,v belong to distinct components of F, the u,v-path in T contains another edge e' between distinct components of F. If e' costs more than e, then $T' = T - e' + e$ is a cheaper spanning tree containing F, which contradicts the choice of T. Hence e' costs the same as e, and T' contains e and is a cheapest spanning tree containing F. Applying this statement at every step of Kruskal's algorithm proves that Kruskal's algorithm finds a minimum weight spanning tree.

2.3.10. *Prim's algorithm produces a minimum-weight spanning tree.* Let v_1 be the initial vertex, let T be the tree produced, and let T^* be an optimal tree that agrees with T for the most steps. Let e be the first edge chosen for T that does not appear in T^*, and let U be the set of vertices in the subtree of T that has been grown before e is added. Adding e to T^* creates a cycle C; since e links U to $\overline{U}$, C must contain another edge e' from U to $\overline{U}$.

Since $T^* + e - e'$ is another spanning tree, the optimality of T^* yields

$w(e') \le w(e)$. Since e' is incident to U, e' is available for consideration when e is chosen by the algorithm; since the algorithm chose e, we have $w(e) \le w(e')$. Hence $w(e) = w(e')$, and $T^* + e - e'$ is a spanning tree with the same weight as T^*. It is thus an optimal spanning tree that agrees with T longer than T^*, which contradicts the choice of T^*.



2.3.11. *Every minimum-weight spanning tree achieves the minimum of the maximum weight edge over all spanning trees.* Let T be a minimum-weight spanning tree, and let T^* be one that minimizes the maximum weight edge. If T does not, then T has an edge e whose weight is greater than the weight of every edge in T^*. If we delete e from T, Then we can find an edge $e^* \in E(T^*)$ that joins the components of $T-e$, since T^* is connected. Since $w(e) > w(e^*)$, the weight of $T - e + e'$ is less than the weight of T, which contradicts the minimality of T. Thus T has the desired property.

2.3.12. *The greedy algorithm cannot guarantee minimum weight spanning paths.* This fails even on four vertices with only three distinct vertex weights. If two incident edges have the minimum weight a, such as $a=1$, the algorithm begins by choosing them. If the two edges completing a 4-cycle with them have maximum weight c, such as $c=10$, then one of those must be chosen to complete a path of weight $2a+c$. However, if the other two edges have intermediate weight b, such as $b=2$, there is a path of weight $2b+a$, which will be cheaper whenever $b < (a+c)/2$. For $n > 4$, the construction generalizes in many possible ways using three weights $a < b < c$. A path of length $n-2$ having weight a for each edge and weight c for the two edges completing the cycle yields a path of weight $(n-2)a+c$ by the greedy algorithm, but if all other weights equal b there is a path of weight $2b+(n-3)a$, which is cheaper whenever $b < (a+c)/2$.

2.3.13. *If T and T' are spanning trees in a weighted graph G, with T having minimum weight, then T' can be changed into T by steps that exchange one edge of T' for one edge of T so that the edge set is always a spanning tree and the total weight never increases.* It suffices to find one such step whenever T' is different from T; the sequence then exists by using induction on the number of edges in which the two trees differ.

Choose any $e' \in E(T') - E(T)$. Deleting e' from T' creates two components with vertex sets U, U'. The path in T between the endpoints of e' must have an edge e from U to U'; thus $T' - e' + e$ is a spanning tree. We want to show that $w(T' - e' + e) \le w(T')$.

Since e is an edge of the path in T between the endpoints of e', the edge e belongs to the unique cycle in T created by adding e' to T. Thus $T + e' - e$ is also a spanning tree. Because $T - e + e'$ is a spanning tree and T has minimum weight, $w(e) \le w(e')$. Thus $T' - e' + e$ moves from T' toward T without increasing the weight.

2.3.14. *When e is a heaviest edge on a cycle G in a connected weighted graph G, there is a minimum spanning tree not containing e.* Let T be a minimum spanning tree in G. If $e \in E(T)$, then $T - e$ has two components with vertex sets U and U'. The subgraph $C - e$ is a path with endpoints in U and U'; hence it contains an edge e' joining U and U'. Since $w(e') \le w(e)$ by hypothesis, $T - e + e'$ is a tree as cheap as T that avoids e.

Given a weighted graph, iteratively deleting a heaviest non-cut-edge produces a minimum spanning tree. A non-cut-edge is an edge on a cycle. A heaviest such edge is a heaviest edge on that cycle. We have shown that some minimum spanning tree avoids it, so deleting it does not change the minimum weight of a spanning tree. This remains true as we delete edges. When no cycles remain, we have a connected acyclic subgraph. It is the only remaining spanning tree and has the minimum weight among spanning trees of the original graph.

2.3.15. *If T is a minimum spanning tree of a connected weighted graph G, then T omits some heaviest edge from every cycle of G.*

Proof 1 (edge exchange). Suppose e is a heaviest edge on cycle C. If $e \in E(T)$, then $T - e$ is disconnected, but $C - e$ must contain an edge e' joining the two components of $T - e$. Since T has minimum weight, $T - e + e'$ has weight as large as T, so $w(e') \ge w(e)$. Since e has maximum weight on C, equality holds, and T does not contain all the heaviest edges from C.

Proof 2 (Kruskal's algorithm). List the edges in order of increasing weight, breaking ties by putting the edges of a given weight that belong to T before those that don't belong to T. The greedy algorithm (Kruskal's algorithm) applied to this ordering L yields a minimum spanning tree, and it is precisely T. Now let C be an arbitrary cycle in G, and let $e_1, \dots, e_k$ be the edges of C in order of appearance in L; $e_k = uv$ is a heaviest edge of C. It suffices to show that e_k does not appear in T. For each earlier edge e_i of C, either e_i appears in T or e_i is rejected by the greedy algorithm because it completes a cycle. In either case, T contains a path between the endpoints of e_i. Hence when the algorithm considers e_k, it has already selected edges that form paths joining the endpoints of each other edge of C. Together, these paths form a u, v-walk, which contains a u, v-path. Hence adding e_k would complete a cycle, and the algorithm rejects e_k.

2.3.16. *Four people crossing a bridge.* Name the people 10, 5, 2, 1, respectively, according to the number of minutes they take to cross when walking alone. To get across before the flood, they can first send $\{1, 2\}$ in time 2. Next 1 returns with the flashlight in time 1, and now $\{5, 10\}$ cross in time 10. Finally, 2 carries the flashlight back, and $\{1, 2\}$ cross together again. The time used is $2 + 1 + 10 + 2 + 2 = 17$. The key is to send 5 and 10 together to avoid a charge of 5.

To solve the problem with graph theory, make a vertex for each possible state. A state consists of a partition of the people into the two banks, along with the location of the flashlight. There is an edge from state A to state B if state A is obtained from state B by moving one or two people (and the flashlight) from the side of A that has the flashlight to the other side. The problem is to find a shortest path from the initial state $(10, 5, 2, 1, F|\varnothing)$ to the final state $(\varnothing|10, 5, 2, 1, F)$. Dijkstra's algorithm finds such a path.

There are many vertices and edges in the graph of states. The path corresponding to the solution in the first paragraph passes through the vertices $(10, 5|2, 1, F)$, $(10, 5, 1, F|2)$, $(1|10, 5, 2, F)$, $(2, 1, F|10, 5)$, $(10, 5, 2, 1)$.

2.3.17. *The BFS algorithm computes $d(u, z)$ for every $z \in V(G)$.* The algorithm declares vertices to have distance k when searching vertices declared to have distance $k - 1$. Since vertices are searched in the order in which they are found, all vertices declared to have distance less than $k - 1$ are searched before any vertices declared to have distance $k - 1$.

We use induction on $d(u, z)$. When $d(u, z) = 0$, we have $u = z$, and initial declaration is correct. When $d(u, z) > 0$, let W be the set of all neighbors of z along shortest z, u-paths. Since $d(u, W) = d(u, z) - 1$, the induction hypothesis implies that the algorithm computes $d(u, v)$ correctly for all $v \in W$. Also, the preceding paragraph ensures that z will not be found before any vertices of W are searched. Hence when a vertex of W is searched, z will be found and assigned the correct distance.

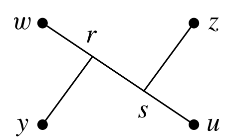
2.3.18. *Use of Breadth-First Search to compute the girth of a graph.* When running BFS, reaching a vertex that is already in the list of vertices already reached creates a second path from the root to that vertex. Following one path and back the other is a closed path in which the edges reaching the new vertex occur only once, so they lie on a cycle.

When the root is a vertex of a shortest cycle, the sum of the two lengths to the reached vertex is the length of that cycle. The sum can never be smaller. Thus we run BFS from each vertex as root until we find a vertex repeatedly, record the sum of the lengths of the two paths, and take the smallest value of this over all choices of the root.

2.3.19. *Computing diameter of trees.* From a arbitrary vertex w, we find a maximally distant vertex u (via BFS), and then we find a vertex v maximally distant from u (via BFS). We show that $d(y,z) \le d(u,v)$ for all $y,z \in V(T)$. Because v is at maximum distance from u, this holds if $u \in \{y,z\}$, so we may assume that $u \notin \{y,z\}$.

We use that each vertex pair in a tree is connected by a unique path. Let r be the vertex at which the w,y-path separates from the w,u-path. Let s be the vertex at which the w,z-path separates from the w,u-path. By symmetry, we may assume that r is between w and s. Since $d(w,u) \ge d(w,z)$, we have $d(s,u) \ge d(s,z)$. Now

$$d(y,z) = d(y,r)+d(r,s)+d(s,z) \le d(y,r)+d(r,s)+d(s,u) = d(y,u) \le d(v,u).$$



2.3.20. *Minimum diameter spanning tree.* An MDST is a spanning tree in which the maximum length of a path is as small as possible. Intuition suggests that running Dijkstra's algorithm from a vertex of minimum eccentricity (a center) will produce an MDST, but this may fail.

a) Construct a 5-vertex example of an unweighted graph (edge weights all equal 1) in which Dijkstra's algorithm can be run from some vertex of minimum eccentricity and produce a spanning tree that does not have minimum diameter. Answer: the chin of the bull.

(Note: when there are multiple candidates with the same distance from the root, or multiple ways to reach the new vertex with minimum distance, the choice in Dijkstra's algorithm can be made arbitrarily.)

b) Construct a 4-vertex example of a weighted graph such that Dijkstra's algorithm cannot produce an MDST when run from any vertex.

2.3.21. *Algorithm to test for bipartiteness.* In each component, run the BFS search algorithm from a given vertex x, recording for each newly found vertex a distance one more than the distance for the vertex from which it is found. By the properties of distance, searching from a vertex v to find a vertex w may discover $d(x,w) = d(x,v) - 1$ or $d(x,w) = d(x,v)$ or $d(x,w) = d(x,v) + 1$ (if w is not yet in the set found).

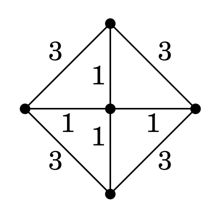
If the second case ever arises, then we have adjacent vertices at the same distance from x, and there is an odd cycle in the graph. Otherwise, at the end we form a bipartition that partions the vertices according to the parity of their distance from x.

2.3.22. *The Chinese Postman Problem in the k-dimensional cube Q_k, with every edge having weight 1.* If k is even, then no duplicate edges are needed, since Q_k is k-regular; total cost is $k2^{k-1}$. If k is odd, then a duplicated edge is needed at every vertex. It suffices to duplicate the matching across the last coordinate. Thus the total cost in this case is $(k+1)2^{k-1}$.

2.3.23. *The Lazy Postman.* The postman's trail must cover every edge and contribute even degree to each vertex except the start P and end H. In the example given, C,D,G,H have the wrong parity. Hence the duplicated edges must consist of two paths that pair these vertices (with least total distance), since this will change the degree parity only for the ends of the paths. If we pair them as DG and CH, then the shortest paths are DEIFG and CBEIH, totaling 18 extra (obviously not optimal since both use EI). If CG and DH, then the paths are (CBEIFG or CPAFG) and DEIH, totaling 18 in either case. If CD and GH, then the paths are CBED and GFIH, totaling 15. Hence the edges in the paths CBED and GFIH are traveled twice; all others are traveled once.

2.3.24. *Chinese Postman Problem.* Solving the Chinese Postman problem on a weighted graph with $2k$ vertices of odd degree requires duplicating the edges in a set of k trails that pair up the vertices of odd degree as endpoints. The only vertices of a trail that have odd degree in the trail are its endpoints. If some u,v-trail T in the optimal solution is not a path, then it contains a u,v-path P. In P, every vertex degree is even, except for the endpoints. Hence using P instead of T to join u and v does not change the parity on any vertex and yields smaller total weight.

Since no edge need be used thrice, the duplicated trails in an optimal solution are pairwise edge-disjoint. As in the example below, they need not be vertex-disjoint. With four vertices of odd degree, two paths are required, and the cheapest way is to send both through the central vertex.



2.3.25. *If G is an n-vertex rooted plane tree in which every vertex has 0 or k children, then $n = tk + 1$ for some integer t.*

Proof 1 (Induction). We use induction on the number of non-leaf vertices. When there are no such vertices, the root is the only vertex, and the formula works with $t = 0$. When the tree T is bigger, find a leaf at maximum distance from the root, and let x be its parent. By the choice of x, all children of x are leaves. Deleting the children of x yields a tree T' with one less non-leaf vertex and k fewer total vertices. By the induction hypothesis, $n(T') = tk + 1$ for some t, and thus $n(T) = (t+1)k + 1$.

Proof 2 (Degree counting). If T has n vertices, then it has $n-1$ edges, and the degree-sum is $2n-2$. If $n > 1$, then the root has degree l, the other $t-1$ non-leaf vertices each have degree $k+1$, and the $n-t$ leaves each have degree 1. Thus $2n-2 = k + (t-1)(k+1) + (n-t)$. This simplifies to $n = tk + 1$.

2.3.26. *A recurrence relation to count the binary trees with $n+1$ leaves.* Let a_n be the desired number of trees. When $n = 0$, the root is the only leaf. When $n > 0$, each tree has some number of leaves, k, in the subtree rooted at the left child of the root, where $1 \le k \le n$. We can root any binary tree with k leaves at the left child and any binary tree with $n-k+1$ leaves at the right child. Summing over k counts all the trees. Thus $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$ for $n > 0$, with $a_0 = 1$. (*Comment:* These are the *Catalan numbers*.)

2.3.27. *A recurrence relation for the number of rooted plane trees with n vertices.* Let a_n be the desired number of trees. When $n = 1$, there is one tree. When $n > 1$, the root has a child. The subtree rooted at the leftmost child has some number of vertices, k, where $1 \le k \le n-1$. The remainder of the tree is a rooted subtree with the same root as the original tree; it has $n-k$ vertices. We can combine any tree of the first type with any tree of the second type. Summing over k counts all the trees. Thus $a_n = \sum_{1=k}^{n-1} a_k a_{n-k}$ for $n > 1$, with $a_1 = 1$. (*Comment:* This is the same sequence as in the previous problem, with index shifted by 1.)

2.3.28. *A code with minimum expected length for messages with relative frequencies 1,2,3,4,5,5,6,7,8,9.* Iteratively combining least-frequent items and reading paths from the resulting tree yields the codes below. Some variation in the codes is possible, but not in their lengths. The average length (weighted by frequency!) is 3.48.

frequency	1	2	3	4	5	5	6	7	8	9
code	00000	00001	0001	100	101	110	111	001	010	011
length	5	5	4	3	3	3	3	3	3	3

2.3.29. *Computation of an optimal code.* Successive combination of the cheapest pairs leads to a tree. For each letter, we list the frequency and the depth of the corresponding leaf, which is the length of the associated codeword. The assignment of codewords is not unique, but the set (with multiplicities) of depths for each frequency is. Given frequencies f_i, with associated lengths l_i and total frequency T, the expected length per character is $\sum f_i l_i / T$. For the given frequencies, this produces expected length

of $(7\cdot 4 + 6\cdot 19 + 5\cdot 21 + 4\cdot 26 + 3\cdot 30)/100 = 4.41$ bits per character, which is less than the 5 bits of ASCII.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	$\varnothing$
9	2	2	4	12	2	3	2	9	1	1	4	2	6	8	2	1	6	4	6	4	2	2	1	2	1	2
3	6	6	5	3	6	5	6	3	7	7	5	6	4	4	6	7	4	5	4	5	6	6	7	6	6	5

2.3.30. *Optimal code for powers of 1/2.*

a) the two smallest probabilities are equal. Let p_n, p_{n-1} be smallest and second smallest probabilities in the distribution. Each probability other than p_n is a multiple of p_{n-1}. If $p_n < p_{n-1}$, then the sum of all the probabilities is not a multiple of p_{n-1}. This contradicts $\sum_{i=1}^{n} p_i = 1$, since 1 is a multiple of p_{n-1}.

b) The expected message length of the optimal (Huffman) code for such a distribu tion is $-\sum p_i \lg p_i$. We use induction on n to prove that each item with probability $(1/2)^k$ is assign to a leaf at length k from the root; this yields the stated formula. For $n = 1$ and $p_1 = 1$, the one item has message length 0, as desired. For larger n, the Huffman tree is obtained by finding the optimal tree for the smaller set $q_1, \ldots, q_{n-1}$ (where $q_{n-1} = p_n + p_{n-1}$ and $q_i = p_i$ for $1 \le i \le n$) and extending the tree at the leaf for q_{n-1} to leaves one deeper for p_{n-1} and p_n. By part (a), $q_{n-1} = 2p_{n-1} = 2p_n$. By the induction hypothesis, the depth of the leaf for q_{n-1} is $-\lg q_{n-1}$, and for $p_1, \ldots, p_{n-2}$ it is as desired. The new leaves for p_{n-1}, p_n have depth $+1 - \lg q_{n-1} = -\lg p_{n-1} = -\lg p_n$, as desired.

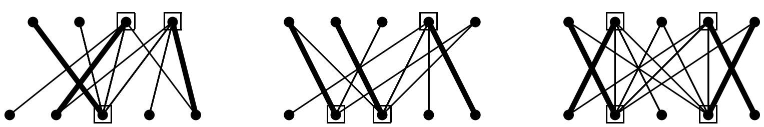
2.3.31. *For every probability distribution $\{p_i\}$ on n messages and every binary code for these messages, the expected length of a code word is at least $-\sum p_i \lg p_i$.* Proof by induction on n. For $n = 1 = p_1$, the entropy and the expected length for the optimal code both equal 0; there is no need to use any digits. For $n > 1$, let W be the words in an optimal code, with W_0, W_1 denoting the sets of code words starting with 0,1, respectively. If all words start with the same bit, then the code is not optimal, because the code obtained by deleting the first bit of each word has smaller expected length. Hence W_0, W_1 are codes for smaller sets of messages. Let q_0, q_1 be the sum of the probabilities for the messages in W_0, W_1. Normalizing the p_i's by q_0 or q_1 gives the probability distributions for the smaller codes. Because the words within W_0 or W_1 all start with the same bit, their expected length is at least 1 more than the optimal expected length for those distributions.

Applying the induction hypothesis to both W_0 and W_1, we find that the expected length for W is at least $q_0[1 - \sum_{i\in W_0} \frac{p_i}{q_0} \lg \frac{p_i}{q_0}] + q_1[1 - \sum_{i\in W_1} \frac{p_i}{q_1} \lg \frac{p_i}{q_1}]$ $= 1 - \sum_{i\in W_0} p_i(\lg p_i - \lg q_0) - \sum_{i\in W_1} p_i(\lg p_i - \lg q_1) = 1 + q_0 \lg q_0 + q_1 \lg q_1 - \sum p_i \lg p_i$. It suffices to prove that $1 + q_0 \lg q_0 + q_1 \lg q_1 \ge 0$ when $q_0 + q_1 = 1$. Because $f(x) = x \lg x$ is convex for $0 < x < 1$ (since $f''(x) = 1/x > 0$), we have $1 + f(x) + f(1-x) \ge 1 + 2f(.5) = 0$.

3.MATCHINGS AND FACTORS

3.1. MATCHINGS AND COVERS

3.1.1. *Examples of maximum matchings.* In each graph below, we show a matching in bold and mark a vertex cover of the same size. We know that the matching has maximum size and the vertex cover has minimum size because the size of every matching in a graph is at most the size of every vertex cover in the graph.



3.1.2. *The minimum size of a maximal matching in C_n is $\lceil n/3 \rceil$.* Suppose that C_n has a matching of size k. The $n-2k$ unmatched vertices fall into k "buckets" between matched edges around the cycle. The matching is maximal if and only if none of the buckets contain two vertices. By the pigeonhole principle, there must be some such bucket if $n-2k \ge k+1$; hence a matching of size k cannot be maximal if $k \le (n-1)/3$.

On the other hand, if $n-2k \le k$ and $k \le n/2$, then we can match two vertices and skip one until we have skipped $n-2k$, which avoids having two vertices in one bucket. Hence C_n has a maximal matching of size $k \le n/2$ if and only if $3k \ge n$. We conclude that the minimum size of a maximal matching in C_n is $\lceil n/3 \rceil$.

3.1.3. *If $S \subseteq V(G)$ is saturated by some matching in G, then S is saturated by some maximum matching.* If M saturates S, then the characterization of maximum matchings implies that a maximum matching M^* can be obtained from M by a sequence of alternating path augmentations. Although edges may be lost in an augmentation, each augmentation continues to saturated all saturated vertices and enlarges the saturated set by 2. Thus S remains saturated in M^*

When $S \subseteq V(G)$ is saturated by some matching in G, it need not be true that S is saturated by every maximum matching. When G is an odd cycle, all maximum matchings omit distinct vertices.

3.1.4. Let G be a simple graph.

$\alpha(G) = 1$ *if and only if G is a complete graph.* The independence number is 1 if and only if no two vertices are nonadjacent.

$\alpha'(G) = 1$ *if and only if G consiste of isolated vertices plus a triangle or nontrivial star.* Deleting the vertices of one edge in such a graph leaves no edges remaining. Conversely, suppose that $\alpha'(G) = 1$, and ignore the isolated vertices. Let v be a vertex of maximum degree. If every edge is incident to v, then G is a star. Otherwise, an edge e not incident to v shares an endpoint with every edge incident to v. Since e has only two endpoints, $d(v) = 2$, and there is only one such edge e. Thus G is a triangle.

$\beta(G) = 1$ *if and only if G is a nontrivial star plus isolated vertices.* The vertex cover number is 1 if and only if one vertex is incident to all edges.

$\beta'(G) = 1$ *if and only if $G = K_2$.* Since every edge covers two vertices, $\beta'(G) = 1$ requires that $n(G) = 2$, and indeed $\beta'(K_2) = 1$.

3.1.5. $\alpha(G) \ge \frac{n(G)}{\Delta(G)+1}$ *for every graph G.* Form an independent set S by iteratively selecting a remaining vertex for S and deleting that vertex and all its neighbors. Each step adds one vertex to S and deletes at most $\Delta(G)+1$ vertices from G. Hence we perform at least $n(G)/(\Delta(G)+1)$ steps and obtain an independent set at least that big.

3.1.6. *If T is a tree with n vertices and independence number k, then $\alpha'(T) = n-k$.* The vertices outside a maximum independent set form a vertex cover of size $n-k$. Since trees are bipartite, the König–Egerváry Theorem then applies to yield $\alpha'(T) = \beta(T) = n - \alpha(T) = n-k$.

3.1.7. *A graph G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H having no isolated vertices.* If G is bipartite, then every subgraph H of G is bipartite, and by König's Theorem the number of edges of H needed to cover $V(H)$ equals $\alpha(H)$ if H has no isolated vertices. If G is not bipartite, then G contains an odd cycle H, and this subgraph H has no isolated vertices and requires $\alpha(H)+1$ edges to cover its vertices.

3.1.8. *Every tree T has at most one perfect matching.*

Proof 1 (contradiction). Let M and M' be perfect matchings in a tree. Form the symmetric difference of the edge sets, $M \triangle M'$. Since the matchings are perfect, each vertex has degree 0 or 2 in the symmetric difference, so every component is an isolated vertex or a cycle. Since the tree has no cycle, every vertex must have degree 0 in the symmetric difference, which means that the two matchings are the same.

Proof 2 (induction). For the basis step, a tree with one vertex has no perfect matching; a tree with two vertices has one. For the induction step, consider an arbitrary tree T on $n > 2$ vertices, and consider a leaf v. In any perfect matching, v must be matched to its neighbor u. The remainder of any matching is a matching in $T - \{u, v\}$. Since each perfect matching in T must contain the edge uv, the number of perfect matchings in T equals the number of perfect matchings in $T - \{u, v\}$.

Each component of $T - \{u, v\}$ is a tree; by the induction hypothesis, each component has at most one perfect matching. The number of perfect matchings in a graph is the product of the number of perfect matchings in each component, so the original T has at most one perfect matching. (More generally, a forest has at most one perfect matching.)

3.1.9. *Every maximal matching in a graph G has size at least $\alpha'(G)/2$.*

Proof 1 (counting and contrapositive). Let M^* be a maximum matching, and let M be another matching. We show that if $|M| < \alpha'(G)/2$, then M is not a maximal matching. Since M saturates $2|M|$ vertices and $|M| < \alpha'(G)/2$, we conclude that M saturates fewer than $\alpha'(G)$ vertices. This means that M cannot saturate a vertex of every edge of M^*, and there is some edge of M^* that can be added to enlarge M.

Proof 2 (augmenting paths). Let M be a maximal matching, and let M^* be a maximum matching. Consider the symmetric difference $F = M \triangle M^*$. Since the number of edges from M and M^* in a component of F differ by at most one, the symmetric difference contains at least $|M^*| - |M|$ augmenting paths. Since M is maximal, each augmenting path must contain an edge of M (an M-augmenting path of length one is an edge that can be added to M). Thus $|M^*| - |M| \le |M|$, and we obtain $|M| \ge |M^*|/2 = \alpha'(G)/2$.

Proof 3 (vertex covers). When M is a maximal matching, then the vertices saturated by M form a vertex cover (if an edge had no vertex in this set, then it could be added to M). Since every vertex cover has size at least $\alpha'(G)$, we obtain $2|M| \ge \beta(G) \ge \alpha'(G)$.

3.1.10. *If M and N are matchings in a graph G and $|M| > |N|$, then there are matchings M' and N' in G such that $|M'| = |M| - 1$, $|N'| = |N| + 1$, $M' \cap N' = M \cap N$, and $M' \cup N' = M \cup N$.* Consider the subgraph F of G consisting of the edges in the symmetric difference $M \triangle N$; this consists of all edges belonging to exactly one of M and N. Since each of M and N is a matching, every vertex has at most one incident edge in M and at most one incident edge in N. Hence the degree of every vertex in F is at most 2.

The components of a graph with maximum degree 2 are paths and cycles. A path or cycle in F alternates edges between M and N. Since $|M| > |N|$, F has a component with more edges of M than of N. Such a component can only be a path P that starts and ends with an edge of M. Form M' from M by replacing $M \cap E(P)$ with $N \cap E(P)$; this reduces the size by one. Form N' from N by replacing $N \cap E(P)$ with $M \cap E(P)$; this increases the size by one. Since we have only switched edges belonging to exactly one of the sets, we have not changed the union or intersection.

3.1.11. *If C and C' are cycles in a graph G, then $C \triangle C'$ decomposes into cycles.* Since even graphs decompose into cycles (Proposition 1.2.27), it suffices to show that $C \triangle C'$ has even degree at each vertex. The set of edges in a cycle that are incident to v has even size (2 or 0). The symmetric difference of any two sets of even size has even size, since always $|A \triangle B| = |A| + |B| - 2|A \cap B|$.

3.1.12. *If C and C' are cycles of length k in a graph with girth k, then $C \triangle C'$ is a single cycle if and only if $C \cap C'$ is a single nontrivial path.*

Sufficiency. If $C \cap C'$ is a single path P, then the other paths in C and C' between the endpoints of P share only their endpoints, and hence $C \triangle C'$ is their union and is a single cycle.

Necessity. We know that $C \cap C'$ must have an edge, since otherwise C and C' are edge-disjoint and $C \triangle C'$ has two cycles. Also we may assume that C and C' are distinct, since otherwise $C \triangle C'$ has no edges.

Suppose that P and P' are distinct maximal paths in $C \cap C'$. Now C is the union of four paths P, Q, P', Q', and C' is the union of four paths P, R, P', R'. Note that Q and Q' may share edges with R and R'. By symmetry, we may assume that P' is no longer than P and that Q and R share the same endpoint of P.

If Q and R also share the same endpoint of P', then $Q \cup R$ and $Q' \cup R'$ both form closed walks in which (by the maximality of P and P') some edge appears only once. If Q and R do not share the same endpoint of P', then $P' \cup Q \cup R$ and $P' \cup Q' \cup R'$ both form closed walks in which the edges of P' appear only once. In each case, the two closed walks each each contain a cycle, and the sum of their lengths is less than $2k$. This yields a cycle of length less than k in G, which is impossible.

Comment: The statement can fail for longer cycles. In the 3-dimensional cube Q_3, there are two 6-cycles through antipodal vertices, and their symmetric difference consists of two disjoint 4-cycles.

3.1.13. *In an X,Y-bigraph G, if $S \subseteq X$ is saturated by a matching M and $T \subseteq Y$ is saturated by a matching M', then there is a matching that saturates both S and T.* Let F be a subgraph of G with edge set $M \cup M'$. Since each vertex has at most one incident edge from each matching, F has maximum degree 2. Each component of F is an alternating path or an alternating cycle (alternating between M and M'). From a component that is an alternating cycle or an alternating path of odd length, we can choose the edges of M or of M' to saturate all the vertices of the component.

Let P be a component of F that is a path of even length. The edge at one end of P is in M; the edge at the other end is in M'. Also P starts and ends in the same partite set. If it starts and ends in X, then the ends cannot both be in S, because only one endpoint of P is saturated by M.

Choosing the edges of M in this component will thus saturate all vertices of S and T contained in $V(P)$. Similarly, choosing the edges of M' from any component of F that is a path of even length with endpoints in Y will saturate all the vertices of $S \cup T$ in that component.

3.1.14. *Matchings in the Petersen graph.*

Deleting any perfect matching leaves C_5+C_5. Deleting a perfect matching leaves a 2-regular spanning subgraph, which is a disjoint union of cycles. Since the Petersen graph has girth 5, the only possible coverings of the vertices by disjoint cycles are $C_5 + C_5$ and C_{10}.

If a 10-cycle exists, with vertices $[v_1, \dots, v_{10}]$ in order, then the remaining matching consists of chords. Two consecutive vertices cannot neighbor their opposite vertices on the cycle, since that creates a 4-cycle. Similarly, the neighbors must be at least four steps away on the cycle. Hence we may assume by symmetry that $v_1 \leftrightarrow v_5$. Now making v_{10} adjacent to any of $\{v_6, v_5, v_4\}$ creates a cycle of length at most 4, so there is no way to insert the remaining edges.

a) The Petersen graph has twelve 5-cycles. Each edge extends to P_4 in four ways by picking an incident edge at each endpoint. Since the graph has diameter two and girth 5, every P_4 belongs to exactly one 5-cycle through an additional vertex. Since there are 15 edges, we have generated 60 5-cycles, but each 5-cycle is generated five times.

b) The Petersen graph has six perfect matchings. Since the Petersen graph has girth five, the five remaining edges incident to any 5-cycle form a perfect matching, and deleting them leaves a 5-cycle on the complementary vertices. Hence the 5-cycles group into pairs of 5-cycles with a matching between them. Since every matching leaves C_5+C_5, every matching arises in this way, and by part (b) there are six of them.

3.1.15. *Matchings in k-dimensional cubes.*

a) For $k \geq 2$, if M is a perfect matching of Q_k, then there are an even number of edges in M whose endpoints differ in coordinate i. Let V_0 and V_1 be the sets of vertices having 0 and 1 in coordinate i, respectively. Each has even size. Since the vertices of V_r not matched to V_{1-r} must be matched within V_r, the number of vertices matched by edges to V_{1-r} must be even.

b) Q_3 has nine perfect matchings. There are four edges in each such matching, with an even number distributed to each coordinate. The possible distributions are $(4,0,0)$ and $(2,2,0)$. There are three matchings of the first type. For the second type, we pick a direction to avoid crossing, pick one of the two matchings in one of the 4-cycles, and then the choice of the matching in the other 4-cycle is forced to avoid making all four edges change the same coordinate. Hence there are $3 \cdot 2 \cdot 1$ perfect matchings of the second type.

3.1.16. *When $k \geq 2$, the k-dimensional hypercube Q_k has at least $2^{(2^{k-2})}$ perfect matchings.*

Proof 1 (induction on k). Let m_k denote the number of perfect matchings. Note that $m_2 = 2$, which satisfies the inequality. When $k > 2$, we can choose matchings independently in each of two disjoint subcubes of dimension $k-1$. The number of such matchings is m_{k-1}^2. By the induction hypothesis, this is at least $(2^{2^{k-3}})^2$, which equals $2^{2^{k-2}}$.

Comment: Since we could choose the two disjoint subcubes in k ways, we can recursively form km_{k-1}^2 perfect matchings in this way, some of which are counted more than once.

Proof 2 (direct construction). Pick two coordinates. There are 2^{k-2} copies of Q_2 in which those two coordinates vary, and two choices of a perfect matching in each copy of Q_2. This yields $2^{2^{k-2}}$ perfect matchings. (Since we can choose the two coordinates in $\binom{k}{2}$ ways, we can generate $\binom{k}{2}2^{2^{k-2}}$ perfect matchings, but there is some repetition.)

3.1.17. *In every perfect matching in the hypercube Q_k, there are exactly $\binom{k-1}{i}$ edges that match vertices with weight i to vertices with weight $i+1$,* where the *weight* of a vertex is the number of 1s in its binary k-tuple name.

Proof 1 (induction on i). Since the vertex of weight 0 must match to a vertex of weight 1, the claim holds when $i=0$. For the induction step, the induction hypothesis yields $\binom{k-1}{i-1}$ vertices of weight $i-1$ matched to vertices of weight i. The remaining vertices of weight i must match to vertices of weight $i+1$. Since $\binom{k}{i} - \binom{k-1}{i-1} = \binom{k-1}{i}$, the claim follows.

Proof 2 (canonical forms). Let M^* be the matching where every edge matches vertices with 0 and 1 in the last coordinate. The number of edges matching weight i to weight $i+1$ is the number of choices of i ones from the first $k-1$ positions, which is $\binom{k-1}{i}$.

It now suffices to prove that every perfect matching M has the same weight distribution as M^*. The symmetric difference of M and M^* is a union of even cycles alternating between M and M^*, plus isolated vertices saturated by the same edge in both matchings. It suffices to show that the weight distribution on each cycle is the same for both matchings.

The edges joining vertices of weights i and $i+1$ along a cycle C alternate appearing with increasing weight and with decreasing weight, since weight changes by 1 along each edge. For the same reason, the number of edges along C from a vertex to the next appearance of a vertex with the same weight is even. Since C alternates between M and M^*, this means that the edges joining vertices of weights i and $i+$ alternate between M and M^*. Hence there is the same number of each type, as desired.

3.1.18. *The game of choosing adjacent vertices, where the last move wins.* Suppose that G has a perfect matching M. Whenever the first player chooses a vertex, the second player takes its mate in M. This vertex is available, because after each move of the second player the set of vertices visited forms a set of full edges in M, and the first player cannot take two vertices at a time. Thus with this strategy, the second player can always make a move after any move of the first player and never loses.

If G has no perfect matching, then let M be a maximum matching in G. The first player starts by choosing a vertex not covered by M. Thereafter, whenever the second player chooses a vertex x, the first player chooses the mate of x in M. The vertex x must be covered by M, else x completes an M-augmenting path using all the vertices chosen thus far. Thus the first player always has a move available and does not lose.

3.1.19. *A family $A_1, \ldots, A_m$ of subsets of Y has a system of distinct representatives if and only if $\left|\bigcup_{j\in S} A_i\right| \ge |S|$ for every $S \subseteq [m]$.* Form an X, Y-bigraph G with $X = \{1, \ldots, m\}$ and $Y = \{y_1, \ldots, y_n\}$. Include the edge iy_j if and only if $y_j \in A_i$. A set of edges in G is a matching if and only if its endpoints in Y form a system of distinct representatives for the sets indexed by its endpoints in X. The family has a system of distinct representatives if and only if G has a matching that saturates X.

It thus suffices to show that the given condition is equivalent to Hall's condition for saturating X. If $S \subseteq X$, then $N_G(S) = \bigcup_{i\in S} A_i$, so $|N(S)| \ge |S|$ if and only if $\left|\bigcup_{i\in S} A_i\right| \ge |S|$.

3.1.20. *An extension of Hall's Theorem using stars with more than two vertices.* We form an X, Y-bigraph G with partite sets $X = x_1, \ldots, x_n$ for the trips and $Y = y_1, \ldots, y_m$ for the people, and edge set $\{x_i y_j\colon$ person j likes trip $i\}$. To fill each trip to its capacity c_i, we seek a subgraph whose components are stars, with degree c_i at x_i.

Form an X', Y-bigraph G' by making n_i copies of each vertex x_i. Now G has the desired stars if and only if G' has a matching that saturates X'. Thus the desired condition for G should become Hall's Condition for G'.

In G', the neighborhoods of the copies of a vertex of x are the same. Hence Hall's Condition will hold if and only if it holds whenever $S \subseteq X'$ consists of all copies of each vertex of X for which it includes any copies. That is, Hall's Condition reduces to requiring $|N(T)| \ge \sum_{x_i\in T} c_i$ for all $T \subseteq X$. This condition is necessary, since the trips in T need this many distinct people. It is sufficient, because it implies Hall's Condition for G'.

3.1.21. *If G is an X, Y-bigraph such that $|N(S)| > |S|$ whenever $\varnothing \ne S \subset X$, then every edge of G belongs to some matching that saturates X.* Let xy be an edge of G, with $x \in X$ and $y \in Y$, and let $G' = G - x - y$. Each set $S \subseteq X - \{x\}$ loses at most one neighbor when y is deleted. Combining this with the hypothesis yields $|N_{G'}(S)| \ge |N_G(S)| - 1 \ge |S|$. Thus G' satisfies Hall's Condition and has a matching that saturates $X - \{x\}$. With the edge xy, this completes a matching in G that contains xy and saturates X.

3.1.22. *A bipartite graph G has a perfect matching if and only if $|N(S)| \ge |S|$ for all $S \subseteq V(G)$.* This conclusion does not hold for non-bipartite graphs. In an odd cycle, we obtain neighbors for a set of vertices by taking the vertices immediately following them on the cycle. Thus $|N(S)| \ge |S|$ for all $S \subseteq V$, but the graph has no perfect matching. Complete graphs of odd order also form counterexamples. For bipartite graphs, we give two proofs.

Proof 1 (graph transformation). Let G' be a bipartite graph consisting of two disjoint copies of G, where each partite set in G' consists of one copy of X and one copy of Y; call these X' and Y'. Then G' has a perfect matching if and only if G has a perfect matching. Since $\left|X'\right| = \left|Y'\right|$, G' has a perfect matching if and only if it has a matching that completely saturates X'.

By Hall's Theorem, G' has a matching saturating X' if and only if $\left|N(S')\right| \ge \left|S'\right|$ for all $S' \subseteq X'$. Given $S' \subseteq X'$, let $T_1 = S' \cap X$ and $T_2 = S' - T_1$. Let $S \subseteq V(G)$ be the set of vertices consisting of T_1 in X plus the vertices of Y having copies in T_2. This establishes a bijection between subsets S' of X' and subsets S of $V(G)$, with $\left|S'\right| = |S|$. Also $\left|N(S')\right| = |N(S)|$, by the construction of G'.

Hence Hall's condition is satisfied for G' if and only if the condition of this problem holds in G. In summary, we have shown

$$[G \text{ has a 1-factor}] \Leftrightarrow G' \text{ has a 1-factor} \Leftrightarrow$$
$$\left|N(S')\right| \ge \left|S'\right| \text{ for all } S' \subseteq X' \Leftrightarrow |N(S)| \ge |S| \text{ for all } S \subseteq V(G).$$

Proof 2 (by Hall's Theorem). *Necessity:* Let M be a perfect matching, and let S be a subset of $V(G)$. Vertices of S are matched to distinct vertices of $N(S)$ by M, so $|N(S)| \ge |S|$. *Sufficiency:* If $|N(S)| \ge |S|$ for all $S \subseteq V$, then $|N(A)| \ge |A|$ for all $A \subseteq X$. By Hall's Theorem, the graph thus has a matching M that saturates X. Thus $|Y| \ge |X|$, and the condition $|N(Y)| \ge |Y|$ yields $|X| \ge CY$. Thus $|Y| = |X|$, and M is a perfect matching.

3.1.23. *Alternative proof of Hall's Theorem.* Given an X, Y-bigraph G, we prove that Hall's Condition suffices for a matching that saturates X. Let $m = |X|$. For $m = 1$, the statement is immediate.

Induction step: $m > 1$. If $|N(S)| > |S|$ for every nonempty proper subset $S \subset X$, select any neighbor y of any vertex $x \in X$. Deleting y reduces the size of the neighborhood of each subset of $X - \{x\}$ by at most 1. Hence Hall's Condition holds in $G' = G - x - y$. By the induction hypothesis, G' has a matching that saturates $X - \{x\}$, which combines with xy to form a matching that saturates X.

Otherwise, $|N(S)| = |S|$ for some nonempty proper subset $S \subset X$. Let $G_1 = G[S \cup N(S)]$, and let $G_2 = G - V(G_1)$. Because the neighbors of vertices in S are confined to $N(S)$, Hall's Condition for G implies Hall's Condition for G_1. For G_2, consider $T \subseteq X - S$. Since $|N_G(T \cup S)| \ge |T \cup S|$, we obtain

$$N_{G_2}(T) = N_G(T \cup S) - N_G(S) \ge |T \cup S| - |S| = |T|.$$

Thus Hall's Condition holds for both G_1 and G_2. By the induction hypothesis, G_1 has a matching that saturates S, and G_2 has a matching that saturates $X - S$. Together, they form a matching that saturates X.

3.1.24. *A square matrix of nonnegative integers is a sum of k permutation matrices if and only if each row and column sums to k.* If A is the sum of k permutation matrices, then each matrix adds one to the sum in each row and column, and each row or column of A has sum k.

For the converse, let A be a square matrix with rows and columns summing to k. We use induction on k to express A as a sum of k permutation matrices. For $k = 1$, A is a permutation matrix.

For $k > 1$, form a bipartite graph G with vertices $x_1, \dots, x_n$ and $y_1, \dots, y_n$ so that the number of edges joining x_i and y_j is $a_{i,j}$. The graph G is bipartite and regular, so by the Marriage Theorem it has a perfect matching. Let $b_{i,j} = 1$ if $x_i y_j$ belongs to this matching and $b_{i,j} = 0$ otherwise; the resulting matrix B is a permutation matrix. Each row and column of B has exactly one 1. Thus $A' = B - A$ is a nonnegative integer matrix whose rows and columns sum to $k - 1$. Applying the induction hypothesis to A' yields $k - 1$ additional permutation matrices that with B sum to A.

3.1.25. *A nonnegative doubly stochastic matrix can be expressed as a convex combination of permutation matrices.* For simplicity, we allow multiples of doubly stochastic matrices and prove a superficially more general statement. We use induction on the number of nonzero entries to prove that if Q is a matrix of nonnegative entries in which every row and every column sums to t, then Q can be expressed as a linear combination of permutation matrices with nonnegative coefficients summing to t.

If Q has exactly n nonzero entries, then Q is t times a permutation matrix, because Q must have at least one nonzero entry in every row and column. If Q has more nonzero entries, begin by defining a bipartite graph G with $x_i \leftrightarrow y_j$ if and only if $Q_{i,j} > 0$. If G has a perfect matching, then the edges $x_i y_{\sigma(i)}$ of the matching correspond to a permutation σ with permutation matrix P.

Let ε be the minimum (positive) value in the positions of Q corresponding to the 1's in P. The matrix $Q' = Q - \varepsilon P$ is a nonnegative matrix with fewer nonzero entries than Q, and row and column sums $t - \varepsilon$. By the induction hypothesis, we can express Q' as a nonnegative combination

$\sum_{i=1}^m c_i P_i$, with $\sum_{i=1}^m c_i = t - \varepsilon$. Hence $Q = \sum c_i P_i + \varepsilon P$. With $c_{m+1} = \varepsilon$ and $P_{m+1} = P$, we have expressed Q in the desired form.

It remains to prove that G has a perfect matching; we show that it satisfies Hall's condition. If S is a subset of X corresponding to a particular set of rows in Q, we need only show that these rows have nonzero entries in at least $|S|$ columns altogether. This follows because the total nonzero amount in the rows S is $t\,|S|$. Since each column contains only a total of t, it is not possible to contain a total of $t\,|S|$ in fewer than $|S|$ columns.

Comment. When the entries of Q are rational, the result follows directly from the Marriage Theorem. Multiplying Q by the least common denominator d of its positive entries converts it to an integer matrix in which all rows and columns sum to d. The entry in position i, j now is the number of edges joining x_i and y_j in a d-regular bipartite graph (multiple edges allowed). The Marriage Theorem implies that the graph has a perfect matching. By induction on d, it can be decomposed into perfect matchings. These matchings correspond to permutation matrices. In the expression of Q as a convex combination of these matrices, we give weight a/d to a permutation matrix arising a times in the list of matchings.

3.1.26. *Achieving columns with all suits.* The cards in an n by m array have m values and n suits, with each value on one card in each suit.

a) It is always possible to find a set of m cards, one in each column, having the m different values. Form a X, Y-bigraph in which X represents the columns and Y represents the values, with r edges from $x \in X$ to $y \in Y$ if value y appears r times in column x. Since each column contains n cards and each value appears in n positions (once in each suit), the multigraph is n-regular. By the Marriage Corollary to Hall's Theorem, every nontrivial regular bipartite graph has a perfect matching. (This applies also when multiple edges are present, which can occur here.) A perfect matching selects m distinct values occurring in the m columns.

(Using Hall's Theorem directly, a set S of k columns contains nk cards. Since there are n cards of each value, S contains cards of at least k values. Hence the graph satisfies Hall's condition and has a perfect matching.)

b) By a sequence of exchanges of cards of the same value, the cards can be rearranged so that each column consists of n cards of distinct suits. Making each column consist of n cards of different suits is equivalent to spreading each suit across all columns. The full result follows by induction on n, with $n = 1$ as a trivial basis step.

For the induction step, when $n > 1$, use part (a) to find cards of distinct values representing the m columns. Then perform at most one exchange for each value to bring the values in a single suit to those positions. Positions within a column are unimportant, so we can treat the other suits as an instance of the problem with $n-1$ suits. We apply the induction hypothesis to fix up the remaining suits.

The problem can always be solved using at most $mn - \sum_{k\le n} \lceil m/k \rceil$ exchanges. The worst case requires at least $\lfloor m/n \rfloor n(n-1)/2$ exchanges.

3.1.27. *The second player can force a draw in a positional game if $a \ge 2b$, where a is the minimum size of a winning set and b is the maximum number of winning sets containing a particular position.* Let P be the set of positions, and let $W_1, \ldots, W_m$ be the winning sets of positions. With $|P| = n$, let G be the bipartite graph on $n + 2m$ vertices with partite sets $P = \{p_1, \ldots, p_n\}$ and $W = \{w_1, \ldots, w_m\} \cup \{w'_1, \ldots, w'_m\}$ by creating two edges $p_i w_j$ and $p_i w'_j$ for each incidence $p_i \in W_j$ of a position in a winning set.

If G has a matching M that saturates W, then Player 2 can use M to force a draw. When the position taken by Player 1 on a given move is matched to one of $\{w_i, w'_i$ in M, Player 2 responds by taking the position matched to the other one of these two elements. Player 1 thus can never obtain all the positions of a winning set. (If Player 1 takes an unmatched position, Player 2 can respond by taking any available position.)

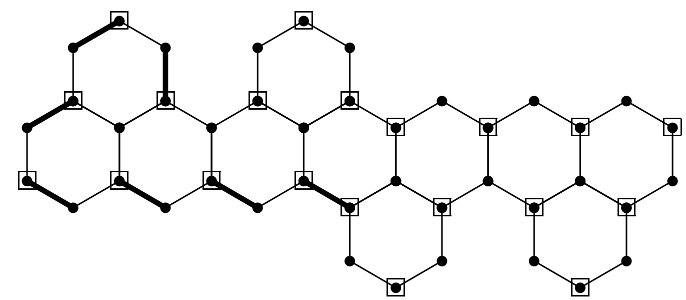
The existence of such a matching follows from $a \ge 2b$ by Hall's condition. If $S \subseteq W$, then S has representatives (w or w') of at least $|S|/2$ winning sets. Since each position appears in at most b winning sets, the number of positions in the union of these winning sets is at least $a(|S|/2)/b \ge |S|$. Thus $|N(S)| \ge |S|$ for every $S \subseteq W$.

3.1.28. *A graph with no perfect matching.*

Proof 1 (vertex cover). The graph has 42 vertices, so a perfect matching would have 21 edges. The marked vertices form a vertex cover of size 20. The edges of a matching must be covered by distinct vertices in a vertex cover, so there is no matching with more than 20 edges.

Proof 2 (Hall's condition). Using two labels X and Y, we obtain a bipartition of the graph. Partite set X consists of the marked vertices in the left half of the picture and the unmarked vertices in the right half. This is an independent set of size 21, and the remaining vertices also form an independent set Y of size 21.

Hall's Condition is a necessary condition for a perfect matching; we show that Hall's Condition does not hold. Let S be the subset of X consisting of the 11 unmarked vertices in the right half of the graph. The neighbors of vertices in S are the 10 marked vertices in the right half of the graph. Thus $|N(S)| < |S|$.



Proof 3 (other dual problems). In every graph $\alpha'(G) + \beta'(G) = n(G)$, so it suffices to show that at least 22 edges are needed to cover $V(G)$. Also $\beta'(G) \ge \alpha(G)$ always, since distinct edges are needed to cover the vertices of an independent set. Thus it suffices to show that G has an independent set of size at least 22. Such a set is given by the unmarked vertices above (the complement of a vertex cover).

Proof 4 (augmenting paths). Having found a matching M of size 20, one can prove that there is no perfect matching by following all possible M-alternating paths from one M-unsaturated vertex to show that none reaches the other unsaturated vertex. In this particular example, this method is not too difficult.

Proof 5 (symmetry and case analysis). The graph has two edges whose deletion leaves two isomorphic components of order 21. Since 21 is odd, a perfect matching must use exactly one of the two connecting edges. By symmetry, we may assume it is the one in bold above. This forces a neighboring vertex of degree 2 to be matched to its other neighbor, introducing other bold edge. Repeating this argument yields a path of bold edges forced into the matching. As soon as this leaves a vertex with no available neighbor, we have proved that a perfect matching cannot be completed.

3.1.29. *Every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$.* Each vertex of G covers at most $\Delta(G)$ edges. Since all edges must be covered in a vertex cover, this yields $\beta(G) \ge e(G)/\Delta(G)$. By the König–Egerváry Theorem, $\alpha'(G) = \beta(G)$ when G is bipartite. Thus $\alpha'(G) \ge e(G)/\Delta(G)$.

Every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k. Such a graph G is a simple bipartite graph with partite sets of size n. Thus $\Delta(G) \le n$, and we compute $\alpha'(G) \ge e(G)/\Delta(G) > (k-1)n/n = k-1$. Thus G has a matching of size k.

3.1.30. *The maximum number of edges in a simple bipartite graph that has no matching with k edges and no star with l edges is $(k-1)(l-1)$.* If G is a bipartite graph having no matching with k edges, then G has a vertex cover using at most $k-1$ vertices. If G is a simple graph having no star with l edges, then each vertex covers at most $l-1$ edges. Hence the vertex cover covers at most $(k-1)(l-1)$ edges, which must be all the edges of G. The bound is achieved by $(k-1)K_{1,l-1}$.

3.1.31. *Hall's Theorem from the König–Egerváry Theorem.* By the König–Egerváry Theorem, an X,Y-bigraph G fails to have a matching that saturates X if and only if G has a vertex cover of size less than $|X|$. Let Q be such a cover, with $R = Q \cap X$ and $T = Q \cap Y$. Because Q is a vertex cover, there is no edge from $X - R$ to $Y - T$, which means that $N(X-R) \subseteq T$. This yields

$$|N(X-R)| \le |T| = |Q| - |R| < |X| - |R| = |X-R|.$$

We have used the König–Egerváry Theorem to show that absence of a matching that saturates X yields a violation of Hall's Condition. Thus Hall's Condition is sufficient for such a matching.

Similarly, if $|N(S)| < |S|$ for some $S \subseteq X$, then $N(S) \cup X - S$ is a vertex cover of size less than $|X|$, and there is no matching of size $|X|$. Hence Hall's Condition also is necessary.

3.1.32. *If G is a bipartite graph with partite sets X, Y, then $\alpha'(G) = |X| - \max_{S \subseteq X}(|S| - |N(S)|)$.* Let $d = \max(|S| - |N(S)|)$. The case $S = \varnothing$ implies that $d \ge 0$. Choose $T \subseteq X$ such that $|T| - |N(T)| = d$. Because saturated vertices of T must have distinct neighbors in any matching and only $|T| - d$ neighbors are available, every matching leaves at least d vertices (of T) unsaturated. Thus $\alpha'(G) \le |X| - d$.

To prove that G has a matching as large as $|X| - d$, we form a new graph G' by adding d vertices to the partite set Y and making all of them adjacent to all of X. This adds d vertices to $N(S)$ for each $S \subseteq X$, which yields $|N_{G'}(S)| \ge |S|$ for all $S \subseteq X$. By Hall's Theorem, G' has a matching saturating all of X. When we delete the new vertices of G', we lose at most d edges of the matching. Hence what remains is a matching of size at least $|X| - d$ in G, as desired.

3.1.33. *König–Egerváry from Exercise 3.1.32.* Always $\alpha'(G) \le \beta(G)$, so it suffices to show that a bipartite graph G has a matching and a vertex cover of the same size. Consider an X,Y-bigraph G in which S is a subset of X with maximum deficiency. By part (a), $\alpha'(G) = |X| - |S| + |N(S)|$.

Let $R = (X - S) \cup (N(S))$. By the definition of $N(S)$, there are no edges joining S and $Y - N(S)$. Therefore, R is a vertex cover of G. The size of R is $|X| - |S| + |N(S)|$, which equals $\alpha'(G)$. Thus G has a matching and a vertex cover of the same size, as desired.

3.1.34. *When G is an X,Y-bigraph with no isolated vertices and the deficiency of a set S is $|S| - |N(S)|$, the graph G has a matching that saturates X if and only if each subset of Y has deficiency at most $|Y| - |X|$.* Using Hall's Theorem, it suffices to show that $|N(S)| \ge |S|$ for all $S \subseteq X$ if and only if $|T| - |N(T)| \le |Y| - |X|$ for all $T \subseteq Y$.

We rewrite the latter condition as $|X| - |N(T)| \le |Y| - |T|$. Since every vertex of $X - N(T)$ has no neighbor in T, we have $N(X - N(T)) \subseteq Y - T$. If Hall's Condition holds, then applying it with $S = X - N(T)$ yields $|X| - |N(T)| = |S| \le |N(S)| \le |Y| - |T|$, which is the desired condition.

Conversely, suppose that $|X| - |N(T)| \le |Y| - |T|$ for all $T \subseteq Y$. Given $S \subseteq X$, let $T = Y - N(S)$. Since T omits all neighbors of vertices in S, we have $S \subseteq X - N(T)$. Now $|S| \le |X| - |N(T)| \le |Y| - |T| = |N(S)|$. Hence Hall's Condition holds.

3.1.35. *A bipartite graph G in $K_{X,Y}$ fails to have $(k+1)K_2$ as an induced subgraph if and only if each $S \subseteq X$ has a subset of size at most k with neighborhood $N(S)$.* For any $S \subseteq X$, let T be a minimal subset of S with neighborhood $N(S)$. By the minimality of T, G has an induced matching of size $|T|$. Hence if G has no induced matching of size k, each $S \subseteq X$ has a subset of size at most k with neighborhood $N(S)$. Conversely, if $k + 1K_2$ does occur as an induced subgraph, then the set of its vertices in X have no subset of size at most k with the same neighborhood.

3.1.36. *If a bipartite graph G has a matching saturating a partite set X of size m, then at most $\binom{m}{2}$ edges of G belong to no matching of size m.* There are at least three distinguishable ways to get the bound. The most direct one, and the one that suggests the extremal graph proving that no smaller bound is possible, considers pairs of vertices.

Proof 1. First note that, after renumbering the vertices so the edges of the given matching are $\{x_i y_i\}$, every edge involving any other vertex of Y belongs to a perfect matching, so it suffices to restrict attention to the subgraph induced by X and $y_1, \ldots, y_m$. Consider the edges $x_i y_j$ and $x_j y_i$, for $j < i \le m$. If both edges are present, then neither belongs to no maximum matching, since they can be exchanged for $x_i y_i$ and $x_j y_j$ in the original matching. Taking one from each such pair bounds the number of unmatchable edges by $\binom{m}{2}$. On the other hand, taking all $x_i y_j$ with $j < i \le m$ yields a graph in which $\{x_i y_i\}$ is the unique maximum matching. If any edge $x_i y_i$ is deleted from this graph, then $|\mathrm{Adj}(\{x_1, \ldots, x_i\})| = i - 1$.

Proof 2. Induction on m. As before, reduce attention to the edges between X and $y_1, \ldots, y_m$. Among X, let x be the vertex of maximum degree in this subgraph. It has degree at most m, and its deletion leaves a graph satisfying the hypotheses for $m - 1$. Allowing for the matched edge involving x, this gives a bound of $\binom{m-1}{2} + m - 1 = \binom{m}{2}$ on the unmatchable edges. Note that x coresponds to x_m in the example above.

Proof 3. Let S be an arbitrary subset of X. If $|N(S)| > |S|$ for every proper nonempty subset S of X, then every edge belongs to a perfect matching. To show this, delete the endpoints of an edge xy. This reduces the adjacency set of any S not containing x by at most 1, so the reduced graph has a perfect matching, and replacing xy yields a perfect matching of the original graph containing xy. So, assume there is an S with $|N(S)| = |S| = s$, and let $N(S) = T$. Then the subgraphs induced by $S \cup T$, $(X - S) \cup T$, and $(X - S) \cup (Y - T)$ partition the edges of G. The first and last have perfect matchings, and an edge there fails to appear in a perfect matching of G if and only if it appears in no perfect matching of the subgraph. No edge of the middle graph appears in a perfect matching of G. By induction and the fact that $|T| = |S|$, the bound on the number of edges that appear in no perfect matching of G is $\binom{s}{2} + s(m-s) + \binom{m-s}{2} = \binom{m}{2}$.

3.1.37. Let G be an X,Y-bigraph having a matching that saturates X.

a) If $S, T \subseteq X$ are sets such that $|N(S)| = |S|$ and $|N(T)| = |T|$, then $|N(S \cap T)| = |S \cap T|$.

Proof 1 (manipulation of sets). Since G has a matching that saturates X, $|N(S \cap T)| \ge |S \cap T|$, and $|N(S \cup T)| \ge |S \cup T|$. Also $N(S \cup T) = N(S) \cup N(T)$ and $N(S \cap T) \subseteq N(S) \cap N(T)$. Together, these statements yield

$$\begin{aligned}|S \cup T| + |S \cap T| &\le |N(S \cup T)| + |N(S \cap T)| \le |N(S) \cup N(T)| + |N(S) \cap N(T)| \\ &= |N(S)| + |N(T)| = |S| + |T|\end{aligned}$$

Since the two ends of this string of expressions are equal, the inequalities along the way hold with equality. In particular, $|N(S \cap T)| = |S \cap T|$ and $|N(S \cup T)| = |S \cup T|$.

Proof 2 (characterization of sets with no excess neighbors). If M is a matching that saturates X, then $|N(S)| = |S|$ if and only if every vertex of $N(S)$ is matched by M to a vertex of S. If $|N(S)| = |S|$ and $|N(T)| = |T|$, then every vertex of $N(S)$ is matched into S and every vertex of $N(T)$ is matched into T. Since $N(S \cap T) \subseteq N(S) \cap N(T)$, we conclude that every vertex of $N(S \cap T)$ is matched into $S \cap T$, and therefore $|N(S \cap T)| = |S \cap T|$.

b) There is a vertex $x \in X$ such that every edge incident to x belongs to some matching that saturates X. We use induction on $|X|$. If $|N(S)| > |S|$ for every nonempty proper subset of X, then Hall's Condition holds for the graph obtained by deleting the endpoints of any edge. Thus each edge can be combined with a matching saturating what remains of X in the graph obtained by deleting its endpoints, so every edge of G belongs to some matching saturating X.

In the remaining case, there is a nonempty proper subset $S \subseteq X$ such that $|N(S)| = |S|$. Let G_1 be the subgraph of G induced by $S \cup N(S)$, and let G_2 be the subgraph obtained by deleting $S \cup N(S)$. As in the proof of Hall's Theorem, the graph G_2 obtained by deleting $S \cup N(S)$ satisfies Hall's Condition (the proof of $\left|N_{G_2}(T)\right| \ge |T|$ follows from $|N_G(T \cup S)| \ge |T \cup S|$). Thus G_2 has a matching saturating $X - S$.

The subgraph G_1 also satisfies Hall's condition, since it retains all neighbors of each vertex of S. By the induction hypothesis, S has a vertex x such that every edge incident to x belongs to a matching in G_1 that saturates S. These matchings can be combined with a single matching in G_2 that saturates $X - S$ to obtain matchings in G that saturate X. Hence the vertex x serves as the desired vertex in G.

It appears that part (a) is not needed to solve part (b).

3.1.38. *Pairing up farms and hunting ranges.* Suppose the unit of area is the size of one range. Let G be the bipartite graph between hunting ranges and farms formed by placing an edge between a hunting range and a farm if the area of their intersection is at least ε, where $\varepsilon = 4/(n+1)^2$ if n is odd and $\varepsilon = 4/[n(n+2)]$ if n is even. We prove that this graph has a perfect matching, which yields the desired assignments.

Let H be the union of some set of k hunting ranges. Let $f_1 \ge \cdots \ge f_n$ be the areas of intersection with H of the farms, and let F be the set of k farms having largest intersection with H. If $f_k = \alpha$, then the area of H is bounded by $\sum_{i=1}^{k-1} g_i + \alpha(n+1-k) \le k - 1 + \alpha(n+1-k)$. It also equals k, so we have $\alpha \ge 1/(n+1-k)$. Since we have k farms meeting H with area at least $1/(n+1-k)$, we find for each farm in F a hunting range contained in H that intersects the farm with area at least $1/[k(n+1-k)] \ge \varepsilon$. Hence any set of k hunting ranges has at least k neighbors in G, which guarantees the matching.

Note that ε is the largest possible guaranteed minimum intersection. Let $k = \lceil n/2 \rceil$. With hunting ranges in equal strips, we can arrange that some set of $k-1$ farms intersects each of the first k hunting ranges with area $1/k$, and the remaining farms intersect each of the first k hunting ranges with area ε, since $(k-1)/k + (n+1-k)\varepsilon = 1$. Now one of the first k hunting ranges must be matched with area ε.

3.1.39. $\alpha(G) \le n(G) - e(G)/\Delta(G)$. Let S be an independent set of size $\alpha(G)$. Since $V(G) - S$ is a vertex cover, summing the vertex degrees in $V(G) - S$ provides an upper bound on $e(G)$. Thus $e(G) \le (n(G) - \alpha(G))\Delta(G)$, which is equivalent to the desired inequality.

If G is regular, then $\alpha(G) \le n(G)/2$. In the previous inequality, set $e(G) = n(G)\Delta(G)/2$.

3.1.40. *If G is a bipartite graph, then $\alpha(G) = n(G)/2$ if and only if G has a perfect matching.* Since $\alpha(G) = n(G) - \beta(G) = n(G) - \alpha'(G)$ by Lemma 3.1.21 and the König–Egerváry Theorem, we have $\alpha(G) = n(G)/2$ if and only if $\alpha'(G) = n(G)/2$.

3.1.41. (corrected statement) *If G is a nonbipartite n-vertex graph with exactly one cycle C, then $\alpha(G) \ge (n-1)/2$, with equality if and only if $G - V(C)$ has a perfect matching.* The cycle C must have odd length, say k. Let e be an edge of C, and let $G' = G - e$. The graph G' is bipartite, so $\alpha(G-e) \ge n/2$. An independent set S in $G - e$ is also independent in G unless it contains both endpoints of e. If $|S| > n/2$, then we can afford to drop one of these vertices. If $|S| = n/2$, then we can take the other partite set instead to avoid the endpoints of e. In each case, $\alpha(G) \ge (n-1)/2$.

If $G - V(C)$ has a perfect matching, then an independent set is limited to $(k-1)/2$ vertices of C and $(n-k)/2$ vertices outside C, so $\alpha(G) \le (n-1)/2$ and equality holds.

For the converse, observe that deleting $E(C)$ leaves a forest F in which each component has a vertex of G. Let H be a component of F, with x being its vertex on C, and let r be its order. If $H - x$ has no perfect matching, then $\alpha'(H-x) \le r/2-1$ (that is, it cannot equal $(r-1)/2$). Now $\beta(H-x) \le r/2 - 1$, by König-Egerváry, and $\alpha(H-x) \ge r/2$, since the complement of a vertex cover is an independent set. Since this independent set does not use x, we can combine it with an independent set of size at least $(n-r)/2$ in the bipartite graph $G - V(H)$ to obtain $\alpha(G) \ge n/2$. Since this holds for each component of F, $\alpha(G) = (n-1)/2$ requires a perfect matching in $G - V(C)$. (This direction can also be proved by induction on $n - k$.)

3.1.42. *The greedy algorithm produces an independent set of size at least $\sum_{v\in V(G)} \frac{1}{d(v)+1}$ in a graph G.* The algorithm iteratively selects a vertex of minimum degree in the remaining graph and deletes it and its neighbors. We prove the desired bound by induction on the number of vertices.

Basis step: $n = 0$. When there are no vertices, there is no contribution to the independent set, and the empty sum is also 0.

Induction step $n > 0$. Let x be a vertex of minimum degree, let $S = \{x\} \cup N(x)$, and let $G' = G - S$. The algorithm selects x and then seeks an independent set in G'. We apply the induction hypothesis to G' to obtain a lower bound on the contribution that the algorithm obtains from G'. Thus the size of the independent set found in G is at least $1 + \sum_{v\in V(G')} \frac{1}{d_{G'}(v)+1}$.

Note that $\{x\} \cup N(x)$ is a set of size $d_G(x) + 1$, and the choice of x as a vertex of minimum degree implies that each vertex in this set contributes at most $d_G(x) + 1$ to the desired sum. Thus

$$\begin{aligned} 1 + \sum_{v\in V(G')} \frac{1}{d_{G'}(v)+1} &= \sum_{v\in S} \frac{1}{d_G(x)+1} + \sum_{v\in V(G)-S} \frac{1}{d_{G-S}(v)+1} \\ &\ge \sum_{v\in S} \frac{1}{d_G(v)+1} + \sum_{v\in V(G)-S} \frac{1}{d_G(v)+1}\ . \end{aligned}$$

Thus the algorithm finds an independent set at least as large as desired.

3.1.43. *Consequences of Gallai's Theorem* (G has no isolated vertices).

a) A maximal matching M is a maximum matching if and only if it is contained in a minimum edge cover. If M is a maximal matching, then the smallest edge cover L containing M adds one edge to cover each M-unsaturated vertex, since no edge covers two M-unsaturated vertices. We have $|L| = |M| + (n - 2|M|) = n - |M|$. By Gallai's Theorem (Theorem 3.1.22), $|M| = \alpha'(G)$ if and only if $|L| = \beta'(G)$.

b) A minimal edge cover L is a minimum edge cover if and only if it contains a maximum matching. As observed in proving Theorem 3.1.22, every minimal edge cover consists of disjoint stars. The largest matching contained in a disjoint union of stars consists of one edge from each component. The size of this matching is $n - |L|$. Hence a minimal edge cover L has size $n - \alpha'(G)$ if and only if L contains a matching of size $\alpha'(G)$.

3.1.44. *If G is a simple graph in which the sum of the degrees of any k vertices is less than $n - k$, then every maximal independent set in G has more than k vertices.* Let S be an independent set. If $|S| \le k$, then the sum of the degrees of the vertices in S is less than $n - k$. This means that some vertex x outside S is not a neighbor of any vertex in S, and hence x can be added to form an independent set containing S. Thus maximal independent sets must have more than k vertices.

3.1.45. *If xy and xz are α-critical edges in G and $y \not\leftrightarrow z$, then G contains an induced odd cycle (through xy and xz).* Let Y, Z be maximum stable sets in $G - xy$ and $G - xz$, respectively. Since Y, Z are not independent in G, we have $x, y \in Y$ and $x, z \in Z$.

Proof 1. Let $H = G[Y \triangle Z]$. Since $x \in Y \cap Z$, H is a bipartite graph with bipartition $Y - Z, Z - Y$. If some component of H has partite sets of different sizes, then we can substitute the larger for the smaller in Y or Z to obtain a stable set in G of size exceeding $\alpha(G)$.

If y and z belong to different components H_y and H_z of H, then let S be the union of $V(H_y) \cap Z$, $V(H_z) \cap Y$, one partite set of each other component of H, and $Y \cap Z$. Since $x \in Y \cap Z$ and $y, z \notin S$, the set S is independent in G. Also, $|S| = |Y| = |Z| > \alpha(G)$. Hence y and z belong to the same component of H. A shortest y,z-path in H is a chordless path of odd length in G, and it completes a chordless odd cycle with zx and xy.

Proof 2. Let $H' = G[(Y \triangle Z)] \cup \{x\}$. Note that $|Y \triangle Z|$ is even, since $|Y| = |Z|$. Let $2k = |Y \triangle Z|$. If H' is bipartite, then it has an independent set of size at least $k+1$, which combines with $Y \cap Z - \{x\}$ to form an independent set of size $\alpha(G) + 1$ in G. Hence H' has an odd cycle. Since H is bipartite, this cycle passes through x. Since $N_{H'}(x) = \{y, z\}$, the odd cycle contains the desired edges.

3.1.46. *A graph has domination number 1 if and only if some vertex neighbors all others.* This is immediate from the definition of dominating set.

3.1.47. *The smallest tree where the vertex cover number exceeds the domination number is P_6.* In a graph with no isolated vertices, every vertex cover is a dominating set, since every vertex is incident to an edge, and at least one endpoint of that edge is in the set. Hence $\gamma(G) \le \beta(G)$. We want a tree where the inequality is strict.

If $\gamma(G) = 1$, then a single vertex is adjacent to all others, and since G is a tree there are no other edges, so $\beta(G) = 1$. Hence we need $\gamma(G) \ge 2$ and $\beta(G) \ge 3$. A tree is bipartite, so $al'(G) = \beta(G) \ge 3$. A matching of size 3 requires at least 6 vertices. There are two isomorphism classes of 6-vertex trees with perfect matchings, and P_6 is the only one having a dominating set of size 2. (Smaller trees can also be excluded by case analysis instead of using the König–Egerváry Theorem.)

3.1.48. $\gamma(C_n) = \gamma(P_n) = \lceil n/3 \rceil$. With maximum degree 2, vertices can dominate only two vertices besides themselves. Therefore, $\lceil n/3 \rceil$ is a lower bound. Picking every third vertex starting with the second (and using the last when n is not divisible by 3) yields a dominating set of size $\lceil n/3 \rceil$.

3.1.49. *In a graph G without isolated vertices, the complement of a minimal dominating set is a dominating set, and hence $\gamma(G) \le n(G)/2$.* Let S be a minimal dominating set. Every vertex of $\overline{S}$ has a neighbor in S, and by minimality this fails when a vertex is omitted from S. For each $x \in S$, there is thus a vertex of $\overline{S}$ whose only neighbor in S is x. In particular, every $x \in S$ has a neighbor in $\overline{S}$, which means that $\overline{S}$ is a dominating set.

Since S and $\overline{S}$ are disjoint dominating sets, one of them has size at most $n(G)/2$.

3.1.50. *If G is a n-vertex graph without isolated vertices, then $\gamma(G) \le n - \beta'(G) \le n/2$.*

Proof 1. Since every edge covers at most two vertices, always $\beta'(G) \ge n/2$. As discussed in the proof of Theorem 3.1.22, the components of a minimum edge cover are stars, and the number of stars is $n - \beta'(G)$. Since the union of these stars is a spanning subgraph, choosing the centers of these stars yields a dominating set.

Proof 2. Since $\alpha'(G) = n - \beta'(G)$ by Theorem 3.1.22, it suffices to show that $\gamma(G) \le \alpha'(G)$. In a maximum matching M, the two endpoints of an edge in M cannot have distinct unsaturated neighbors. Also the unsaturated neighbors all have saturated neighbors. Therefore, picking from each edge of M the endpoint having unsaturated neighbor(s) (or either if neither has such a neighbor) yields a dominating set of size $\alpha'(G)$.

Construction of n-vertex graphs with domination number k, for $1 \le k \le n/2$. Form G from a matching of size k by selecting one vertex from each edge and adding edges to make these vertices pairwise adjacent.

3.1.51. *Domination in an n-vertex simple graph G with no isolated vertices.*

a) $\lceil n/(1+\Delta(G) \rceil \le \gamma(G) \le n - \Delta(G)$. Each vertex takes care of itself and at most $\Delta(G)$ others; thus $\gamma(G)(1+\Delta(G)) \ge n$. For the upper bound, note that the set consisting of all vertices *except* the neighbors of a vertex of maximum degree is a dominating set.

b) $(1 + \operatorname{diam} G)/3 \le \gamma(G) \le n - \lceil 2\operatorname{diam} G/3 \rceil$. Let P be a shortest u,v-path, where $d(u,v) = \operatorname{diam} G$. Taking the vertices at distances $1, 4, 7, \ldots$ along P from u yields a set that dominates all the vertices of P (v is also needed if $\operatorname{diam} G$ is divisible by 3. Even if all the vertices off P are needed to augment this to a dominating set, we still have used at most $n - \lceil 2\operatorname{diam} G/3 \rceil$ vertices.

For the lower bound, the vertices at distances $0, 3, 6, \ldots$ from u along P must be dominated by distinct vertices in a dominating set; a vertex dominating two of them would yield a shorter u,v-path. This yields the lower bound.

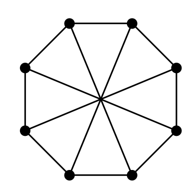
Both bounds hold with equality for the path P_n.

3.1.52. *If the diameter of G is at least 3, then $\gamma(\overline{G}) \le 2$.* Let u and v be two vertices such that $d_G(u,v) = 3$. The set $\{u, v\}$ is a dominating set in $\overline{G}$, because u and v have no common neighbors in G. For $x \in V(G) - \{u, v\}$, at least one of $\{u, v\}$ is nonadjacent to x in G and therefore adjacent to it in $\overline{G}$.

3.1.53. *Examples with specified domination number.*

A $5k$-vertex graph with domination number $2k$ and minimum degree 2. Begin with kC_5 and add edges to form a cycle using one vertex from each 5-cycle. Two vertices must be used from each original 5-cycle, and this suffices for a dominating set.

A 3-regular graph G with $\gamma(G) = 3n(G)/8$. In a 3-regular graph, each vertex dominates itself and three others. In the 8-vertex graph below, deletion of any vertex and its neighbors leaves P_4, which cannot be dominated by one additional vertex. Hence the domination number is 3.



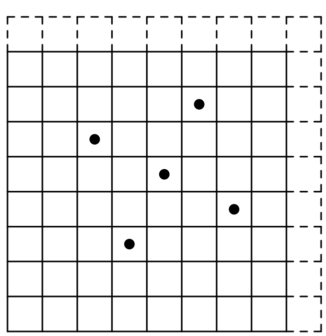
3.1.54. *The Petersen graph has domination number 3 and total domination number 4.* Each vertex dominates itself and three others, so at least three vertices are needed. Since the graph has diameter 2, the neighbors of a single vertex form a dominating set.

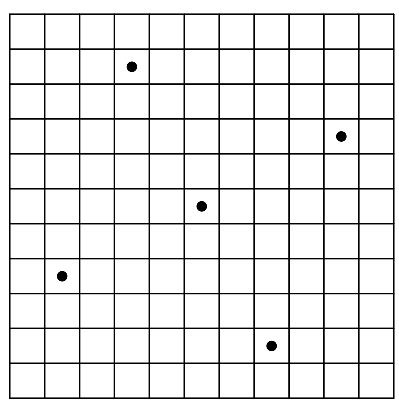
A total dominating set S must include a neighbor of every vertex in S. Hence S must contain two adjacent vertices. This pair leaves four undominated vertices. Adding a neighbor of the original pair dominates at most three of these, since the graph is 3-regular. Hence $|S| \ge 4$. One vertex and its neighbors form a total dominating set of size four.

3.1.55. *Dominating sets in the hypercube Q_4.* Since Q_4 is 4-regular, each vertex dominates itself and four others. Now $n(Q_4) = 16$ yields $\gamma(Q_4) \ge \lceil 16/5 \rceil = 4$. Since $\{0000, 0111, 1100, 1011\}$ is an independent dominating set and $\{0000, 0001, 1110, 1111\}$ is a total dominating set, the domination, independent domination, and total domination numbers all equal 4.

Adding two vertices to this total dominating set of size 4 completes a connected dominating set of size 6. We show there is no smaller connected dominating set. A connected 5-vertex subgraph contains two incident edges. Let S be the set of three vertices in two such edges. The set T of vertices undominated by S has size 6. Each neighbor of a vertex of S dominates at most two vertices in T. Each vertex of T dominates at most three vertices in T, except for one vertex that dominates itself and four others (For example, if $S = \{0000, 0001, 0010\}$, then the high-degree vertex of $Q_4[T]$ is 1111.) To dominate T with only two additional vertices, we must therefore use the high-degree vertex of T. However, its distance to S is 3, so it cannot be used to complete a connected set of size 5.

3.1.56. *Five pairwise non-attacking queens can control an 8-by-8 chessboard.* As shown below, they can also control a 9-by-9 chessboard. Five queens still suffice for an 11-by-11 chessboard, but this configuration does not exist on the 8-by-8 board.





3.1.57. *An n-vertex tree with domination number 2 in which the minimum size of an independent dominating set is $\lfloor n/2 \rfloor$.* Consider the tree of diameter 3 with two central vertices u and v in which one central vertex has $\lfloor (n-2)/2 \rfloor$ leaf neighbors and the other has $\lceil (n-2)/2 \rceil$ leaf neighbors. The set $\{u, v\}$ is a dominating set, but these cannot both appear in an independent dominating set. If u does not appear in a dominating set, then all its leaf neighbors must appear. We also must include at least one vertex from the set consisting of v and its leaf neighbors, since these are not dominated by the other leaves. Hence the independent dominating set must have at least $\lfloor (n-2)/2 \rfloor + 1$ vertices.

3.1.58. *Every $K_{1,r}$-free graph G has an independent dominating set of size at most $(r-2)\gamma(G) - (r-3)$.* Let S be a minimum dominating set in G. Let S' be a maximal independent subset of S. Let $T = V(G) - R$, where R is the set $N(S') \cup S'$ of vertices dominated by S'. Let T' be a maximal independent subset of T.

Since T' contains no neighbor of S', $S' \cup T'$ is independent. Since S' is a maximal independent subset of S, every vertex of $S - S'$ has a neighbor in S'. Similarly, T' dominates $T - T'$. Hence $S' \cup T'$ is a dominating set.

It remains to show that $\left|S' \cup T'\right| \le (r-1)\gamma(G) - (r-3)$. Each vertex of $S - S'$ has at most $r-2$ neighbors in T', since it has a neighbor in S', and $S' \cup T'$ is independent, and G is $K_{1,r}$-free. Since S is dominating, each vertex of T' has at least one neighbor in $S - S'$. Hence $\left|T'\right| \le (r-2)\left|S - S'\right|$, which yields $\left|S' \cup T'\right| \le (r-2)\left|S\right| - (r-3)\left|S'\right|$. Since $|S| = \gamma(G)$ and $\left|S'\right| \ge 1$, we conclude that $\left|S' \cup T'\right| \le (r-2)\gamma(G) - (r-3)$.

3.1.59. *In a graph G of order n, the minimum size of a connected dominating set is n minus the maximum number ℓ of leaves in a spanning tree.* For the upper bound, deleting the leaves in a spanning tree with ℓ leaves yields a connected dominating set of size $n - \ell$.

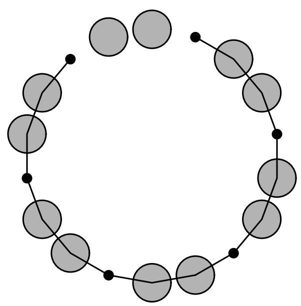
For the lower bound, we form a spanning tree T of G by taking a spanning tree in the subgraph induced by a connected dominating set S and adding each remaining vertex as a neighbor of one of these. Thus $|S|$ is n minus the number of leaves in T. Since this number of leaves is at most ℓ, we have $|S| \le n = \ell$.

3.1.60. *A graph with minimum degree k and no connected dominating set of size less than $3n(G)/(k+1) - 2$.* Form G from a cyclic arrangement of $3m$ pairwise-disjoint cliques of sizes $\lceil k/2 \rceil, \lfloor k/2 \rfloor, 1, \lceil k/2 \rceil, \lfloor k/2 \rfloor, 1, \cdots$ in order by making each vertex adjacent to every vertex in the clique before it and the clique after it. Each vertex is adjacent to all but itself in its own clique and the two neighboring cliques, so G is k-regular.

A graph has a connected dominating set of size r if and only if it has a spanning tree with at most r non-leaf vertices (Exercise 3.1.59). Hence we show that spanning trees in G must have many non-leaves. Let S be the set of vertices in the cliques of size 1 in the construction.

In a subgraph of G having all pairs of nearest vertices in S joined by paths through the two intervening cliques, there is a cycle. If at least two such pairs are not joined by such paths, then the subgraph is disconnected. Therefore, every spanning tree of G contains a path P directly connecting all but one of the successive pairs, as shown below.

At least one endpoint of P must be a non-leaf. If one endpoint of P is a leaf, then the other endpoint of P and some vertex in the untouched clique next to it must be non-leaves. In either case, we have obtained at least $3m-2$ non-leaves. Since $m = n/(k+1)$, we have the desired bound.



3.2. ALGORITHMS AND APPLICATIONS

3.2.1. *A weighted graph with four vertices where the maximum weight matching is not a maximum size matching.* Let $G = P_4$, and give the middle edge greater weight than the sum of the other weights.

3.2.2. *Use of the Hungarian Algorithm to test for the existence of a perfect matching in a bipartite graph G.* Given that the partite sets of G have size n, form a weighted matching problem on $K_{n,n}$ in which the edges of G have weight 1 and the edges not in G have weight 0. There is a perfect matching in G if and only if the solution to the weighted matching problem is n.

3.2.3. *Multiplicity of stable matchings.* With men u, v and women a, b, there may be two stable matching. Suppose the preferences are $u : a > b$, $v : b > a$, $a : v > u$, $b : u > v$. If both men get their first choices, then they prefer no one to their assigned partner, so the matching is stable. The same argument applies when the women get their first choices. However,

the matchings with men getting their first choices and women getting their first choices are different.

3.2.4. *Stable matchings under proposal algorithm.* Consider the preference orders listed below.

Men $\{u, v, w, x, y, z\}$	Women $\{a, b, c, d, e, f\}$
$u: a > b > d > c > f > e$	$a: z > x > y > u > v > w$
$v: a > b > c > f > e > d$	$b: y > z > w > x > v > u$
$w: c > b > d > a > f > e$	$c: v > x > w > y > u > z$
$x: c > a > d > b > e > f$	$d: w > y > u > x > z > v$
$y: c > d > a > b > f > e$	$e: u > v > x > w > y > z$
$z: d > e > f > c > b > a$	$f: u > w > x > v > z > y$

When men propose, the steps of the algorithm are as below. For each round, we list the proposals by u, v, w, x, y, z in order, followed by the resulting rejections. Round 1: a, a, c, c, c, d; $a \times v$, $c \times w$, $c \times y$. Round 2: a, b, b, c, d, d; $b \times v$, $d \times z$. Round 3: a, c, b, c, d, e; $c \times x$. Round 4: a, c, b, a, d, e; $a \times u$. Round 5: b, c, b, a, d, e; $b \times u$. Round 6: d, c, b, a, d, e; $d \times u$. Round 7: c, c, b, a, d, e; $c \times u$. Round 8: f, c, b, a, d, e; stable matching.

When women propose, the steps of the algorithm are as below. For each round, we list the proposals by a, b, c, d, e, f in order, followed by the resulting rejections. Round 1: z, y, v, w, u, u; $u \times e$. Round 2: z, y, v, w, v, u; $v \times e$. Round 3: z, y, v, w, x, u; stable matching.

Note that the pairs uf and vc occur in both results, and in all other cases the women are happier when the women propose and the men are happier when the men propose.

3.2.5. *Maximum weight transversal.* For each matrix below, we underscore a maximum weight transversal, and the labels on the rows and columns form a cover whose total cost equals the weight of the transversal.

For every position (i, j), the label on row i plus the label on column j is at least the entry in position (i, j) in the matrix. Hence the labeling is feasible for the dual problem. Equality between the sum of the labels in a feasible labeling and the sum of the entries of a transversal implies that the transversal is one of maximum weight and feasible labeling is one of minimum weight, because *every* feasible labeling has sum as large as the weight of *every* matching (since the positions in the matching must be covered disjointly by the labels).

$$\begin{array}{cc} & \begin{matrix}0&0&1&0&2\end{matrix} \\ \begin{matrix}4\\3\\4\\7\\8\end{matrix} & \begin{pmatrix}\underline{4}&4&4&3&6\\1&1&\underline{4}&3&4\\1&\underline{4}&5&3&5\\5&6&4&7&\underline{9}\\5&3&6&\underline{8}&3\end{pmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}3&1&2&1&0\end{matrix} \\ \begin{matrix}7\\6\\6\\5\\5\end{matrix} & \begin{pmatrix}7&8&\underline{9}&8&7\\8&7&6&\underline{7}&6\\9&6&5&4&\underline{6}\\\underline{8}&5&7&6&4\\7&\underline{6}&5&5&5\end{pmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}0&0&1&1&2\end{matrix} \\ \begin{matrix}3\\7\\3\\7\\4\end{matrix} & \begin{pmatrix}1&2&3&4&\underline{5}\\6&\underline{7}&8&7&2\\1&3&\underline{4}&4&5\\3&6&2&\underline{8}&7\\\underline{4}&1&3&5&4\end{pmatrix}\end{array}$$

Review of the "Hungarian Algorithm" for maximum weighted matching in the assignment problem. Find a feasible vertex labeling for the dual, i.e. weights $l(v)$ such that $l(x_i) + l(y_i) \geq w(ij)$. (This can be done by using the maximum in each row as the row label, with 0's for the columns.) Subtract out to find the "excess value" matrix $l(x_i) + l(y_j) - w(ij)$. Find a maximum matching and minimum cover in the equality subgraph (0's in the excess matrix). If this is a perfect matching, its value equals the dual value $\sum l(v)$ being minimized, hence is optimal. If not, let S be the set of rows not in the cover, T the set of columns in the cover, and ε the minimum excess value in the uncovered positions. Subtract ε from the row labels in S, add ε to the row labels in T, readjust the excess matrix, and iterate. (Note: any minimum cover can be used, and we know from the proof of the König–Egerváry Theorem that we can obtain a minimum cover by using $T \cup (X - S)$, where T and S are the subsets of Y and X reachable by alternating paths from the unsaturated vertices in the row-set X.)

If the matching was not complete, then $|S| > |T|$ and $\sum l(v)$ decreases, which guarantees the finiteness of the algorithm. The positions are of four types, corresponding to edges from S to T, $X-S$ to T, S to $Y-T$, and $X-S$ to $Y-T$. The change to the excess in the four cases is $0, +\varepsilon, -\varepsilon, 0$, respectively. Note that ε was defined to be the minimum excess corresponding to edges from S to $Y-T$, so every excess remains positive. For the first matrix above, the successive excess matrices computed in the algorithm could look like those below. These are not unique, because different matchings could be chosen in the equality subgraphs. The entries in the matching and the rows and columns in the cover ($X - S$ and T) are indicated with underscores.

$$
\begin{array}{c}
 \\ 6\\4\\5\\9\\ \underline{8}
\end{array}
\!\!\begin{array}{c}
\begin{array}{ccccc} 0 & 0 & \underline{0} & 0 & \underline{0}\end{array}\\
\begin{pmatrix} 2&2&2&3&0\\ 3&3&\underline{0}&1&0\\ 4&1&0&2&\underline{0}\\ 4&3&5&2&0\\ 3&5&2&\underline{0}&5 \end{pmatrix}
\end{array}
\rightarrow
\begin{array}{c}
 \\ 5\\ \underline{3}\\ \underline{4}\\8\\ \underline{8}
\end{array}
\!\!\begin{array}{c}
\begin{array}{ccccc} 0 & 0 & 1 & 0 & \underline{1}\end{array}\\
\begin{pmatrix} 1&1&2&2&0\\ 2&2&\underline{0}&0&0\\ 3&\underline{0}&0&1&0\\ 3&2&5&1&\underline{0}\\ 3&5&3&\underline{0}&6 \end{pmatrix}
\end{array}
\rightarrow
\begin{array}{c}
 \\ 4\\3\\4\\7\\8
\end{array}
\!\!\begin{array}{c}
\begin{array}{ccccc} 0 & 0 & 1 & 0 & 2\end{array}\\
\begin{pmatrix} \underline{0}&0&1&1&0\\ 2&2&\underline{0}&0&1\\ 3&\underline{0}&0&1&1\\ 2&1&4&0&\underline{0}\\ 3&5&3&\underline{0}&7 \end{pmatrix}
\end{array}
$$

3.2.6. *Finding a transversal of minimum weight.* Let the rows correspond to vertices $x_1, \ldots, x_5$, the columns to vertices $y_1, \ldots, y_5$, and let the weight of edge $x_i y_j$ be the value in position ij. Optimality of the answer can be proved by exhibiting an optimal matching and exhibiting a feasible labeling for the dual problem that has the same total value.

Alternatively, finding a minimum transversal is the same as finding a minimum weight perfect matching in the corresponding graph, which corresponds to a maximum weight matching in the weighted graph obtained by subtracting all the weights from a fixed constant. In the example given, we could subtract the weights from 13, and then the answer would be $5 \cdot 13$ minus the maximum weight of a transversal in the resulting matrix.

In the direct approach, the dual problem is to maximize $\sum l(v)$ subject to $l(x_i) + l(y_j) \leq w(x_i y_j)$. Subtracting the labels from the weights yields the "reduced cost" matrix. At each iteration, we determine the equality subgraph and ε as before, but this time add ε to the labels of vertices in S (rows not in the cover) and subtract ε from the labels of vertices in T (columns in the cover). Since $|S| > |T|$, $\sum l(v)$ increases. Every matching has weight at least $\sum l(v)$. When G_l contains a complete matching, $\min \sum w$ and $\max \sum l$ are attained and equal.

In the matrix below, the underscored positions form a minimum-weight transversal; the weight is 30. In the dual problem, the indicated labeling has total value 30, and the labels $l(x_i)$ and $l(y_j)$ sum to at most the matrix entry $w_{i,j}$. Hence these solutions are optimal.

$$
\begin{array}{c}
 \\ 4\\2\\4\\5\\4
\end{array}
\!\!\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 3 & 5 & 2\end{array}\\
\begin{pmatrix} \underline{4}&5&8&10&11\\ 7&6&\underline{5}&7&4\\ 8&\underline{5}&12&9&6\\ 6&6&13&10&\underline{7}\\ 4&5&7&\underline{9}&8 \end{pmatrix}
\end{array}
$$

3.2.7. *The Bus Driver Problem.* Bus drivers are paid overtime for the time by which their routes in a day exceed t. There are n bus drivers, n morning routes with durations $x_1, \ldots, x_n$, and n afternoon routes with durations $y_1, \ldots, y_n$. Assign to the edge $a_i b_j$ the weight $w_{i,j} = \max\{0, x_i + y_j - t\}$. The problem is then to find the perfect matching of minimum total weight. Index the morning runs so that $x_1 \geq \cdots \geq x_n$. Index the afternoon runs so that $y_1 \geq \cdots \geq y_n$. A feasible solution matches a_i to $b_{\sigma(i)}$ for some permutation σ of $[n]$. If there exists $i < j$ with $\sigma(i) > \sigma(j)$, then we have

$$\alpha = w_{i,\sigma(i)} + w_{j,\sigma(j)} = \max\{0, x_i + y_{\sigma(i)} - t\} + \max\{0, x_j + y_{\sigma(j)} - t\}$$

$$\beta = w_{i,\sigma(j)} + w_{j,\sigma(i)} = \max\{0, x_i + y_{\sigma(j)} - t\} + \max\{0, x_j + y_{\sigma(i)} - t\}$$

It suffices to prove that $\alpha \geq \beta$, because then there exists a minimizing permutation with no inversion. The nonzero terms in the maximizations have the same sum for each pair. Also,

$$x_i + y_{\sigma(i)} - t \geq x_i + y_{\sigma(j)} - t \geq x_j + y_{\sigma(j)} - t$$

$$x_i + y_{\sigma(i)} - t \geq x_j + y_{\sigma(i)} - t \geq x_j + y_{\sigma(j)} - t$$

If the central terms in the inequalities are both positive, then α is at least their sum, which equals β. If both are nonpositive, then $\alpha \ge 0 = \beta$. If the first is positive and the second nonpositive, then

$$\alpha = x_i + y_{\sigma(i)} - t \ge x_i + y_{\sigma(j)} - t = \beta.$$

If instead the second is positive, then

$$\alpha = x_i + y_{\sigma(i)} - t \ge x_j + y_{\sigma(i)} - t = \beta.$$

3.2.8. *When the weights in a matrix are the products of nonnegative numbers associated with the rows and columns, a maximum weight transversal is obtained by pairing the row having the kth largest row weight with the column having the kth largest column weight, for each k.* We show that all other pairings are nonoptimal. If the weights are not matched in order, then there exist indices i, j such that $a_i > a_j$ but the weight b matched with a_i is less than the weight b' matched with a_j. To show that switching these assignments increases the total weight, we compute

$$\begin{aligned} a_i b' + a_j b &= a_i b + a_i(b' - b) + a_j(b - b') + a_j b' \\ &= a_i b + a_j b' + (a_i - a_j)(b' - b) > a_i b + a_j b' \end{aligned}$$

When the weights in a matrix are the sums of nonnegative numbers associated with the rows and columns, every transversal has the same weight. Since a transversal uses one element in each row and each column, $w_{i,j} = a_i + b_j$ means that every transversal has total weight $\sum a_i + \sum b_j$.

3.2.9. *One-sided preferences.* There are k seminars and n students, each student to take one seminar. The ith seminar will have k_i students, where $\sum k_i = n$. Each student ranks the seminars; we seek a stable assignment where no two students can both improve by switching.

Form an X, Y-bigraph where X is the set of students and Y has k_i vertices for each vertex i. For each edge from student x_j to a vertex representing the ith seminar, let the weight be k minus the rank of the ith seminar in the preference of x_j.

A maximum weight matching in this weighting is a stable assignment, since if two students can both improve by trading assignments, then the result would be a matching of larger weight.

3.2.10. *Weighted preferences need not be stable.* Consider men $\{x_1, x_2, x_3\}$ and women $\{y_1, y_2, y_3\}$. Each assigns $3 - i$ points to the ith person in his or her preference list. Hence we indicate a preference order by a triple whose entries are $0, 1, 2$ in some order, with the position of the integer i being the index of the person of the opposite sex to whom this person assigns i points.

In the matrix below, the preference vectors of the men and women label the rows and coloumns, respectively. An entry in the matrix is the sum

of the points assigned to that potential edge by the two people. The underlined diagonal is the only matching that uses the maximum entry in each row and column, so it is the only maximum-weight matching. However, it is not a stable matching, because man x_1 and woman y_1 prefer each other to their assigned mates.

$$\begin{array}{c} \\ 120 \\ 021 \\ 120 \end{array} \begin{array}{c} \begin{array}{ccc} 120 & 120 & 210 \end{array} \\ \begin{pmatrix} 2 & 3 & 3 \\ 1 & 4 & 2 \\ 2 & 2 & 0 \end{pmatrix} \end{array}$$

The example can be extended for all larger numbers of men and women by adding pairs who are each other's first choice and are rated last by the six people in this example.

3.2.11. *In the result of the Gale-Shapley Proposal Algorithm with men proposing, every man receives a mate at least as high on his list as in any other stable matching.* We prove that under the G-S Algorithm with men proposing, no man is ever rejected by any woman who is matched to him in any stable matching. This yields the result, since each man's sequence of proposals proceeds downward from the top of his list, and he can only wind up with a woman less desirable than his most desirable match over all stable matchings if he is rejected by some women who matches him in some stable matching.

Consider the first time when some man x is rejected by a woman a to whom he is matched in some stable matching M. The rejection occurs because a has a proposal from a man y higher than x on her list. In M, man y is matched to some woman b. Since M is stable, y cannot prefer a to b. Thus b appears above a on the list for y. But now the decreasing property of proposals from men implies that y has proposed to b in the G-S Algorithm before proposing to a. If y is now proposing to a, then y was previously rejected by b. Since y is matched to b in M, this contradicts our hypothesis that a rejecting x was the first rejection involving a pair that occurs in some stable matching.

3.2.12. *The Stable Roommates Problem defined by the preference orderings below has no stable matching.* There are only three matchings to consider: $ab|cd$, $ac|bd$, and $ad|bc$. In each, two non-paired people prefer each other to their current roommates. The problematic pairs are $\{b, c\}$, $\{a, b\}$, $\{a, c\}$ in the three matchings, respectively.

$$\begin{aligned} a &: b > c > d \\ b &: c > a > d \\ c &: a > b > d \\ d &: a > b > c \end{aligned}$$

3.2.13. In the stable roommates problem, suppose that each individual declares a top portion of the preference list as "acceptable". Define the *acceptability graph* to be the graph whose vertices are the people and whose edges are the pairs of people who rank each other as acceptable. Prove that all sets of rankings with acceptability graph G lead to a stable matching if and only if G is bipartite. (Abeledo–Isaak [1991]).

In the stable roommates problem with each individual declaring a top portion of the preference list as "acceptable", and the acceptability graph being the graph on the people whose edges are the mutually acceptable pairs, all sets of rankings with acceptability graph G allow stable matchings if and only if G is bipartite. If G is bipartite, then we view the two partite sets as the two groups in the classical stable matching problem (isolated vertices may be added to make the partite sets have equal size). The unacceptable choices for an individual x may be put in any order, since they are all (equally) unacceptable, so we can ensure that all choices for x that are in the same partite set appear at the bottom of the preference order for x. In the outcome of the Gale-Shapley Proposal Algorithm, there is no pair (x, a) from opposite partite sets such that a and x prefer each other to their assigned mates. Also no x prefers an individual in its own partite set to the person assigned to x, since all individuals in its own partite set are unacceptable. Hence the stable matching produced for the bipartite version is also stable in the original problem.

If G is not bipartite, then G has an odd cycle $[x_1, \ldots, x_k]$. Define a set of rankings such that x_i prefers x_{i+1} to x_{i-1} (indices modulo k), and x_i prefers x_{i-1} to all others. The preferences of people not on the cycle are irrelevant. Since the cycle has odd length, the people on the cycle cannot be paired up using edges of the cycle. Given a candidate matching M, we may assume by symmetry that x_1 is not matched to x_2 or to x_k in M. Now x_1 prefers x_2 to M-mate of x_1, and x_2 prefers x_1 to the M-mate of x_2 (which might be x_3). Hence the matching M is not stable. Thus there is no stable matching for these preferences, which means that this acceptability graph does not always permit a stable matching.

3.2.14. *In the Proposal Algorithm with men proposing, no man is every rejected by all the women.*

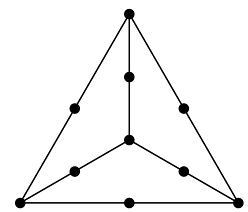
Proof 1. By Theorem 3.2.18, the Proposal Algorithm succeeds, so it ends with each men being accepted before being rejected by all women.

Proof 2. Once a woman has received a proposal, she thereafter receives a proposal on each round, since the key observation is her sequence of "maybe"s is nondecreasing in her list. If a round has j rejections and $n-j$ "maybe"s, then the $n-j$ unrejected men are distinct, since men propose to exactly one woman on each round.

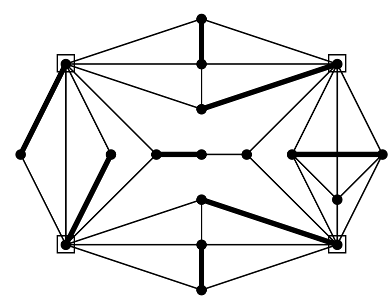
When a man has been rejected by k women, those k women have received proposals, and thereafter by the remarks above they always receive proposals from k distinct men. In particular, when a man has been rejected by $n-1$ women, on the next round they receive proposals from $n-1$ distinct men other than him, and he proposes to the remaining women, so the algorithm ends successfully on that step.

3.3. MATCHINGS IN GENERAL GRAPHS

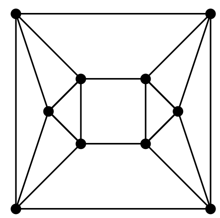
3.3.1. *The graph G below has no 1-factor.* Deleting the four vertices with degree 3 leaves six isolated vertices; thus $o(G-S) > |S|$ for this set S.



3.3.2. *The maximum size of a matching in the graph G below is 8.* A matching of size 8 is shown. Since $n(G) = 18$, it suffices to show that G has no perfect matching. For this we present a set S such that $o(G-S) > |S|$, violating Tutte's condition. Such a set S is marked. (Note: the smallest vertex cover has size 9, so duality using vertex cover is not adequate.)



3.3.3. *k-factors in the 4-regular graph below.* The full graph is a 4-factor, and the spanning subgraph with no edges is a 0-factor. There is a 2-factor consisting of the outer 4-cycle and the 6-cycle on the remaining vertices. Since these cycles have even length, taking alternating edges from both cycles yields a 1-factor. Deleting the edges of the 1-factor leaves a 3-factor.



3.3.4. *A k-regular bipartite graph is r-factorable if and only if r divides k.* The edges incident to a single vertex demonstrate necessity. For sufficiency, a k-regular bipartite graph has a perfect matching, and hence by induction on k is 1-factorable; take unions of the 1-factors in groups of r.

3.3.5. *Join of graphs G and H.* As long as G and H have at least one vertex each, $G \vee H$ is connected (it has $K_{n(G),n(H)}$ as a spanning subgraph).

In forming $G \vee H$, every vertex of G gains $n(H)$ neighbors in H, and every vertex in H gains $n(G)$ neighbors in G. Hence $\Delta(G \vee H) = \max\{\Delta(G) + n(H), \Delta(H) + n(G)\}$.

3.3.6. *A tree T has a perfect matching if and only if $o(T - v) = 1$ for every $v \in V(T)$. Necessity.* Let M be a perfect matching in T in which u is the vertex matched to v. Each component of $T - v$ not containing u must have a perfect matching and hence even order. The component containing u is matched by M except for u, so it has odd order.

Sufficiency. **Proof 1** (construction of matching). Suppose that $o(T - v) = 1$ for all $v \in V(T)$. Each vertex has a neighbor in one component of odd order. We claim that pairing each w to its neighbor in the odd component of $T - w$ yields a matching. It suffices to prove that if u is the neighbor of v in the unique odd component T_1 of $T - v$, then v is the neighbor of u in the unique odd component T_2 of $T - u$. Since $o(T - v) = 1$, the components of $T - v$ other than T_1 have even order. The subtree T_2 consists of these components and edges from these to v. Hence T_2 includes some even vertex sets and $\{v\}$, and T_2 thus has odd order.

Proof 2 (induction on $n(T)$). The claim is immediate for $n(T) = 2$. If $n(T) > 2$ and $o(T - v) = 1$ for all v, then the neighbor w of any leaf u has only one leaf neighbor. Let $T' = T - \{u, w\}$. The components of $T' - v$ are the same as the components of $T - v$, except that one of them in $T - v$ includes $\{u, w\}$ and the corresponding component of $T' - v$ omits them. Hence the parities are the same, and $o(T' - v) = 1$ for all $v \in V(T')$. By the induction hypothesis, T' has a perfect matching, and adding the edge uw to this completes a perfect matching in T.

(Comment: It is also possible to do the induction step by deleting an arbitrary vertex, but it is then a bit more involved to prove that every

component T' of the forest left by matching v to its neighbor in the odd component of $T - v$ satisfies the condition $o(T' - x) = 1$ for all x.

Proof 2a (induction and extremality). The basis again is $n(T) = 2$. For $n(T) > 2$, let P be a longest path. Let x be an endpoint of P, with neighbor y. Since $o(T - y) = 1$ and P is a longest path, $d_T(y) = 2$. Deleting x and y yields a tree T' such that $o(T' - v) = o(T - v) = 1$ for all $v \in V(T')$, since x and y lie in the same component of $T' - v$. Hence the induction hypothesis yields a perfect matching in T', which combines with xy to form a perfect matching in T.

Proof 3 (Tutte's Condition). By Tutte's Theorem, it suffices to prove for all $S \subseteq V(T)$ that $o(T - S) \le |S|$. We prove this by induction on $|S|$. Since $o(T - v) = 1$, we have $n(T)$ even, and hence $o(T - \varnothing) = 0$. When $|S| = 1$, the hypothesis $o(T - v) = 1$ yields the desired inequality for $S = \{v\}$.

For the induction step, suppose that $|S| > 1$. Let T' be the smallest subtree of T that contains all of S. Note that all leaves of T' are elements of S. Let v be a leaf of T', and let $S' = S - \{v\}$. By the induction hypothesis, $o(T - S') \le |S'| = |S| - 1$. It suffices to show that when we delete v from $T - S'$, the number of odd components increases by at most 1.

Let T'' be the component of $T - S'$ containing v. Deleting v from $T - S'$ replaces T'' with the components of $T'' - v$. We worry only if $T'' - v$ has at least two odd components. Since v is a leaf of T', all of S' lies in one component of $T - v$. Hence the components of $T'' - v$ are the same as the components of $T - v$ except for the one component of $T - v$ containing S'.

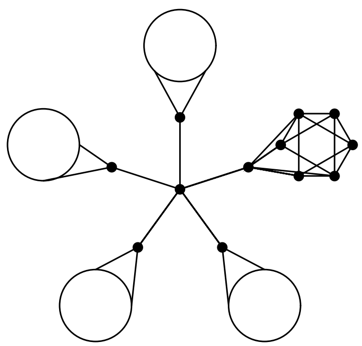
Since $o(T - v) = 1$, we can have two odd components in $T'' - v$ only if the one odd component of $T - v$ is a component of $T'' - v$ and the component of $T'' - v$ that is not a component of $T - v$ is also odd. Since the remaining components of $T'' - v$ are even, this means that T'' itself has odd order (it includes v and the two odd components of $T'' - v$). Therefore, the replacement of T'' with the components of $T'' - v$ increases the number of odd components only by one. We conclude that $o(T - S) \le o(T - S') + 1 \le |S'| + 1 = |S|$, which completes the induction step.

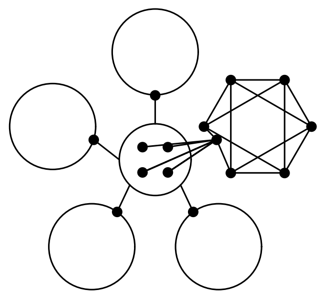
3.3.7. *There exist k-regular simple graphs with no perfect matching.* When k is even, K_{k+1} is a k-regular graph with no perfect matching, since it has an odd number of vertices. When k is odd, there are two usual types of constructions.

Construction 1. Begin with k disjoint copies of K_{k+1}. Delete $(k-1)/2$ disjoint edges from each copy, which drops the degree of $k - 1$ vertices in each copy to $k - 1$. Add a new vertex v_i to the ith copy, joining it to each of these vertices of degree $k - 1$. Add one final vertex x joined to $v_1, \ldots, v_k$. The graph has been constructed to be k-regular. Deleting x leaves k components of order $k + 2$ (odd); hence the graph fails Tutte's condition and has no perfect matching.

A slight variation is to start with k copies of $K_{k-1,k}$, add a matching of size $(k-1)/2$ to the larger side in each copy, and join the leftover vertices from each larger side to a final vertex x.

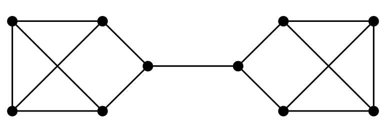
Construction 2. Begin with k disjoint copies of K_{k+1}. Subdivide one edge in each copy, which introduces k new vertices of degree 2. To raise their degree to k, add an independent set of $k-2$ additional vertices in the center joined to each of these k vertices. Deleting the $k-2$ vertices in the center violates Tutte's condition.





3.3.8. *No graph with a cut-vertex is 1-factorable.* Suppose v is a cut-vertex of G. If G is 1-factorable, then G has even order, and $G-v$ has a component H of odd order. For any 1-factor using an edge incident to v whose other endpoint is not in H, the vertices of H cannot all be matched. The contradiction implies there is no 1-factorization.

A 3-regular simple graph having a 1-factor and connectivity 1.



3.3.9. *Every graph G with no isolated vertices has a matching of size at least $n(G)/(1+\Delta(G))$.* We use induction on the number of edges. In the induction step, we will delete an edge whose endpoints have degree at least 2 (other edge deletions would isolate a vertex). This tells us what we need to cover in the basis step.

Basis step: every edge of G is incident to a vertex of degree 1. In such a graph, every component has at most one vertex of degree exceeding 1, and thus each component is a star. We form a matching using one edge from each component. Since the number of vertices in each component is 1

plus the degree of the central vertex, the number of components is at least $n(G)/(1+\Delta(G))$.

Induction step: G has an edge e whose endpoints have degree at least 2. Since $G' = G - e$ has no isolated vertex, we can apply the induction hypothesis to obtain $\alpha'(G) \ge \alpha'(G') \ge n(G')/(1+\Delta(G')) \ge n(G)/(1+\Delta(G))$.

3.3.10. *The maximum possible value of $\beta(G)$ in terms of $\alpha'(G)$ is $2\alpha'(G)$.* If G has a maximal matching of size k, then the $2k$ endpoints of these edges form a set of vertices covering the edges, because any uncovered edge could be added to the matching. Hence $\beta(G) \le 2\alpha'(G)$. A graph consisting of k disjoint triangles has $\alpha' = k$ and $\beta = 2k$, so the inequality is best possible. These values also hold for the graph K_{2k+1}, since we cannot omit two vertices from a vertex cover of K_{2k+1}. More generally, every disjoint union of cliques of odd order satisfies $\beta(G) = 2\alpha'(G)$.

3.3.11. *A graph G has a matching that saturates a set $T \subseteq V(G)$ if and only if for all $S \subseteq V(G)$, the number of odd components of $G-S$ contained in $G[T]$ is at most $|S|$.*

Necessity. Saturating T requires saturating each vertex in the odd components of $G[T]$, which uses a vertex of S for each such component.

Sufficiency. Form G' by adding a set U of $n(G)$ new vertices adjacent to each other and to every vertex of $G-T$. We claim that G' satisfies Tutte's Condition. Each $S' \subseteq V(G')$ that contains all of U has size at least $n(G)$. Since $G'-S'$ has at most $n(G)$ vertices, it has at most $|S'|$ odd components.

When $U \not\subseteq S'$, what remains of G' outside of T is a single component. Letting $S = V(G) \cap S'$, the number of odd components in $G'-S'$ is thus at most one more than the number of odd components of $G-S$ contained in T. This yields $o(G'-S') \le |S'| + 1$, which suffices since $n(G')$ is even.

Since G' satisfies Tutte's Condition, G' has a perfect matching. The edges used to saturate T all lie in G, since no edges were added from T to U. Hence these edges form a matching in G that saturates T.

3.3.12. *Extension of König–Egerváry to general graphs.* A *generalized cover* of G is a collection of vertex subsets $S_1, \dots, S_k$ and T such that each S_i has odd size and every edge of G has one endpoint in T or both endpoints in some S_i. The *weight* of a generalized cover is $|T| + \sum(|S_i|-1)/2$.

The minimum weight $\beta^*(G)$ of a generalized cover equals the maximum size $\alpha'(G)$ of a matching. Always $\alpha'(G) \le \beta^*(G)$, because a matching uses at most $(|S_i|-1)/2$ edges within S_i and at most $|T|$ edges incident to T, and there are no edges not of this type when $S_1, \dots, S_k$ and T form a generalized cover. For equality, it suffices to exhibit a generalized cover with weight equal to $\alpha'(G)$.

The Berge–Tutte formula says that $2\alpha'(G) = \min_T\{n - d(T)\}$, where $d(T) = o(G-T) - |T|$ is the deficiency of a vertex set T. Let T be a *maximal* set among those having maximum deficiency. For this choice of T, there are no components of (positive) even order in $G-T$, since we could add to T a leaf of a spanning tree of such a component to obtain a larger set T' with the same deficiency. Let $S_1, \ldots, S_k$ be the vertex sets of the components of $G-T$. By construction, this is a generalized cover. Because $2\alpha'(G) = n - d(T)$, we have $k = |T| + d(T)$. Thus

$$\beta^*(G) = |T| + \sum_{i=1}^k (|S_i| - 1)/2 = (n + |T| - k)/2 = \alpha'(G).$$

3.3.13. *Proof of Tutte's Theorem from Hall's Theorem.* Given a graph G such that $o(G-S) \le |S|$ for all $S \subseteq V(G)$, we prove that G has a perfect matching. Let T be a maximal vertex subset such that $o(G-T) = |T|$.

a) Every component of $G-T$ is odd, and $T \ne \varnothing$. If $G-T$ has an even component C, then let v be a leaf of a spanning tree of C. Now $|T \cup \{v\}| = |T| + 1 = o(G-T) + 1 = o(G-(T\cup\{v\}))$, which contradicts the maximality of T. Thus $G-T$ has no even components.

Since $o(G-\varnothing) \le 0$, the graph G has no odd components. Since $G-T$ has no even components, we have $|T| > 0$, and $G-T$ is smaller than G.

b) If C is a component of $G-T$, then Tutte's Condition holds for every subgraph of C obtained by deleting one vertex. Since $C-x$ has even order, a violation requires $o(C-x-S) \ge |S| + 2$. Adding this inequality to $|T| = o(G-T)$ and $|\{x\}| = 1$ yields

$$|T \cup x \cup S| \le o(G-T) - 1 + o(C-x-S) = o(G-T-x-S),$$

which contradicts the maximality of T.

c) The bipartite graph H formed from G by contracting the components of $G-T$ (and deleting edges within T) satisfies Hall's Condition for a matching that saturates the partite set opposite T. There is an edge from a vertex $t \in T$ to a component C of $G-T$ if and only if $N_G(t)$ contains a vertex of C. For $A \subset \mathbf{C}$, let $B = N_H(A)$. The elements of A are odd components of $G-B$; hence $|A| \le o(G-B)$. Since Tutte's condition yields $o(G-B) \le |B|$, we have $|N_H(A)| \ge |A|$.

d) The final proof. By Hall's Theorem and part (c), H has a matching that saturates $\mathbf{C}$. This matching yields $o(G-T) = |T|$ pairwise disjoint edges from odd components of $G-T$ to T. By part (a), these are all the components of $G-T$ These edges saturates one vertex from each component of $G-T$. By part (c) and the induction hypothesis, the vertices remaining in each component of $G-T$ are saturated by a perfect matching of that subgraph. The union of the matchings created is a perfect matching of G.

3.3.14. *If G is a simple graph with $\delta(G) \ge k$ and $n(G) \ge 2k$, then $\alpha'(G) \ge k$.* Let $n = n(G)$. By the Berge–Tutte Formula, it suffices to show that the deficiency $o(G-S) - |S|$ is at most $n - 2k$ for every $S \subseteq V(G)$. We prove this by contradiction; suppose that $o(G-S) - |S| > n - 2k$.

Let $s = |S|$. We have $o(G-S) > n - 2k + s$. Thus there are more than $n - 2k + s$ vertices outside S. Together with S, we have $n > n - 2k + 2s$. Thus $s < k$. With $s < k$, a vertex outside S has fewer than k neighbors in S, and $\delta(G) \ge k$ implies that no odd components of $G-S$ are single vertices.

Indeed, every component of $G-S$ has at least $1 + k - s$ vertices. Thus we can improve our earlier inequality: $(1+k-s)(n-2k+s+1) + s \le n$. This simplifies to $(k-s)(n-2k+s-1) < 0$. Since $n \ge 2k$, both factors on the left are positive, which yields a contradiction.

3.3.15. *Every 3-regular graph G with at most two cut-edges has a 1-factor.* Since G has at most two cut-edges, at most two odd components of $G-S$ have one edge to S; the remainder have at least three edges to S (using the parity of degrees). With $\left|[S, \overline{S}]\right| = m$, this yields $3|S| \ge m \ge 3o(G-S) - 4$. Thus $|S| \ge o(G-S) - 4/3$. Since $n(G)$ is even, $|S|$ and $o(G-S)$ have the same parity, which means that $o(G-S)$ exceeds $|S|$ only if is greater by at least 2. This contradicts $o(G-S) \le |S| + 4/3$. Hence Tutte's condition holds, and Tutte's Theorem implies that G has a 1-factor.

3.3.16. *If G is k-regular and remains connected when any $k-2$ edges are deleted, then G has a 1-factor.* By Tutte's Theorem, it suffices to show that $o(G-S) \le |S|$ for every $S \subseteq V(G)$. This follows for $S = \varnothing$ from the assumption that $n(G)$ is even; hence we may assume that $S \ne \varnothing$. Let H be an odd component of $G-S$, and let m be the number of edges joining H to S. In the subgraph H, the sum of the degrees is $kn(H) - m$. Since this must be even and $n(H)$ is odd, k and m must have the same parity.

By the hypothesis, there are at least $k-1$ edges between H and S. The requirement of equal parity thus yields $m \ge k$. Summing over all odd components of $G-S$ yields at least $k \cdot o(G-S)$ edges between S and $V(G) - S$. Since the degree sum of the vertices in S is exactly $k|S|$, we obtain $k \cdot o(G-S) \le k|S|$, or $o(G-S) \le |S|$.

3.3.17. *Under the conditions of Exercise 3.3.16, each edge belongs to some 1-factor in G.* We want to show that $G' = G - x - y$ has a 1-factor. By Tutte's Theorem, since G has even order, it suffices to show that $o(G'-S') \le \left|S'\right| + 1$ for all $S' \subset V(G')$. Equivalently, $o(G-S) \le |S| - 1$ for all $S \subseteq V(G)$ that contain $\{x, y\}$.

Let l be the number of edges between S and an odd component H of $G-S$; the hypothesis yields $l \ge k-1$. The sum $kn(H) - m$ of the vertex degrees in H must be even, but $n(H)$ is odd, so k and m must have the same parity; we conclude that $l \ge k$. Summing over all odd components of $G-S$, we have $m \ge k \cdot o(G-S)$, where m is the number of edges between S and the rest of the graph. Since G is k-regular and $G[S]$ contains the edge xy, we have $m \le k|S| - 2$. Thus $o(G-S) \le |S| - 2/k$. Since $o(G-S)$ and $|S|$ are integers, we have the needed inequality.

3.3.18. *Construction of a k-regular graph with no 1-factor (when k is odd), such that deleting any $k-3$ edges leaves a connected graph.* We make the graph simple and connected under the deletion of any $k-3$ vertices, which is a stronger requirement.

The Tutte set S will have size $k-2$, leaving k components in $G-S$. Each component of $G-S$ consists of $K_{k-2,k-1}$ plus a cycle added through the vertices in the larger partite set. This gives those vertices degree k. Add a matching from the vertices in the smaller partite set to S. Now G is k-regular and has a Tutte set, so it has no 1-factor.

When any $k-3$ vertices are deleted to form G', what remains in each component H of $G-S$ is connected, due to the spanning biclique. Also some edge from H to S remains. If exactly one vertex of S remains, then G' is now connected. If more than one vertex of S remains, then any two are connected in G' by a path through some component of $G-S$.

3.3.19. *Every 3-regular simple graph with no cut-edge decomposes into copies of P_4 (the 4-vertex path).* By Petersen's Theorem, a 3-regular simple graph G with no cut-edge has a 1-factor M. Deleting the edges of M from G leaves a 2-factor H, which is a disjoint union of cycles. Choose an orientation D for the 2-factor so that each vertex has one edge in and one edge out (that is, make the components of the 2-factor into directed cycles).

We let each edge of M be the central edge in a copy of P_4. The other two edges in the copy of P_4 containing $uv \in M$ are the edges leaving u and v in D; let these be uw and vz. These three edges form P_4 if $w \neq z$. We have $w \neq z$ because each vertex has only one entering edge in D.

The central edges of these P_4's are precisely the $n(G)/2$ edges of M. Each edge of H appears in exactly one of the constructed P_4's, since every edge outside M follows exactly one vertex in D. Thus the copies of P_4 formed in this way are pairwise edge-disjoint and cover $E(G)$.

3.3.20. *A 3-regular simple graph G has a 1-factor if and only if it decomposes into copies of P_4.*

Necessity. Deleting the edges of a 1-factor M from G leaves a 2-factor H, which is a disjoint union of cycles. Choose an orientation D for the 2-factor by choosing a consistent orientation around each cycle.

Let each edge of M be the central edge in a copy of P_4. The other two edges in the copy of P_4 containing $uv \in M$ are the edges leaving u and v in D; let these be uw and vz. These three edges form P_4 when $w \neq z$, which holds since each vertex has only one entering edge in D.

The central edges of these P_4s are the $n(G)/2$ edges of M. Each edge of the 2-factor also appears in exactly one constructed P_4, since each such edge is the tail of exactly one vertex in D. Thus these copies of P_4 are pairwise edge-disjoint and cover $E(G)$.

Sufficiency. A P_4-decomposition of a 3-regular graph G has $n(G)/2$ subgraphs, since $e(G) = 3n(G)/2$ and $e(P_4) = 3$. No edge-disjoint copies of P_4 have a common internal vertex v, since that would give v degree at least 4. Hence the middle edges in the subgraphs of the decomposition form a matching of size $n(G)/2$ and hence a 1-factor.

3.3.21. *If G is a $2m$-regular graph, and T is a tree with m edges and diameter less than the girth of G, then G decomposes into copies of T.* We prove a stronger result. Consider an arbitrary labeling of $V(T)$ with $\{1, \ldots, m+1\}$. We prove by induction on m that G has a T-decomposition such that each vertex of G appears in $m+1$ copies of T, once with each label. Call this a *labeled T-decomposition* of G. The trivial necessary degree conditions are satisfied because the sum of the vertex degrees in T is $2m$. There will be $n(G)$ copies of T, independent of the value of m.

For $m = 0$ (or $m = 1$), the claim is immediate. For $m \geq 1$, let H be a 2-factor of G, and let i be a leaf of T, with neighbor j. Note that the distance in T from j to any other vertex of T is at most $\operatorname{diam}(T) - 1$. The induction hypothesis guarantees a labeled $T-i$-decomposition of $G-E(H)$. For each vertex w in each cycle in H, we add to the copy of $T-i$ with j at w by adding the edge to the next vertex in the cycle, which will then receive label i. This vertex does not already appear in this copy of $T-i$, because the girth of G exceeds the diameter of T.

3.3.22. *Hall's Theorem follows from Tutte's Theorem.* Given an X,Y-bigraph G, let H be the graph obtained from G by adding one vertex to Y if $n(G)$ is odd and then adding edges to turn Y into a clique.

a) G has a matching of size $|X|$ if and only if H has a 1-factor. Each edge of a matching in G has one vertex of X and one vertex of Y. Since $H[Y]$ is a clique, we can pair the remaining vertices arbitrarily to obtain a 1-factor in H from a matching of size $|X|$ in G. Conversely, if H has a 1-factor, it must use $|X|$ edges to saturate X, since $H[X]$ is an independent set. These edges from the desired matching in G.

b) If G satisfies Hall's Condition ($|N(S)| \geq |S|$ for all $S \subseteq X$), then H satisfies Tutte's Condition ($o(H-T) \leq |T|$ for all $T \subseteq V(H)$). Since $H[Y \cap T]$ is a clique, the odd components obtained by deleting T are the vertices of X whose neighbors all lie in T and perhaps the one large remaining component. Let $S = \{x \in X \colon N(x) \subseteq T \cap Y\}$. Since G satisfies Hall's Condition, $|S| \leq |T \cap Y| \leq |T|$. Thus $o(H-T) \leq |T| + 1$. Since $n(H)$ is even, $o(H-T)$ and $|T|$ have the same parity, and thus $o(H-T) \leq |T|$. Thus H satisfies Tutte's Condition.

c) Tutte's Theorem implies Hall's Theorem. The necessity of Hall's condition is immediate (any subset of X must have as many neighbors as elements to be completely matched). For sufficiency as a consequence of

Tutte's Theorem, we form H from G as described above. Since G satisfies Hall's Condition, part (b) implies that H satisfies Tutte's Condition. Tutte's Theorem now implies that H has a 1-factor. Part (a) now implies that G has a matching saturating X.

3.3.23. Consider a connected claw-free graph G of even order.

a) In a spanning tree of G generated by Breadth-First Search, any two vertices with a common parent other than the root are adjacent. Let r be the root, and let u, v be children of s, with $s \neq r$. In a Breadth-First Search tree, the path from the root to any vertex is a shortest such path. We have $d(u,r) = d(v,r) = d(s,r)+1$. An edge from u or v to the parent t of s would establish a shorter path to the root of length $d(s,r)$. Hence there is no such edge. To avoid having s, t, u, v induce a claw, u and v must be adjacent.

b) G has a 1-factor. We use induction on $n(G)$. Basis step ($n(G) = 2$): G must be K_2.

Induction step ($n(G) > 2$): Let T be a Breadth-First Search tree from r. Let u be a vertex at maximum distance from r, and let s be the parent of v. If s has no other child, then $T' = T - \{u, s\}$ is connected. if s has another child v, then let $T' = T - \{u, v\}$.

In each case, T' is connected, so $G'[V(T')]$ is a smaller connected claw-free graph of even order. The induction hypothesis guarantees a perfect matching in G'. To this matching we add the edge between the two vertices we deleted to obtain G' (in the first case, the edge us exists because s is the parent of u; in the second case, the edge uv exists by part (a).)

3.3.24. *Maximum number of edges with no 1-factor. A maximal n-vertex graph with no 1-factor consists of m vertices of degree $n-1$, with the remaining $n-m$ vertices inducing a union of $m+2$ cliques of odd order.* Since adding an edge cannot reduce α' or increase it by more than one, we may assume that $\alpha'(G) = n/2 - 1$. Hence $\max_{S\subseteq V}\{o(G-S) - |S|\} = 2$; the maximum matching omits 2 vertices. Let S be a set achieving equality, so $o(G-S) = |S|+2$. Each component of $G-S$ must induce a clique, $G-S$ has no component of even order (else add edges from even to odd components), and vertices of S have degree $n-1$, all because adding the edges that would be missing if any of these failed would not reduce the deficiency of S. This completely describes the maximal graphs.

The maximum number of edges in a graph with minimum degree $k < n/6 - 2$ and no 1-factor is $\binom{k}{2} + k(n-k) + \binom{n-2k-1}{2}$. We assume n is even. Let G be a maximal n-vertex graph with no 1-factor. Let $f(k) = \binom{k}{2} + k(n-k) + \binom{n-2k-1}{2}$. We first observe that there is a maximal graph having no 1-factor that has $f(k)$ edges and minimum degree k; the graph is $K_k \vee ((k+1)K_1 + K_{n-2k-1})$. This is an example of the structure above with $m = k$. This construction is valid when $n - 2k - 1 > 0$, which requires only $k \le n/2 - 1$. Nevertheless, $f(k)$ is not always the maximum size of a graph H with minimum degree k and no 1-factor. For $n \ge 8$, we can build counterexamples when $k < n/2 - 1$ and k is at least $n/6$ (approximately). The smallest counterexample occurs when $n = 8$ and $k = 2$. We have $f(2) = 16$ and $f(3) = 18$. We obtain a graph with minimum degree 2, no 1-factor, and 17 edges by deleting one edge from $K_3 \vee (5K_1)$.

Suppose H has a vertex of degree $k < n/6-2$ and has no perfect matching; we obtain an upper bound on $e(H)$. Augment H by adding edges to obtain a maximal supergraph G having no 1-factor; note that $\delta(G) = l \ge k$. By direct computation, $f(t+1) - f(t) = 3t - n + 4$. If $t \ge n/2$, then there is no graph with minimum degree at least t that has no 1-factor. Since $f(t)$ is a parabola centered at $t = (n-4)/3$ and $\delta(G) < n/2$, we have $f(k) \ge f(l)$ if $k < n/6 - 2$. Therefore, it suffices to prove that if $\delta(G) = k$ and G has the form described above, then $e(G) \le f(k)$.

If v does not have degree $n-1$, then v belongs to a clique of size $d(v) - m + 1$ in $G - S$, which is odd. If any two components of $G-S$ have sizes $p \ge q \ge k - m + 3$, then we gain $2p$ and lose $2q - 4$ edges by moving two vertices from the smaller to the larger clique in $G - S$, still maintaining the same minimum degree. Hence for fixed m, $e(G)$ is maximized by using cliques of size $k - m + 1$ for all but one component of $G - S$. Now the degree sum in G is $m(n-1) + (k-m+1)(m+1)k + [n-(k-m+1)(m+1)][n-(k-m+1)(m+1)]$. If $m < k/2$, then replacing m by $k - m$ increases this, since only the first term changes. Hence we may assume $m \ge k/2$. If $k > m \ge k/2 - 1$, then we can increase m by moving two vertices from a small clique to S, since $2(m+1) \ge 2(k-m-1)$ guarantees that we can then take vertices from the other small components to make new components of size $k - m - 1$. This increases the number of edges (computation omitted), so we may assume $m = k$. Now $e(G) = f(k)$.

3.3.25. *A graph G is factor-critical if and only if $n(G)$ is odd and $o(G-S) \le |S|$ for all nonempty $S \subseteq V(G)$. Necessity.* Factor-critical graphs are those where every subgraph obtained by deleting one vertex has a 1-factor. Thus factor-critical graphs have odd order. Given a nonempty subset S of $V(G)$, let v be a member of S, and let $G' = G - v$. Since G' has a 1-factor and $G - S = G' - S'$, we have $o(G-S) = o(G'-S') \le |S'| = |S| - 1$. Thus the desired inequality holds for S.

Sufficiency. Suppose that $n(G)$ is odd and $o(G-S) \le |S|$ for all nonempty $S \subseteq V(G)$. Given a vertex $v \in V(G)$, let $G' = G - v$. Consider $S' \subseteq V(G')$, and let $S = S' \cup \{v\}$. Since G has odd order, the quantities $|S|$ and $o(G-S)$ have different parity. The hypothesis thus yields $o(G-S) \le |S| - 1$. Since $G' - S' = G - S$, we have $o(G'-S') = o(G-S) \le |S| - 1 = |S'|$. By Tutte's Theorem, $G - v$ has a 1-factor.

3.3.26. *If M is a matching in a graph G, and u is an M-unsaturated vertex, and G has no M-augmenting path that starts at u, then u is unsaturated in some maximum matching in G.* We use induction on the difference between $|M|$ and $\alpha'(G)$. If the difference is 0, then already u is unsaturated in some maximum matching. If M is not a maximum matching, then there is an M-augmenting path P. Since u is not an endpoint of P, then $M \triangle E(P)$ is a larger matching M' that does not saturate u.

If no M'-augmenting path starts at u, then by the induction hypothesis u is unsaturated in some maximum matching. Suppose that an M'-augmenting path P' starts at u. Since no M-augmenting path starts at u, P' shares an edge with P. Form an M-alternating path by following P' from u until it first reaches a vertex of P; this uses no edges of $M' - M$. Then follow P to the end in whichever direction continues the M-alternating path. Since both endpoints of P are M-unsaturated, this completes an M-augmenting path starting at u, which contradicts the hypothesis.

3.3.27. *Proof of Tutte's 1-Factor Theorem from correctness of the Blosson Algorithm.*

a) If G has no perfect matching, and M is a maximum matching in G, and S and T are the sets generated when running the Blossom Algorithm from u, then $|T| < |S| \le o(G - T)$. Since M is a maximum matching, no M-augmenting path is found. At the start of the algorithm, $|T| = 0 < 1 = |S|$. As the algorithm proceeds, exploring a vertex $v \in S$ leads to consideration of a neighbor y of v that is not in T. If y is unsaturated in M, then an M-augmenting path is found; this case is forbidden by hypothesis. If y is saturated but not yet in S, then y is added to T and its mate is added to S. This augments both T and S by one element, maintaining $|S| = |T| + 1$.

Finally, if y is saturated and lies in S, then a blossom is established. Every matched edge in the blossom has one vertex of T and one vertex of S, and the blossom is shrunk into the single vertex of S on it that is not matched to another vertex along the blossom. Hence the "current" S and T shrink by the same number of vertices. Therefore, we always maintain $|S| = |T| + 1$.

Now consider the second inequality. When the algorithm ends, the current S is an independent set with no edges to vertices that have not been reached, because if an edge has been found between vertices of S, then a blossom would have been shrunk, and an edge to an unreached vertex would have been explored. Hence deleting T leaves at least $|S|$ odd components. (There may be more among the unreached vertices; the algorithm does not explore unsaturated edges from T.)

These isolated vertices in the final S correspond to odd components of $G - T$ in the original graph, because shrinking of a blossom loses an even number of vertices. Since all edges leaving the blossom are explored, the final T disconnects everything in the blossom from what is outside of it. Since also the blossom corresponds to an odd number of vertices in the original graph, deleting T from the original graph must leave an odd component among these vertices. One such set exists for each vertex in the final S (at the end of the algorithm).

b) Proof of Tutte's 1-Factor Theorem. If $o(G - T) \le |T|$ for every vertex subset T, then the algorithm cannot end by finding a set T such that $|T| < |S| \le o(G - T)$. Hence it can only end by finding an augmentation, and this continues until a 1-factor is found.

3.3.28. *a) Reduction of the f-factor problem to the f-solubility problem.* It suffices to prove that G has an f-factor if and only if the graph H obtained by replacing each edge by a path of length 3 is f'-soluble, where f' is the extension of f obtained by defining f' to equal 1 on all the new vertices.

Suppose that x, a, b, y is the path in H representing the edge xy in G. If G has an f-factor, then H is f'-soluble by giving weights $1, 0, 1$ or $0, 1, 0$ to the successive edges on the path depending on whether xy is or is not in the f-factor.

Conversely, if H is f'-soluble, then because every edge of H is incident to a vertex with $f' = 1$, every edge is used with weight 1 or 0, and the weights along the 3-edge path representing an edge xy must be $1, 0, 1$ or $0, 1, 0$. At a vertex x there must be exactly $f(x)$ paths of the first type, so we obtain an f-factor of G by using the edges corresponding to these paths.

b) Reduction of f-solubility to 1-factor. Let H be the graph formed from G by replacing each vertex $v \in V(G)$ with an independent set of $f(v)$ vertices. Now G is f-soluble if and only if H has a 1-factor; collapsing or expanding the vertices turns the solution of one problem into the solution of the other.

3.3.29. *Tutte's f-factor condition and graphic sequences.* For disjoint sets Q, T, let $e(Q, T)$ denote the number of edges from Q to T. For a function h defined on $V(G)$, let $h(S) = \sum_{v \in S} h(v)$ for $S \subseteq V(G)$.

For $f \colon V(G) \to \mathbb{N}_0$, Tutte proved that G has an f-factor if and only if

$$q(S, T) + f(T) - d_{G-S}(T) \le f(S)$$

for all choices of disjoint subsets $S, T \subseteq V(G)$, where $q(S, T)$ is the number of components Q of $G - S - T$ such that $e(Q, T) + f(V(Q))$ is odd. *a)* **The Parity Lemma**. *The quantity $\delta(S, T)$ has the same parity as $f(V)$ for disjoint sets $S, T \subseteq V(G)$, where $\delta(S, T) = f(S) - f(T) + d_{G-S}(T) - q(S, T)$.* We use the observation that the parity of the number of odd values in a set of integers equals the parity of the sum of the set.

We fix S and use induction on $|T|$. When $T = \varnothing$, we have $f(T) - d_{G-S}(T) = 0$. Also $m(Q, T) = 0$ for each component Q of $G - S - T$, so we can sum over the components to obtain $q(S, \varnothing) \equiv f(\overline{S}) \pmod 2$. Signs don't matter, so $\delta(S, \varnothing) \equiv f(\overline{S}) - f(S) \equiv f(V(G)) \pmod 2$.

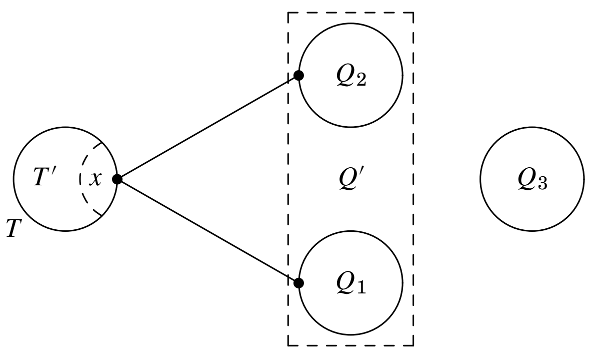
For $T \neq \varnothing$, we compare $\delta(S, T)$ and $\delta(S, T - x)$; it suffices to show that the difference is even. Let $T' = T - x$. In computing the difference, the contributions of $-f(S)$ cancel, as do the sums over T'. This leaves

$$\delta(S, T) - \delta(S, T') = q(S, T) - q(S, T') + f(x) - d_{G-S}(x).$$

The contributions to $q(S, T)$ and $q(S, T')$ from components of $G - S - T$ having no neighbors of x also cancel. The components having neighbors of x combine with x to form one large component in $G - S - T'$ with vertex set $Q' \cup \{x\}$. Our initial observation about parity yields

$$q(S, T) - q(S, T') \equiv f(Q') + m(Q', T) - [f(Q' \cup \{x\}) + m(Q' \cup \{x\}, T')] \pmod 2.$$

The edges from x to Q' count in $m(Q', T)$, the edges from x to T' count in $m(Q' \cup \{x\}, T')$, and the edges from Q' to T' count in both. Thus $m(Q', T) - m(Q' \cup \{x\}, T') \equiv d_{G-S}(x) \pmod 2$, and we have $q(S, T) - q(S, T') \equiv f(x) + d_{G-S}(x) \pmod 2$. This has the same parity as the rest of the difference, yielding $\delta(S, T) - \delta(S, T') \equiv 0 \pmod 2$.



b) Let $d_1, \ldots, d_n$ be nonnegative integers with $\sum d_i$ even and $d_1 \geq \cdots \geq d_n$. If $G = K_n$ and $f(v_i) = d_i$, then G has an f-factor if and only if $\sum_{i=1}^{k} d_i \leq (n - 1 - s)k + \sum_{i=n+1-s}^{n} d_i$ for all k, s with $k + s \leq n$. Such an f-factor exists if and only if $f(V) = \sum d_i$ is even and Tutte's condition holds. Since $d_{K_n - S}(v) = n - 1 - |S|$ for all $v \in T$, the f-factor condition requires that $f(T) \leq f(S) + (n - 1 - |S|)\,|T| - q(S, T)$ for any disjoint sets $S, T \subseteq V(G)$. With $|T| = k$ and $|S| = s$, the condition becomes $f(T) \leq f(S) + (n - 1 - s)k - q(S, T)$.

Necessity. Applied with the k vertices of largest degree in T and the s

vertices of smallest degree in S and using $q(S, T) \geq 0$, the f-factor condition yields the desired inequality.

Sufficiency. It suffices to establish inequality shown above to be equivalent to the f-factor condition. Since $K_n - S - T$ is connected, always $q(S, T) \leq 1$. Since $\sum f(v_i)$ is even, the two sides of the inequality have the same parity, by the Parity Lemma. It therefore suffices to prove that $f(T) \leq f(S) + (n - 1 - s)k$ when S and T are disjoint. With $s = |S|$ and $k = |T|$, it suffices to prove the inquality when T consists of the k vertices of largest degree and S consists of the s vertices of smallest degree. It then becomes the given inequality for $d_1, \ldots, d_n$.

c) Nonnegative integers $d_1, \ldots, d_n$ with $d_1 \geq \cdots \geq d_n$ are the vertex degrees of a simple graph if and only if $\sum d_i$ is even and $\sum_{i=1}^{k} d_i \leq k(k - 1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ for $1 \leq k \leq n$. Any realization can be viewed as an f-factor of K_n, where $f(v_i) = d_i$. Thus it suffice to show that this condition is equivalent to the condition in part (b).

For fixed $|S|, |T|$, this inequality is always satisfied if and only if it is satisfied when $T = \{x_i, \ldots, x_k\}$ and $S = \{x_{n+1-s}, \ldots, x_n\}$, in which case it becomes $\sum_{i=1}^{k} f_i \leq \sum_{i=n+1-s}^{n} f_i + (n - 1 - s)k = (n - 1)k + \sum_{i=n+1-s}^{n} (f_i - k)$. This is always satisfied if and only if it is satisfied when the right side attains its minimum over $0 \leq s \leq n - k$, which happens when $n + 1 - s = \min\{i\colon i > k \text{ and } f_i < k\}$. Since $(n - 1)k = k(k - 1) + (n - k)k$, the value is then $k(k - 1) + \sum_{i=k+1}^{n} \min\{k, f_i\}$.

4.CONNECTIVITY AND PATHS

4.1. CUTS AND CONNECTIVITY

4.1.1. *Statements about connectedness.*

a) Every graph with connectivity 4 is 2-connected—TRUE. If the minimum number of vertices whose deletion disconnects G is 4, then deletion of fewer than two vertices leaves G connected. Also, K_5 is 2-connected.

b) Every 3-connected graph has connectivity 3—FALSE. Every graph with connectivity greater than 3, such as $K_{4,4}$, is 3-connected.

c) Every k-connected graph is k-edge-connected—TRUE. Always $\kappa'(G) \geq \kappa(G)$, which means that every disconnecting set of edges in a k-connected graph has size at least k.

d) Every k-edge-connected graph is k-connected—FALSE. The graph consisting of two $k+1$-cliques sharing a single vertex ($K_1 \vee 2K_k$) is k-edge-connected but not k-connected.

4.1.2. *If e is a cut-edge of G, then e contains a cut-vertex of G unless e is a component of G.* If e is a component of G, then the vertices of e are not cut-vertices; deleting a vertex of degree 1 cannot disconnect a graph. Otherwise, e has an endpoint v with degree greater than one; we claim v is a cut-vertex. Let u be the other endpoint of e, and let w be another neighbor of v. Then G has a u,w-path through v, but every u,w path in G uses e, so G has no u,w-path in $G-v$.

4.1.3. *If a simple graph G is not a complete graph and is not k-connected, then G has a separating set of size $k-1$.*

Proof 1 (verifying definition). Since G' arises by adding edges to G, it is connected. If G' has a cut-vertex v, then v is also a cut-vertex in G, since $G-v$ is a spanning subgraph of $G'-v$. Since neighbors of v in G are adjacent in G', they cannot be in different components of $G'-v$. Hence $G'-v$ has only one component.

Proof 2 (weak duality). Let x,y be vertices of G. Since G is connected, it has an x,y-path $v_0,\ldots,v_k$. In G', both $v_0,v_2,v_4,\ldots,v_k$ and $v_0,v_1,v_3,\ldots,v_k$ are x,y-paths, and they are internally disjoint. Thus at least two vertices must be deleted to separate x and y.

4.1.4. *A graph G is k-connected if and only if $G \vee K_r$ is $k+r$-connected.* To separate $G \vee K_r$, one must delete all of the added vertices, since they are adjacent to all vertices. Since deleting them leaves G, a set is a separating set in $G \vee K_r$ if and only if it contains the r vertices outside G and the remainder is a separating set in G. Thus the minimum size of a separating set in $G \vee K_r$ is r more than the minimum size of a separating set in G.

4.1.5. *If G' is obtained from a connected graph G by adding edges joining pairs of vertices whose distance in G is 2, then G' is 2-connected.*

Proof 1 (definition of 2-connected). Since G' is obtained by adding edges to G, G' is also connected. If G' has a cut-vertex v, then v is also a cut-vertex in G, since $G-v$ is a spanning subgraph of $G'-v$. By construction, neighbors of v in G are adjacent in G', and hence they cannot be in different components of $G'-v$. Hence $G'-v$ has only one component.

Proof 2 (weak duality). Let x,y be vertices of G. Since G is connected, it has an x,y-path $v_0,\ldots,v_k$. In G', both $v_0,v_2,v_4,\ldots,v_k$ and $v_0,v_1,v_3,\ldots,v_k$ are x,y-paths, and they are internally disjoint. Thus at least two vertices must be deleted to separate x and y.

Proof 3 (induction on $n(G)$). When $n(G)=3$, $G'=K_3$, which is 2-connected. When $n(G)>3$, let v be a leaf of a spanning tree in G. Since $G-v$ is connected and $G'-v=(G-v)'$, the induction hypothesis implies that $G'-v$ is 2-connected. Since v has at least two neighbors in G', the Expansion Lemma implies that G' also is 2-connected.

4.1.6. *A connected graph with blocks $B_1,\ldots,B_k$ has $\left(\sum_{i=1}^{k} n(B_i)\right)-k+1$ vertices.* We use induction on k. Basis step: $k=1$. A graph that is a single block B_1 has $n(B_1)$ vertices.

Induction step: $k>1$. When G is not 2-connected, there is a block B that contains only one of the cut-vertices; let this vertex be v, and index the blocks so that $B_k=B$. Let $G'=G-(V(B)-\{v\})$. The graph G' is connected and has blocks $B_1,\ldots,B_{k-1}$. By the induction hypothesis, $n(G')=\left(\sum_{i=1}^{k-1} n(B_i)\right)-(k-1)+1$. Since we deleted $n(B_k)-1$ vertices from G to obtain G', the number of vertices in G is as desired.

4.1.7. *The number of spanning trees of a connected graph is the product of the numbers of spanning trees of each of its blocks.* We use induction on k. Basis step: $k=1$. In a graph that is a single block, the spanning trees of the graph are the spanning trees of the block.

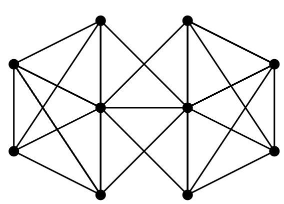
Induction step: $k>1$. Let v be a cut-vertex of G. The graph G is the union of two graphs G_1,G_2 that share only v. A subgraph is a spanning tree of G if and only if it is the union of a spanning tree in G_1 and a spanning tree in G_2. Since we can combine any spanning tree of G_1 with any spanning tree of G_2 to make a spanning tree of G, the number of spanning trees of G is the product of the number in G_1 and the number in G_2.

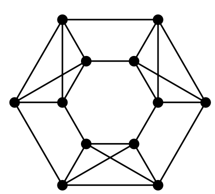
Also the blocks of G are the blocks of G_1 together with the blocks of G_2. Applying the induction hypothesis, we take the product of the numbers of spanning trees in the blocks of G_1 and multiply it by the product of the numbers of spanning trees in the blocks of G_2 to obtain the number of spanning trees of G.

4.1.8. *For the graph G on the left below, $\kappa(G) = 2$, $\kappa'(G) = 4$, and $\delta(G) = 4$. For the graph H on the right, $\kappa(H) = \kappa'(H) = \delta(H) = 4$.* The vertices all have degree 4, except that the vertices in the "center" of the drawing of G have degree 7.

In G, these two vertices form a separating set, and the graph has no cut-vertex, so $\kappa(G) = 2$. By Corollary 4.1.13, if there is an edge cut with fewer than four edges, it must have at least five vertices on each side. Proposition 4.1.12 states that $\left|[S, \overline{S}]\right| = [\sum_{v\in S} d(v)] - 2e(G[S])$. Since one of the sides has at least one of the vertices of degree 7, we may assume that $\sum_{v\in S} d(v) \ge 23$. To obtain $\left|[S, \overline{S}]\right| \le 3$, this requires $2e(G[S]) \ge 20$. Thus the subgraph induced by S must be K_5, but G does not contain K_5.

In H, it suffices to show that there is no separating set S of size 3, since $\kappa(H) \le \kappa'(H) \le \delta(H) = 4$. To show this, let x, y be two nonadjacent vertices of $H - S$. By a small case analysis, one shows that in each direction around the central portion of the graph, there are two x, y-paths sharing no internal vertices. Thus four vertices must be deleted to break all x, y-paths.

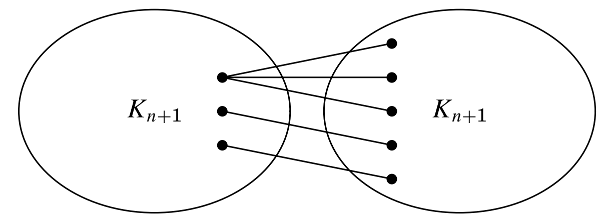




4.1.9. *Given nonnegative integers with $l \le m \le n$, there is a simple graph with $\kappa = l$, $\kappa' = m$, and $\delta = n$.* Begin with two disjoint copies of K_{n+1}. This yields $\delta = n$, and we will add a few more edges. Pick l vertices from the first clique and m vertices from the second. Add m edges between them in such a way that each of the special vertices belongs to at least one of the new edges. The construction is illustrated below with $m = 3$ and $l = 5$.

Deleting the m special edges disconnects the graph, as does deleting the l special vertices in the first copy of K_{n+1}. Since we are using $n + 1$ vertices in each of the complete subgraphs, $l \le m \le n$ guarantees that the

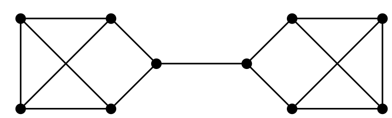
minimum degree remains n and that there really are two components remaining after the deletions. No smaller set disconnects the graph, because the connectivity of the complete subgraphs is n.



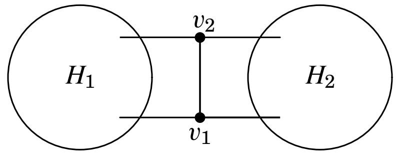
4.1.10. *The graph below is the smallest 3-regular simple graph with connectivity 1.* Since the graph below is 3-regular and has connectivity 1, it suffices to show that every 3-regular simple graph with connectivity 1 has at least 10 vertices.

Proof 1 (case analysis). Let v be a cut-vertex of a 3-regular simple graph G with connectivity 1. Each component H of $G - v$ has one or two neighbors of v. Since the neighbors of v have degree 3, H also has a vertex u not adjacent to v. Since $d(u) = 3$, $n(H) \ge 4$. Since G has at least two such components plus v, we have $n(G) \ge 4 + 4 + 1 = 9$. By the degree-sum formula, no 3-regular graph has order 9, so $n(G) \ge 10$.

Proof 2 (using edge-connectivity). Since $\kappa = \kappa'$ for 3-regular graphs, we seek the smallest 3-regular connected graph G having a cut-edge e. The graph $G - e$ has two components, each having one vertex of degree 2 and the rest of degree 3. Since it has a vertex of degree 3, such a component has at least four vertices. Since it has an even number of vertices of degree 3, each component has at least five vertices.



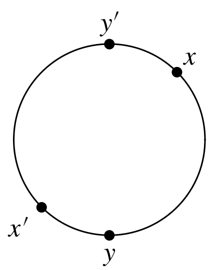
4.1.11. *$\kappa' = \kappa$ when $\Delta(G) \le 3$.* Let S be a minimum vertex cut ($|S| = \kappa(G)$). Since $\kappa(G) \le \kappa'(G)$ always, we need only provide an edge cut of size $|S|$. Let H_1 and H_2 be two components of $G - S$. Since S is a minimum vertex cut, each $v \in S$ has a neighbor in H_1 and a neighbor in H_2. Since $\Delta(G) \le 3$, v cannot have two neighbors in H_1 and two in H_2. For each such v, delete the edge to a member of $\{H_1, H_2\}$ in which v has only one neighbor. These $\kappa(G)$ edges break all paths from H_1 to H_2 except in the case drawn below, where a path can come into S via v_1 and leave via v_2. Here we simply choose the edge to H_1 for each v_i.



4.1.12. *A k-regular k-connected graph when k is odd.* For $n > k = 2r + 1$ and $r \ge 1$, we show that the Harary graph $H_{k,n}$ is k-connected. The graph consists of n vertices $v_0, \dots, v_{n-1}$ spaced equally around a circle, with each vertex adjacent to the r nearest vertices in each direction, plus the "special" edges $v_i v_{i+\lfloor n/2 \rfloor}$ for $0 \le i \le \lfloor (n-1)/2 \rfloor$. When n is odd, $v_{\lfloor n/2 \rfloor}$ has two incident special edges.

To prove that $\kappa(G) = k$, consider a separating set S. Since $G - S$ is disconnected, there are nonadjacent vertices x and y such that every x, y-path passes through S. Let $C(u, v)$ denote the vertices encountered when moving from u to v clockwise along the circle (omitting u and v). The cut S must contain r consecutive vertices from each of $C(x, y)$ and $C(y, x)$ in order to break every x, y-path (otherwise, one could start at x and always take a step in the direction of y). Hence $|S| \ge k$ unless S contains exactly r consecutive vertices in each of $C(x, y)$ and $C(y, x)$.

In this case, we claim that there remains an x, y-path using a special edge involving x or y. Let x' and y' be the neighbors of x and y along the special edges, using v_0 as the neighbor when one of these is $v_{\lfloor n/2 \rfloor}$. Label x and y so that $C(x, y)$ is smaller than $C(y, x)$ (diametrically opposite vertices require n even and are adjacent). Note that $\left|C(x', y')\right| \ge |C(x, y)| - 1$ (the two sets have different sizes when n is odd if $x = v_i$ and $y = v_j$ with $0 \le j < \lfloor n/2 \rfloor \le i \le n - 1$). Because $|C(x, y)| \ge r$, we have $\left|C(x', y')\right| \ge r - 1$. Therefore, when we delete r consecutive vertices from $C(y, x)$, all of $C(y, x') \cup \{x'\}$ or $\{y'\} \cup C(y', x)$ remains. Therefore at least one of the two x, y-paths with these sets as the internal vertices remains in $G - S$.



4.1.13. *Numerical argument for edge-connectivity of $K_{m,n}$.*

a) Size of $[S, \overline{S}]$. Let X and Y be the partite sets of $K_{m,n}$, with $|X| = m$ and $|Y| = n$. Consider $S \subseteq V(K_{m,n})$ such that $|S \cap X| = a$ and $|S \cap Y| = b$.

Now $[S, \overline{S}]$ consists of the edges from $S \cap X$ to $\overline{S} \cap Y$ and from $S \cap Y$ to $\overline{S} \cap X$. There are $a(n - b) + b(m - a)$ such edges.

b) $\kappa'(K_{m,n}) = \min\{m, n\}$. If $m + n = 1$, then the answer is 0, by convention, as desired. Otherwise, we assume that $m \le n$ and consider an edge cut $[S, \overline{S}]$. In the notation of part (a), we have $0 \le a \le m$ and $0 \le b \le n$ and $0 < a+b < m+n$. If $0 < a < m$, then $a(n-b)+b(m-a) \ge (n-b)+b = n \ge m$. If $a = 0$, then $b > 0$ and $a(n - b) + b(m - a) = bm \ge m$. If $a = m$, then $b < n$ and $a(n - b) + b(m - a) = m(n - b) \ge m$. Thus $a(n - b) + b(m - a) \ge m$ in all cases, with equality when $a = 0$ and $b = 1$.

c) $K_{3,3}$ has no edge cut with seven edges. Since $K_{3,3}$ has six vertices, every connected spanning subgraph of $K_{3,3}$ has at least five edges. Hence deleting any five or more edges of $K_{3,3}$ leaves a disconnected subgraph. No set of seven edges is an edge cut, because 7 does not occur in the set of values of $a(3 - b) + b(3 - a)$. Writing this in the form $3(a + b) - 2ab$, achieving 7 requires $a + b \ge 3$. Also, a, b must also have different parity, since $2ab$ even implies that $a + b$ must be odd to obtain 7. The remaining cases, $(1, 2)$, $(0, 3)$, $(2, 3)$, do not yield 7.

4.1.14. *If G is a connected graph and for every edge e there are cycles C_1 and C_2 such that $E(C_1) \cap E(C_2) = \{e\}$, then G is 3-edge-connected.* It suffices to show that no set of two edges disconnects G. Consider $e, e' \in E(G)$. Since G has two cycles through e', $G - e'$ is connected. Since G has two cycles that share only e, at least one of these cycles still exists in $G - e'$. Therefore, e lies on a cycle in $G - e'$ and is not a cut-edge of $G - e'$. We have proved that deleting both e' and e leaves a connected subgraph. The argument holds for each edge pair, so G is 3-edge-connected.

The Petersen graph satisfies this condition (hence is 3-edge-connected).

Proof 1 (symmetry and disjointness description). The underlying set [5] is in the disjointness definition of the Petersen graph can be permuted to turn each edge into any other. Hence it suffices to prove that the condition holds for one edge. In particular, the edge $(12, 34)$ is the only common edge in the two 5-cycles $(12, 34, 51, 23, 45)$ and $(12, 34, 52, 14, 35)$.

Proof 2 (properties of the graph). Alternatively, let x and y be the endpoints of an edge in the Petersen graph. Since the girth is 5, the neighbors u, v of x and w, z of y form an independent set of size 4. Let a be the unique common neighbor of u and w, and let b be the common neighbor of v and y; these are distinct since the girth is 5. Since a, b, x, y, u, v, w, z are eight distinct vertices, we have constructed cycles with vertices (u, x, y, w, a) and (v, x, y, z, b) that share only xy.

4.1.15. *The Petersen graph is 3-connected.* Since the Petersen graph G is 3-regular, it suffices by Theorem 4.1.11 to prove that G is 3-edge-connected. Let $[S, \overline{S}]$ be a minimum edge cut. If $\left|[S, \overline{S}]\right| < 3$, then

$[\sum_{v\in S} d(v)] - 2e(G[S]) \le 2$, by Proposition 4.1.12. We may compute this from either side of the cut, so we may assume that $|S| \le |\overline{S}|$.

Since G has no cycle of length less than 5, when $|S| < 5$ we have $e(G[S]) \le |S| - 1$. This yields $3|S| - 2(|S|-1) \le 2$, which simplifies to $|S| \le 0$. This is impossible for nonempty S. For $|S| = 5$, we obtain $3|S| - 2|S| \le 2$, which again is false. Hence no edge cut has size less than 3.

4.1.16. *The Petersen graph has an edge cut of size m if and only if $3 \le m \le 12$.* Since the graph has 10 vertices, we consider edge cuts of the form $[S,\overline{S}]$ for $1 \le |S| \le 5$. Since $|[S,\overline{S}]| = \sum_{v\in S} d(v) - 2e(G[S]) = 3|S| - 2e(G[S])$, we consider the number of edges in $G[S]$. Since the girth is 5, all induced subgraphs with at most four vertices are forests.

The independent sets with up to four vertices yield cuts of sizes $3, 6, 9, 12$. Deleting two adjacent vertices and their neighbors leaves $2K_2$, so there induced subgraphs with two to four vertices that have one edge, yielding cuts of sizes $4, 7, 10$. Deleting the vertices of a P_3 and their neighbors leaves $2K_1$, so there are induced subgraphs with three to five vertices that have two edges, yielding cuts of sizes $5, 8, 11$.

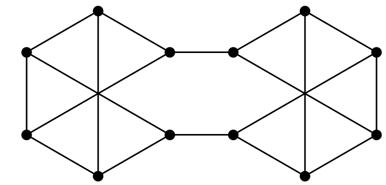
Let $e(S)$ denote $e(G[S])$. An edge cut of size less than 3 requires $3|S| - 2e(S) \le 2$, or $e(S) \ge (3/2)|S| - 1$. Since $e(S) \le |S| - 1$ when $|S| \le 4$, we combine the two inequalities to obtain $|S| \le 0$, which is impossible. (For $|S| = 5$, $e(S) \le |S|$ yields $|S| \le 2$, again a contradiction.)

Similarly, an edge cut of size more than 12 requires $2e(S) \le 3|S| - 13$. With $|S| \le 5$, this yields $2e(S) \le 2$, but there is no 5-vertex induced subgraph with only one edge.

4.1.17. *Deleting an edge cut of size 3 in the Petersen graph isolates a vertex.* Proposition 4.1.12 yields $|[S,\overline{S}]| = 3|S| - 2e(G[S])$. Thus $|[S,\overline{S}]| = 3$ requires S and $\overline{S}$ to have odd size. Let S be the smaller side of the cut. When $|S| = 5$, the induced subgraph has at most 5 edges, and the cut has size at least $3\cdot 5 - 2\cdot 5 = 5$. When $|S| = 3$, the induced subgraph has at most 2 edges, and the cut has size at least $3\cdot 3 - 2\cdot 2 = 5$. Hence $|S| = 1$ for a cut of size 3.

4.1.18. *Every triangle-free simple graph with minimum degree at least 3 and order at most 11 is 3-edge-connected.* Let $[S,\overline{S}]$ be an edge cut of size less than 3, with $|S| \le |\overline{S}|$. Let $k = |S|$. Since $\delta(G) \ge 3$ and $[S,\overline{S}] \le 2$, the Degree-Sum Formula yields $e(G[S]) \ge (3k-2)/2$. Since $G[S]$ is triangle-free, Mantel's Theorem (Section 1.3) yields $e(G[S]) \le \lfloor k^2/4 \rfloor$. Hence $k^2/4 \ge (3k-2)/2$. For positive integer k, this inequality is valid only when $k \ge 6$. Since the smaller side of the cut has at most five vertices, we obtain a contradiction, and there is no edge cut of size at most 2.

The bound of 11 is sharp. The 12-vertex 3-regular triangle-free graph below is not 3-edge-connected.



4.1.19. *a) If $\delta(G) \ge n-2$ for a simple n-vertex graph G, then $\kappa(G) = \delta(G)$.* If $\delta = n-1$, then $G = K_n$, which has connectivity $n-1$. If $\delta = n-2$, then when u and v are nonadjacent the other $n-2$ vertices are all common neighbors of u and v. It is necessary to delete all common neighbors of some pair of vertices to separate the graph, so $\kappa \ge n-2 = \delta$.

b) Construction of graphs with $\delta = n-3$ and $\kappa < \delta$. For any $n \ge 4$, let $G = K_n - E(C_4)$; i.e., G is formed by deleting the edges of a 4-cycle from a clique. The subgraph induced by these four vertices is $2K_2$, so deleting the other $n-4$ vertices of G disconnects the graph. However, G has 4 vertices of degree $n-3$ and $n-4$ of degree $n-1$, so $\kappa(G) < \delta(G)$.

4.1.20. *Every simple n-vertex graph G with $\delta(G) \ge (n+k-2)/2$ is k-connected, and this is best possible.* We do not consider $k = n$, because we have adopted the convention that no n-vertex graph is n-connected. To see that the result is best possible, consider $K_{k-1} \vee (K_{\lfloor (n-k+1)/2 \rfloor} + K_{\lceil (n-k+1)/2 \rceil})$. This graph has a separating set of size $k-1$, and its minimum degree is $k - 1 + \lfloor (n-k+1)/2 \rfloor - 1 = \lfloor (n+k-3)/2 \rfloor$. There are several ways to prove that $\delta \ge (n+k-2)/2$ ensures k-connectedness.

Proof 1 (stronger statement, common neighbors). If $x \not\leftrightarrow y$, then x, y have a total of at least $n+k-2$ edges to the $n-2$ other vertices, which means they have at least k common neighbors (using $|A \cap B| = |A| + |B| - |A \cup B|$ for $A = N(x)$ and $B = N(y)$). Thus at least k vertices must be deleted to make some vertex unreachable from another.

Proof 2 (contradiction). If G is not k-connected, then the deletion of some $k-1$ vertices S leaves a disconnected subgraph H. Consider $v \in V(H)$; since v has at most $k-1$ neighbors in S, we have $d_H(v) \ge \delta(G) - k + 1 \ge (n-k)/2$. Therefore, each component of H has at least $1 + (n-k)/2$ vertices. Since H has at least two components, H has at least $n-k+2$ vertices. However, $n = n(G) = n(H) + |S| \ge (n-k+2) + (k-1) > n$. The contradiction implies that G is k-connected.

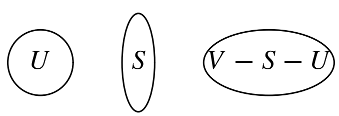
Proof 3 (induction on k). For $k = 1$, $\delta(G) \ge (n-1)/2$ forces every pair of nonadjacent vertices to have degree-sum at least $n-1$; hence they have a common neighbor among the remaining $n-2$ vertices, and G is connected. For $k > 1$, let v be a vertex of a minimum separating set S.

Deleting v removes at most one edge to each other vertex, so $\delta(G-v) \geq [(n-1)+(k-1)-2]/2$. Using the induction hypothesis, we conclude that $G-v$ is $(k-1)$-connected. Since $S-v$ separates $G-v$, we have $|S-v| \geq k-1$ and hence $|S| \geq k$, and G is k-connected.

4.1.21. *If G is a simple n-vertex graph with $n \geq k+l$ and $\delta(G) \geq \frac{n+l(k-2)}{l+1}$, and $G-S$ has more than l components, then $|S| \geq k$.* Proof by contradiction. Suppose $G-S$ has more than l components and $|S| = k-1$ (if there is a smaller cut, we can add to it from components of the remainder that have at least 2 vertices until the cut reaches size $k-1$). Let H be a smallest component of $G-S$; we have $n(H) \leq n-k+1)/(l+1)$. A vertex of H has at most $(n-k+1)/(l+1)-1$ neighbors in H and $k-1$ neighbors in S, which yields $\delta(G) \leq \frac{n-(k-2)-1}{l+1} + (k-2) = \frac{n-l(k-2)-1}{l+1}$.

To prove that the result is best possible, partition $n-k+1$ vertices into $l+1$ sets of sizes $\lfloor \frac{n-k+1}{l+1} \rfloor$ and $\lceil \frac{n-k+1}{l+1} \rceil$. Place cliques on these sets, and form the join of this graph with K_{k-1}. The minimum degree is $\frac{n-l(k-2)-1}{l+1}$.

4.1.22. *a) If the vertex degrees $d_1 \leq \cdots \leq d_n$ of a simple graph G satisfy $d_j \geq j+k$ whenever $j \leq n-1-d_{n-k}$, then G is $(k+1)$-connected.* Let S be a vertex cut; we will prove that $|S| \geq k+1$. Let U be the set of vertices in the smallest component of $G-S$, and let $j = |U|$. Only vertices of S can have degree exceeding $n-1-j$. Since there are at most $|S|$ such vertices, $d_{n-|S|} \leq n-1-j$. If $|S| \leq k$, then $j \leq n-1-d_{n-|S|} \leq n-1-d_{n-k}$, and the hypothesis applies. If $v \in U$, then $d_G(v) \leq j-1+|S|$. Since this yields j vertices with degree at most $j-1+|S|$, we have $d_j \leq j-1+|S|$. Since the hypothesis applies, $d_j \geq j+k$, and we conclude that $|S| \geq k+1$.

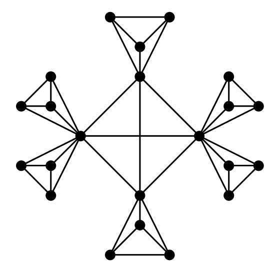


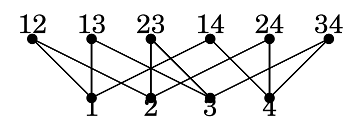
b) The result is sharp. Let $G = K_k \vee (K_j + K_i)$, where $i+j+k = n$; we may assum that $j \leq i$. There are j vertices of degree $j+k-1$, i vertices of degree $i+k-1$, and k vertices of degree $n-1$. When i and j are positive, $\kappa(G) = k$. Since $j \leq i$ and G has k vertices of degree $n-1$, we have $d_j = j+k-1$ and $d_{n-k} = n-j-1$. Thus the condition in part (a) does not hold. However, it just barely fails, since $d_{j'} = j+k-1 \geq j'+k$ for $j' < j$. Thus the result is sharp in the sense that it cannot be weakened by applying the requirement only when $j \leq n-2-d_{n-k}$.

4.1.23. *If $n(G)$ is even, $\kappa(G) \geq r$, and G is $K_{1,r+1}$-free, then G has a 1-factor.* We verify Tutte's 1-factor condition. When $|S| = \varnothing$, the only component of $G-S$ has even order. When $1 \leq |S| \leq r-1$, there is only one component of $G-S$. For $|S| \geq r$, we prove that $G-S$ has at most $|S|$ components.

Each component H of $G-S$ sends edges to at least r distinct vertices in S, since $\kappa(G) = r$. For each such H, choose edges to r distinct vertices in S. Given $v \in S$, we have chosen at most one edge from v to each component of $G-S$. If $G-S$ has more than $|S|$ components, then we have chosen more than $r\,|S|$ edges to S. By the pigeonhole principle, some $x \in S$ appears in more than r of these edges. Since we chose at most one edge from x to each component of $G-S$, the chosen edges containing x have endpoints in distinct components of $G-S$, which creates the forbidden induced $K_{1,r+1}$.

This result is best possible: it is not enough to assume that G is r-edge-connected or that G is $r-1$-connected. Both graphs below have even order, no induced $K_{1,4}$, and no 1-factor (deleting a set of size 4 leaves 6 odd components). The graph on the left is 3-edge-connected, and the graph on the right is 2-connected.





4.1.24. *Degree conditions for $\kappa' = \delta$ in a simple n-vertex graph G.*

a) $\delta(G) \geq \lfloor n/2 \rfloor$ implies $\kappa'(G) = \delta(G)$, and this is best possible. If $\kappa'(G) < \delta(G)$ and F is a minimum edge cut, then the components of $G-F$ have more than $\delta(G)$ vertices (Proposition 4.1.10). Since $\delta(G) \geq \lfloor n/2 \rfloor$, this yields $n(G) \geq 2(\lfloor n/2 \rfloor + 1) \geq n+1$, which is impossible. Hence $\kappa'(G) = \delta(G)$.

To show that the inequality $\delta \geq \lfloor n/2 \rfloor$ cannot be weakened when $n \geq 3$, consider $G = K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$ (the disjoint union of two cliques). This G is disconnected, so $\kappa'(G) = 0$, and $\delta(G) = \lfloor n/2 \rfloor - 1$. The smallest case where this yields $\kappa'(G) < \delta(G)$ is $n = 4$, $\delta = 1$, $G = 2K_2$.

b) $\kappa'(G) = \delta(G)$ if each nonadjacent pair of vertices has degree sum at least $n-1$, and this is best possible. The example $G = K_{m+1} + K_{n-m-1}$ shows that $n-1$ cannot be replaced by $n-2$ in the hypothesis; the conclusion fails spectacularly with $\kappa'(G) = 0$ even though $d(x)+d(y) = m-1+n-m-1 = n-2$ when $x \nleftrightarrow y$. To prove the claim, suppose $[S, \overline{S}]$ is a minimum edge cut, with size $k = \kappa'(G) < \delta(G)$. This forces $|S|, |\overline{S}| > \delta(G)$ (Proposition 4.1.10). With degree-sum at least $n(G)-1$, any two nonadjacent vertices have a common neighbor. Hence if S has a vertex x with no neighbor in $\overline{S}$, then every vertex in $\overline{S}$ must have a neighbor in S. Now $|\overline{S}| > \delta(G)$ imples $k > \delta(G)$. Otherwise, every vertex of S has a neighbor in $\overline{S}$, and now $|S| > \delta(G)$ implies $k > \delta(G)$. We conclude that $k = \delta(G)$.

4.1.25. *$\kappa'(G) = \delta(G)$ for diameter 2.* Suppose that G is a simple graph with diameter 2. Let $[S, \overline{S}]$ be a minimum edge cut with $s = |S| \le |\overline{S}|$, and let $k = |[S, \overline{S}]| = \kappa'(G)$.

a) Every vertex of S has a neighbor in $\overline{S}$. If S has a vertex x with no neighbor in $\overline{S}$, then $d(x) \le s - 1 < n/2$, and diam $G = 2$ implies that every vertex of $\overline{S}$ has a neighbor in S. In this case $\delta(G) < n/2 \le k$. Hence every vertex of S has a neighbor in $\overline{S}$.

b) $\kappa'(G) = \delta(G)$. Since $\kappa'(G) \le \delta(G)$ always, part (a) yields $s \le \delta(G)$. Each vertex of S has at least $\delta(G) - s + 1$ neighbors in $\overline{S}$, so $k \ge s(\delta(G) - s + 1)$. Since $\kappa'(G) \le \delta(G)$, we have $0 \ge (s-1)(\delta(G) - s)$, which requires $s = 1$ or $s \ge \delta(G)$. We conclude that $s = 1$ or $s = \delta(G)$.

Consider the case $s = \delta(G)$. Since we have proved that each vertex of S has a neighbor in $\overline{S}$, from $k \le \delta(G)$ we conclude that each vertex of S has exactly one neighbor in $\overline{S}$. Hence each vertex of S has $\delta(G) - 1$ neighbors in S. We conclude that S induces a clique and that $\kappa'(G) = \delta(G)$.

4.1.26. *A set F of edges in G is an edge cut if and only if F contains an even number of edges from every cycle in G. Necessity.* A cycle must wind up on the same side of an edge cut that it starts on, and thus it must cross the cut an even number of times.

Sufficiency. Given a set F that satisfies the intersection condition with every cycle, we construct a set $S \subseteq V(G)$ such that $F = [S, \overline{S}]$. Each component of $G - F$ must be all in $G[S]$ or all in $G[\overline{S}]$, but we must group them appropriately. Define a graph H whose vertices correspond to the components of $G - F$; for each $e \in F$, we put an edge in H whose endpoints are the components of $G - F$ containing the endpoints of e.

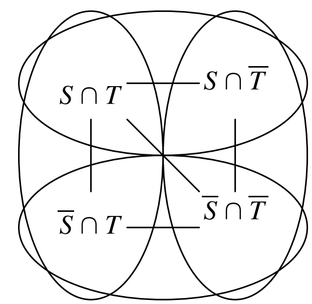
We claim that H is bipartite. From a cycle C in H, we can obtain a cycle C' in G as follows. For $v \in V(C)$ let e, f be the edges of C incident to v (not necessarily distinct), and let x, y be the endpoints of e, f in the component of $G - F$ corresponding to v. We expand v into an x, y-path in that component. Since C visits each vertex at most once, the resulting C' is a cycle in G. The number of edges of F in C' is the length of C. Hence the length of C is even.

We conclude that H is bipartite. Let S be the set of vertices in the components of $G - F$ corresponding to one partite set in a bipartition of H. Now F is the edge cut $[S, \overline{S}]$.

4.1.27. *Every edge cut is a disjoint union of bonds.* Using induction on the size of the cut, it suffices to prove that if $[S, \overline{S}]$ is not a bond, then $[S, \overline{S}]$ is a disjoint union of smaller edge cuts. First suppose G is disconnected, with components $G_1, \ldots, G_k$. If $[S, \overline{S}]$ cuts more than one component, we express $[S, \overline{S}]$ as a union of edge cuts that cut only one component: let the ith cut be $[S \cap V(G_i), \overline{S \cap V(G_i)}]$. This cut consists of the edges of $[S, \overline{S}]$ in G_i, because the vertices of the other components are all on one side of the cut. Hence we may assume that $[S, \overline{S}]$ cuts only one component of G or (equivalently) that G is connected.

An edge cut of a connected graph is a bond if and only if the subgraphs induced by the sets of the vertex partition are connected. Hence if $[S, \overline{S}]$ is not a bond, we may assume that $G[S]$ is not connected. Let $\{G_i\}$ be the components of the induced subgraph $G[S]$, and let $S_i = V(G_i)$. Since there are no edges between components of $G[S]$, $[S, \overline{S}]$ is the disjoint union of the edge cuts $[S_i, \overline{S_i}]$. Since G is connected, each of these is non-empty, so we have expressed $[S, \overline{S}]$ as a disjoint union of smaller edge cuts.

4.1.28. *The symmetric difference of two edge cuts is an edge cut.* The symmetric difference of $[S, \overline{S}]$ and $[T, \overline{T}]$ is $[U, \overline{U}]$, where $U = (S \cap T) \cup (\overline{S} \cap \overline{T})$ and $\overline{U} = (S \cap \overline{T}) \cup (\overline{S} \cap T)$, as sketched below. The other edges of the union are those within U or within $\overline{U}$ and appear in both of the original edge cuts.



4.1.29. *A spanning subgraph H of G is a spanning tree of G if and only if $G - E(H)$ contains no bond of G and adding any edge of H creates a subgraph containing exactly one bond of G.*

Necessity: If H is a spanning tree, then H is connected, so $G - E(H)$ contains no edge cut. Also, $H - e$ is disconnected, with exactly two components having vertex sets S and $\overline{S}$. Let G' be the subgraph obtained by adding e to $G - E(H)$. Note that G' contains all of $[S, \overline{S}]$ (and perhaps additional edges). Since $S, \overline{S}$ induce connected subgraphs of G, $[S, \overline{S}]$ is a bond of G (Proposition 4.1.15). Since adding any edge of $[S, \overline{S}]$ to $H - e$ creates a spanning tree of G, an edge cut contained in G' must include all of $[S, \overline{S}]$, and hence $[S, \overline{S}]$ is the only bond in G'.

Sufficiency: Suppose that $G - E(H)$ contains no bond of G and each subgraph obtained by adding one edge of H contains exactly one bond. Now H is obtained by deleting a set of edges from G that does not disconnect G, and H is connected. Similarly, deleting any additional edge from G does contain a bond, so each $H - e$ is disconnected. Hence H is a tree.

4.1.30. *The graph with vertex set $\{1, \ldots, 11\}$ in which $i \leftrightarrow j$ if and only if i and j have a common factor bigger than 1 has six blocks.* Vertices $1, 7, 11$ are isolated and hence are blocks by themselves. The remaining vertices form a single component with blocks that are complete subgraphs. The vertex sets of these are $\{3, 6, 9\}$, $\{2, 4, 6, 8, 10\}$, and $\{5, 10\}$. Vertices 6 and 10 are cut-vertices.

4.1.31. *The maximum number of edges in a simple n-vertex cactus G is $\lfloor 3(n-1)/2 \rfloor$.* A cactus is a connected graph in which every block is an edge or a cycle. The bound is achieved by a set of $\lfloor (n-1)/2 \rfloor$ triangles sharing a single vertex, plus one extra edge to a leaf if n is even.

Proof 1 (induction on the number of blocks). Let k be the number of blocks. If $k = 1$, then $e(G) = n(G) - 1$ if $n(G) \le 2$, and $e(G) = n(G)$ if $n(G) > 2$. In either case, $e(G) \le \lfloor 3(n(G)-1)/2 \rfloor$.

A graph that has more than one block is not a single block, so it has a cut-vertex v. Let S be the vertex set of one component of $G - v$. Let $G_1 = G[S \cup \{v\}]$, and let $G_2 = G - S$. Both G_1 and G_2 are cacti, and every block of G is a block in exactly one of $\{G_1, G_2\}$. Thus each has fewer blocks than G, and we can apply the induction hypothesis to obtain $e(G_i) \le \lfloor 3(n(G_i)-1)/2 \rfloor$.

If $|S| = m$, then $n(G_1) = m + 1$ and $n(G_2) = n(G) - m$, since v belongs to both graphs. We thus have

$$e(G) = e(G_1) + e(G_2) \le \left\lfloor \tfrac{3(m+1-1)}{2} \right\rfloor + \left\lfloor \tfrac{3(n(G)-m-1)}{2} \right\rfloor \le \left\lfloor \tfrac{3(n(G)-1)}{2} \right\rfloor.$$

Proof 2 (summing over blocks). Let G be a simple n-vertex cactus with k blocks that are cycles and l blocks that are single edges. When we describe G by starting with one block and iteratively adding neighboring blocks, each time we add a block the number of vertices increases by one less than the number of vertices in the block, since one of those vertices (the shared cut-vertex) was already in the graph. If the blocks are $B_1, \ldots, B_{k+l}$, then $n(G) = (\sum n(B_i)) - (k + l - 1)$.

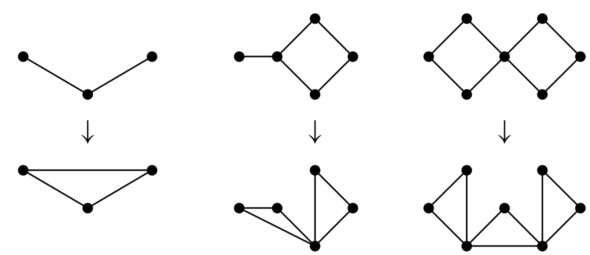
On the other hand, $e(G) = \sum e(B_i)$. We have $e(B_i) = n(B_i)$ if B_i is a cycle, and $e(B_i) = n(B_i) - 1$ if B_i is an edge. Therefore,

$$e(G) = \sum e(B_i) = (\sum n(B_i)) - l = n(G) + k - 1.$$

This implies that we maximize the number of edges by maximizing k, the number of blocks that are cycles. Viewing the cactus again as grown by adding blocks, observe that we add at least two vertices every time we add a block that is a cycle, since cycles have at least three vertices. Starting from a single vertex, the maximum number of cycles we can form is thus $\lfloor (n-1)/2 \rfloor$. This bound on k yields $e(G) \le \lfloor 3(n-1)/2 \rfloor$.

Proof 3 (local change). If the blocks are all triangles, except for at most one that is K_2 or one that is a 4-cycle, then the number of edges equals the given formula. Hence it suffices to show that a cactus not having this description cannot have the maximum number of edges.

If a block is a cycle of length more than 4, then deleting one edge e and replacing it with two edges joining the endpoints of e to another vertex on the cycle creates a new cactus on the same vertices having one more edge. If each of two blocks is a single edge or a 4-cycle, then the blocks can be rearranged by "cutting and pasting" so that the sizes of the blocks are the same as before, but these two special blocks share a vertex. Now a change can be made to increase the number of edges as shown below.



Proof 4 (spanning trees). An n-vertex cactus is a connected graph, so it has a spanning tree with $n - 1$ edges. Each additional edge completes a cycle using at least two edges in the tree. Each edge of the tree is used in at most one such cycle. Hence there are at most $(n-1)/2$ additional edges, and the total number of edges is at most $n + \lfloor (n-1)/2 \rfloor$.

4.1.32. *Every vertex of G has even degree if and only if every block of G is Eulerian. Sufficiency.* If every block is Eulerian, then each vertex receives even degree from each block containing it. The blocks partition the edges, so the total degree at each vertex is even.

Necessity. Since every block is connected, it suffices to show that each vertex has even degree in each block. Certainly this holds for a vertex appearing in only one block. For a cut-vertex v, let G' be the subgraph consisting of one component of $G - v$ together with its edges to v. Each block containing v appears in one such subgraph. Every vertex of G' other than v has even degree in G', since it retains all of its incident edges from G. By the Degree-Sum Formula, also v has even degree in G'. Hence v has even degree in the block of G containing v that is contained in G'.

4.1.33. *A connected graph is k-edge-connected if and only if each of its blocks is k-edge-connected.* We show that a set F of edges is a disconnecting set in a graph G if and only if it disconnects some block. If deleting F leaves each block of G connected, then the full graph remains connected. If deleting F disconnects some block B, then the remainder of G cannot contain a path between distinct components of $B - F$, because then B would not be a maximal subgraph having no cutvertex.

With this claim, the edge-connectivity of G is the minimum of the edge-connectivities of its blocks, which yields the desired statement.

4.1.34. *The block-cutpoint tree.* Given a graph G with connectivity 1, let $B(G)$ be the bipartite graph whose partite sets correspond to the blocks and the cut-vertices of G, with $x \leftrightarrow B$ if B is a block of G containing x.

a) $B(G)$ is a tree. If $G = K_2$, then $B(G) = K_1$. Otherwise G has at least two blocks, and every cut-vertex belongs to a block. Hence to show $B(G)$ is connected it suffices to establish a B, B'-path in $B(G)$, where B, B' are blocks of G. Since G is connected, G has a u, v-path, for any choice of vertices $u \in B$, $v \in B'$. This path visits some sequence of blocks from B to B', moving from one to the next via a cut-vertex of G belonging to both of them. This describes a B, B'-path in $B(G)$.

We prove by contradiction that $B(G)$ also has no cycles and hence is a tree. Suppose x is a cut-vertex of G on a cycle C in $B(G)$. Let B, B' be the neighbors of x on C. The B, B' path $C - x$ provides a route from $B - x$ to $B' - x$ without using x. This is impossible, since when B, B' are two blocks of G containing cut-vertex x, every path between $B - x$ and $B' - x$ in G must pass through x.

b) If G is not a block, then at least two blocks of G each contain exactly one cut-vertex of G. Each cut-vertex of G belongs to at least two blocks of G. Hence the leaves of $B(G)$ all arise from blocks of G, not cut-vertices of G. If G is not a block, then $B(G)$ has at least two leaves, and the leaves of $B(G)$ are the desired blocks in G.

c) G has exactly $k + \sum_{v \in V(G)} (b(v) - 1)$ blocks, where k is the number of components of G and $b(v)$ is the number of blocks containing v.

Proof 1 (explicit count). Since we can count the blocks separately in each component, it suffices to show that a connected graph has $1 + \sum (b(v) - 1)$ blocks. Select a block in a connected graph G and view it as a root; this corresponds to the 1 in the formula. Each vertex v in this block leads us to $b(v) - 1$ new blocks. For each new block, each vertex v other than the one that leads us there leads us to $b(v) - 1$ new blocks. This process stops when we have counted $b(v) - 1$ for each vertex of G.

This tree-like exploration gives the desired count of blocks as long as two facts hold: 1) no two blocks intersect in more than one vertex, and 2) no block can be reached in more than one way from the root. These guarantee that we don't count blocks more than once. If either happens, we get a cycle of blocks, $B_1, \ldots, B_n, B_1$, with $n \ge 2$, so that successive blocks share a vertex. Then there is no vertex whose deletion will disconnect the

subgraph that is the union of these blocks, which is impossible since blocks are maximal subgraphs with no cut-vertex.

Proof 2 (induction on the number of blocks). We need only prove the formula for connected graphs, since both the number of blocks and the value of the formula are sums over the components of G. If G is a block, then every vertex of G appears in one block, and the formula holds.

If G has a cutvertex, then by part (a) this component has a block B containing only one cutvertex, u. Delete all vertices of $B - u$ to obtain a graph G'. The blocks of G' are the blocks of G other than u, u appears in one less block than before, and all other terms of the formula are the same except that for G' we have left out the value 0 for the other vertices of B. The induction hypothesis now yields

$$\begin{aligned}\#blocks(G) = \#blocks(G') + 1 &= [1 + \sum_{v \in V(G')} (b_{G'}(v) - 1)] + 1 \\ &= 1 + \sum_{v \in V(G')} (b(v) - 1).\end{aligned}$$

d) Every graph has fewer cut-vertices than blocks. In the formula of part (c), there is a positive contribution for each cut-vertex. Thus the number of blocks is bigger than the number of cut-vertices, each yielding a term that contributes at least one to the sum.

4.1.35. *If H and H' are distinct maximal k-connected subgraphs of G, then H and H' have at most $k - 1$ vertices in common.* Proof by contradiction; suppose H and H' share at least k vertices. Consider $F = H \cup H'$, and let S be an arbitrary subset of $V(F)$ with fewer than k vertices. It suffices to show that $F - S$ is connected, because then $H \cup H'$ is k-connected, contradicting the hypothesis that H and H' are *maximal* k-connected subgraphs. Since $|S| < k$ and H, H' are k-connected, $H - S$ and $H' - S$ are connected. If H, H' share at least k vertices, then some common vertex x remains, and every vertex that remains has a path to x in $H - S$ or $H' - S$.

4.1.36. *Algorithm 4.1.23 correctly computes blocks of graphs.* We use induction on $n(G)$. For K_2, the algorithm correctly identifies the single block (K_1 is a special case). For larger graphs, it suffices to show that the first set identified as a block is indeed a block B sharing one vertex w with the rest of the graph, since when $w \ne x$ the remaining blocks are the blocks of the graph obtained by deleting $B - w$ from G, and the sets identified as blocks in running the algorithm on $G_(B - w)$ are the sets identified as blocks in the remainder of running the algorithm on G.

When the vertex designated as ACTIVE is changed from v to its parent, w, we check whether any vertex in the subtree T' rooted at v has a neighbor above w. This is easy to do, given that in step 1B when we mark an edge to an ancestor explored, we record for the vertices on the path in T between them that there is an edge from a descendant to an ancestor. When w becomes active again, we check whether it was ever so marked.

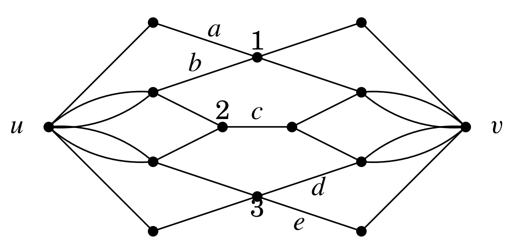
With v and w as above, in any rooted subtree of T' there is an edge from a descendant of the root to an ancestor of the root. Hence no proper subset of $T'(v)$ induces a block, because an additional vertex can be added via a path to an ancestor and then down through T, without introducing a cut-vertex. On the other hand, since there is no edge joining T' to an ancestor of w, then w is a cut-vertex, and hence $G[V(T') \cup \{w\}]$ is a maximal subgraph having no cut-vertex.

4.1.37. *An algorithm to compute the strong components of a digraph.* The algorithm is the same as Algorithm 4.1.23, except that all edges mentioned there are treated as directed edges, from tail to head in the order named there, and "block" changes to "strong component".

The proof that the algorithm works is essentially the same as Exercise 4.1.36. If there is a path from S to S that visits a vertex outside S, then S cannot be the vertex set of a strong component. When w becomes active from below with no edge from a descendant to an ancestor, all edges involving $V(T') \cup \{w\}$ and the remaining vertices are directed in toward $V(T') \cup \{w\}$. Thus a strong component is discovered.

4.2. k CONNECTED GRAPHS

4.2.1. *In the graph below, $\kappa(u, v) = 3$ and $\kappa'(u, v) = 5$.* Deleting the vertices marked 1, 2, 3 or the edges marked a, b, c, d, e makes v unreachable from u. These prove the upper bounds. Exhibiting a set of three pairwise internally disjoint u, v-paths proves $\kappa(u, v) \geq 3$, since distinct vertices must be deleted to cut the paths. Exhibiting a set of five pairwise edge-disjoint u, v-paths proves $\kappa'(u, v) \geq 5$, since distinct edges must be deleted to cut the paths. Lacking colors, we have not drawn these paths.

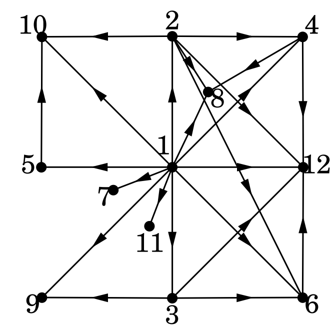


4.2.2. *If G is 2-edge-connected and G' is obtained from G by subdividing an edge of G, then G' is 2-edge-connected.* Let G' be obtained by subdividing an edge e, introducing a new vertex w. A graph is 2-edge-connected if and only if every edge lies on a cycle. This holds for G. If also holds for G', since every cycle in G containing e can be replaced with a cycle using the two edges incident to w instead of e.

Every graph having a closed-ear decomposition is 2-edge-connected. A cycle is 2-edge-connected; we show that adding ears and closed ears preserves 2-edge-connectedness. An ear or closed ear can be added by adding an edge joining existing endpoints or a double edge joining an old vertex to a new vertex, following by subdividing to lengthen the ear.

We have shown that subdivision preserves 2-edge-connectedness. The other operations preserve old cycles. When we add an edge, the new edge form a cycle with a path joining its endpoints. When we add two edges with the same endpoints, together they form a cycle. Hence the additions also preserve 2-edge-connectedness.

4.2.3. *An example of digraph connectivity.* In the digraph G with vertex set [12] defined by $i \to j$ if and only if i divides j, $\kappa(1, 12)$ is undefined and $\kappa'(1, 12) = 5$. Because $1 \to 12$, there is no way to make 12 unreachable from 1 by deleting other vertices. Because there are pairwise edge-disjoint paths from 1 to 12 through 2,3,4,6 and directly, it is necessary to delete at least five edges to make 12 unreachable from 1. Deleting the five edges entering 12 accomplishes this.



4.2.4. *If P is a u, v-path in a 2-connected graph G, then there need not be a u, v-path internally disjoint from P.* The graph $G = K_4 - uv$ with $V(G) = \{u, v, x, y\}$ is 2-connected (connected and no cut-vertex), but it has no u, v-path internally disjoint from the u, v-path P that visits vertices u, x, y, v in order.

4.2.5. *If G be a simple graph, and H is the graph with vertex set $V(G)$ such that $uv \in E(H)$ if and only if u, v appear on a common cycle in G, then H is* *a complete graph if and only if G is 2-connected.* A graph G is 2-connected if and only if for all $u, v \in V(G)$, there is a cycle containing u and v.

4.2.6. *A simple graph G is 2-connected if and only if G can be obtained from C_3 by a sequence of edge additions and edge subdivisions.* We have shown that edge addition and edge subdivision preserve 2-connectedness, so the condition is sufficient. For necessity, observe that every 2-connected graph has an ear decomposition. The initial cycle arises from C_3 by edge subdivisions, and then each ear addition consists of an edge addition followed by edge subdivisions.

4.2.7. *If xy is an edge in a digraph G, then $\kappa(G - xy) \ge \kappa(G) - 1$.* Since every separating set of G is a separating set of $G - xy$, we have $\kappa(G - xy) \le \kappa(G)$. Equality holds unless $G - xy$ has a separating set S that is smaller than $\kappa(G)$ and hence is not a separating set of G. Since $G - S$ is strongly connected, $G - xy - S$ has two induced subdigraphs $G[X]$ and $G[Y]$ such that $X \cup Y = V(G)$ and xy is the only edge from X to Y.

If $|X| \ge 2$, then $S \cup \{x\}$ is a separating set of G, and $\kappa(G) \le \kappa(G - xy) + 1$. If $|Y| \ge 2$, then again the inequality holds. In the remaining case, $|S| = n(G) - 2$. Since we have assumed that $|S| < \kappa(G)$, $|S| = n(G) - 2$ implies that $\kappa(G) \ge n(G) - 1$, which holds only when each ordered pair of distinct vertices is the head/tail for some edge. Thus $\kappa(G - xy) = n(G) - 2 = \kappa(G) - 1$, as desired.

4.2.8. *A graph is 2-connected if and only if for every ordered triple (x, y, z) of vertices, there is an x, z-path through y.* If G is 2-connected, then for any $y \in V(G)$ and set $U = \{x, z\}$, there is a y, U-fan. The two paths of such a fan together form an x, z-path through y. Conversely, if the condition holds, then clearly G is connected. Furthermore, G has no cut-vertex, because for any vertex x and any pair y, z, the condition as stated implies that $G - x$ has an y, z-path.

4.2.9. *A graph G with at least 4 vertices is 2-connected if and only if for every pair of disjoint sets of vertices $X, Y \subset V(G)$ with $|X|, |Y| \ge 2$, there exist two completely disjoint paths P_1, P_2 in G such that each path has an endpoint in X and an endpoint in Y and no internal point in X or Y.*

Sufficiency: If we apply the condition with X, Y being the endpoints of an arbitrary pair of edges, we find that every pair of edges lies on a cycle, so G is 2-connected. Alternatively, if G were disconnected or had a cut-vertex v, then we could select X and Y from separate components (of $G - v$), but then every path between X and Y passes through v.

Necessity: Form a graph G' be add an edge within X, if none exists, and within Y, if none exists. Since we only add edges, G' is still 2-connected. Hence there is a cycle containing an arbitrary pair of edges in G'; in particular, containing an edge within X and one within Y. For each portion of this cycle between the two edges, take the path between the last time it uses a vertex of X and the first time it uses a vertex of Y. This yields the desired completely disjoint paths in G.

4.2.10. (•) A *greedy ear decomposition* of a 2-connected graph is an ear decomposition that begins with a longest cycle and iteratively adds a longest ear from the remaining graph. Use a greedy ear decomposition to prove that every 2-connected claw-free graph G has $\lfloor n(G)/3 \rfloor$ pairwise-disjoint copies of P_3. (Kaneko–Kelmans–Nishimura [2000])

Comment. The proof takes many steps and several pages. It is too difficult for inclusion in this text, and the exercise will be deleted in the next edition.

4.2.11. *For a connected graph G with at least three vertices, the following are equivalent.*

A) G is 2-edge-connected.
B) Every edge of G appears in a cycle.
C) G has a closed trail containing any specified pair of edges.
D) G has a closed trail containing any specified pair of vertices.

A⇔B. A connected graph is 2-edge-connected if and only if it has no cut-edges. Cut-edges are precisely the edges belonging to no cycles.

A⇒D. By Menger's Theorem, a 2-edge-connected graph G has two edge-disjoint x, y-paths, where $x, y \in V(G)$. Following one path and returning on the other yields a closed trail containing x and y. (Without using Menger's Theorem, this can be proved by induction on $d(x, y)$.)

D⇒B. Let xy be an edge. D yields a closed trail containing x and y. This breaks into two trails with endpoints x and y. At least one of them, T, does not contain the edge xy. Since T is an x, y-walk, it contains an x, y-path. Since T does not contain xy, this path completes a cycle with xy.

B⇒C. Choose $e, f \in E(G)$; we want a closed trail through e and f. Subdivide e and f to obtain a new graph G', with x, y being the new vertices. Subdividing an edge does not destroy paths or cycles, although it may lengthen them. Thus G' is connected and has every edge on a cycle, because G has these properties. Because we have already proved the equivalence of B and D, we know that G' has a closed trail containing x and y. Replacing the edges incident to x and y on this trail with e and f yields a closed trail in G containing e and f.

C⇒D. Given a pair of vertices, choose edges incident to them. A closed trail containing these edges is a closed trail containing the original vertices.

4.2.12. *$\kappa(G) = \kappa'(G)$ when G is 3-regular, using Menger's Theorem.* By Menger's Theorem, for each x, y there are $\kappa'(G)$ pairwise edge-disjoint x, y-paths. Since G is 3-regular, these paths cannot share internal vertices (that would force four distinct edges at a vertex). Hence for each x, y there are $\kappa'(G)$ pairwise internally disjoint x, y-paths. This implies that $\kappa(G) \geq \kappa'(G)$, and it always holds that $\kappa(G) \leq \kappa'(G)$.

4.2.13. Given a 2-edge-connected graph G, define a relation R on $E(G)$ by $(e, f) \in R$ if $e = f$ or if $G - e - f$ is disconnected.

a) $(e, f) \in R$ if and only if e and f belong to the same cycles. Suppose that $(e, f) \in R$. If $e = f$, then e and f belong to the same cycles. If $G - e - f$ is disconnected, then f is a cut-edge in $G - e$, whence f belongs to no edges in $G - e$, and thus every cycle in G containing f must also contain e. Since similarly e is a cut-edge in $G - f$, we conclude also that f belongs to every cycle containing e. Thus e and f belong to the same cycles.

If $G - e - f$ is connected, then f is not a cut-edge in $G - e$ and thus belongs to a cycle in $G - e$; this is a cycle in G that does not contain e.

b) R is an equivalence relation on $E(G)$. The reflexive property holds by construction: $(e, e) \in R$ for all $e \in E(G)$. The symmetric property holds because $G - f - e$ is disconnected if $G - e - f$ is disconnected. The transitive property holds by part (a): if $(e, f) \in R$ and $(f, g) \in R$, then e and f belong to the same cycles, and f and g belong to the same cycles, and thus e and g belong to the same cycles (those containing f), and therefore $(e, g) \in R$.

c) Each equivalence class is contained in a cycle. We prove the stronger statement that a cycle contains an element of a class if and only if it contains the entire class. If some cycle contains some element e of the class and omits some other element f, then e and f do not belong to the same cycles, which contradicts (a).

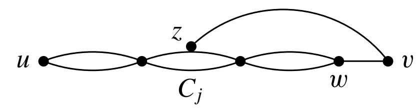
d) For each equivalence class F, $G - F$ has no cut-edge. If e is a cut-edge in $G - F$, then e lies in no cycle in $G - F$, so every cycle in G containing e contains some element of F. By the stronger statement in (c), every such cycle contains all of F. Deleting a single edge $f \in F$ breaks all cycles containing F. Thus $G - e - f$ is disconnected, which yields $(e, f) \in R$, which prevent e and f from being in different classes.

4.2.14. *A graph G is 2-edge-connected if and only if for all $u, v \in V(G)$ there is a u, v-necklace in G,* where a *u, v-necklace* is a list of cycles $C_1, \ldots, C_k$ such that $u \in C_1$, $v \in C_k$, consecutive cycles share one vertex, and non-consecutive cycle are disjoint. The condition is sufficient, because a u, v-necklace has two edge-disjoint u, v-paths, and these cannot both be cut by deleting a single edge. Conversely, suppose that G is 2-edge-connected. We obtain a u, v-necklace.

Proof 1 (induction on $d(u, v)$). Basis step ($d(u, v) = 1$): A u, v-path in $G - uv$ combines with the edge uv to form a u, v-necklace in G.

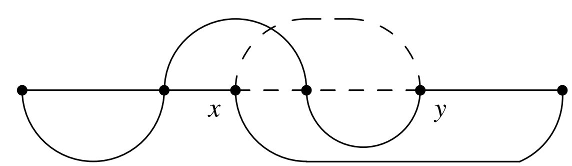
Induction step ($d(u, v) > 1$). Let w be the vertex before v on a shortest u, v-path; note that $d(u, w) = d(u, v) - 1$. By the induction hypothesis, G has a u, w-necklace. If v lies on this u, w-necklace, then the cycles up to the one containing v form a u, v-necklace.

Otherwise, let R be a u, v-path in $G - wv$; this exists since G is 2-edge-connected. Let z be the last vertex of R on the u, w-necklace; let C_j be the last cycle containing z in the necklace. The desired u, v-necklace consists of the cycles before C_j in the u, w-necklace together with a final cycle containing v. The final cycle consists of the remainder of R from z to v, the edge vw, a path from w to C_j in the u, v-necklace, and the path on C_j from there to z that contains the vertex of $C_j \cap C_{j-1}$. The choice of z guarantees that this is a cycle.



Comment. There is also a proof by induction on the number of ears in an ear decomposition, but showing that all pairs still have necklaces when an open ear is added still involves a discussion like that above. Another inductive proof involves showing that the union of a necklace from u to w and an necklace from w to v contains a necklace from u to v.

Proof 2 (extremality). Since G is 2-edge-connected, there exist two edge-disjoint u, v-paths. Among all such pairs of paths, choose a pair P_1, P_2 whose lengths have minimum sum. Let S be the set of common vertices of P_1 and P_2. If the vertices of S occur in the same order on P_1 and P_2, then $P_1 \cup P_2$ is a u, v-necklace. Otherwise, let x, y be the first vertices of P_1 in S that occur in the opposite order on P_2, with x before y in P_1 and after y in P_2. In the figure, P_1 is the straight path. Form two new u, v-paths: Q_1 consists of the portion of P_1 up to x and the portion of P_2 after x, and Q_2 consists of the portion of P_2 up to y and the portion of P_1 after y. Neither of Q_1, Q_2 uses any portion of P_1 or P_2 between x and y, so we have found edge-disjoint u, v-paths with shorter total length. This contradiction completes the proof.



4.2.15. *If G is a 2-connected graph and $v \in V(G)$, then v has a neighbor u such that $G - u - v$ is connected.*

Proof 1 (structure of blocks). Because G is 2-connected, $G - v$ is connected. If $G - v$ is 2-connected, then we may let u be any neighbor of v. If $G - v$ is not 2-connected, let B be a block of $G - v$ containing exactly one cutvertex of $G - v$, and call that cutvertex x. Now v must have a neighbor in $B - x$, else $G - x$ is disconnected, with $B - x$ as a component. Let u be a neighbor of v in $B - x$. Since $B - u$ is connected, $G - v - u$ is connected.

Proof 2 (extremality) If v has no such neighbor, then for every $u \in N(v)$, the graph $G - v - u$ is disconnected. Choose $u \in N(v)$ such that $G - v - u$ has as small a component as possible; let H be the smallest component of $G - v - u$. Since G is 2-connected, v and u have neighbors in every component of $G - v - u$. Let x be a neighbor of v in H. If $G - v - x$ is disconnected, then it has a component that is a proper subgraph of H. This contradicts the choice of u, so $G - v - x$ is connected.

4.2.16. *If G is a 2-connected graph, and T_1 and T_2 are two spanning trees of G, then T_1 transforms into T_2 by a sequence of operations in which a leaf is removed and reattached using another edge of G.* Let T be a largest tree contained in both T_1 and T_2; this is nonempty, since each single vertex is such a tree. We use induction on the number of vertices of G omitted by T. If none are omitted, then $T_1 = T_2$ and the sequence has length 0. If one vertex is omitted, then it is a leaf in both T_1 and T_2, and a single reattachment suffices.

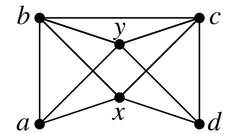
Otherwise, for $i \in \{1, 2\}$ let $x_i y_i$ be an edge of T_i with $x_i \in V(T)$ and $y_i \notin V(T)$. If $y_1 \neq y_2$, then enlarge $T + x_1y_1 + x_2y_2$ to a spanning tree T' of G. Since T' shares more with T_1 than T does, the induction hypothesis yields a sequence of leaf exchanges that turns T_1 into T'. Similarly, it yields a sequence that turns T' into T_2. Together, they complete the desired transformation.

Hence we may assume that $y_1 = y_2$ (this may be necessary even when T omits many vertices of G). We generate another edge x_3y_3 with $x_3 \in V(T)$ and $y_3 \in V(G - y_1)$ (this is possible since G is 2-connected). Now enlarge $T + x_1y_1 + x_3y_3$ to a spanning tree T' and $T + x_2y_2 + x_3y_3$ to a spanning tree T''. For each pair (T_1, T'), (T', T''), or (T'', T_2), there is now a common subtree consisting of T and one additional edge. Hence we can use the induction hypothesis to turn T_1 into T', then T' into T'', and finally T'' into T_2, completing the desired transformation.

(Note: Induction also yields the statement that the common subtree T is never changed during the transformation.)

4.2.17. *The smallest graph with connectivity 3 having a pair of nonadjacent vertices joined by 4 internally-disjoint paths.* "Smallest" usually means least number of vertices, and within that the least number of edges. Let x, y be the nonadjacent pair joined by 4 internally disjoint paths. Each such path has at least one vertex and two edges, so we have at least four more vertices $\{a, b, c, d\}$. We construct a graph achieving this. Since G must be 3-connected, $G - \{x, y\}$ is connected, so if we add no more vertices we

must have a tree on the other four vertices. We add the path a, b, c, d. To complete the prove, we need only show that the graph we have constructed has connectivity 3. Deleting $\{b, x, y\}$ separates a from $\{c, d\}$. To see that G is 3-connected, observed that for each v, $G - v$ contains a spanning cycle and hence is 2-connected, so G is 3-connected.



4.2.18. *If a graph G has no isolated vertices and no even cycles, then every block of G is an edge or a cycle.* A block with two vertices is an edge (if there are no even cycles, then there are no multiple edges). A block H with more than two vertices is 2-connected and has an ear decomposition. If H is not a single cycle, then the addition of the first ear to the first cycle creates a subgraph in which a pair of vertices is connected by three pairwise internally-disjoint paths. By the pigeonhole principle, two of the paths have length of the same parity (both odd or both even), and their union is an even cycle. Hence H must be a single cycle.

4.2.19. *Membership in common cycles.*

a) Two distinct edges lie in the same block of a graph if and only if they belong to a common cycle. Choose $e, f \in E(G)$. If e and f lie in a cycle, then this cycle forms a subgraph with no cut-vertex; by the definition of block, the cycle lies in a single block. Conversely, consider edges e and f in a block B. If e and f have the same endpoints, then they form a cycle of length 2. Otherwise, B has at least three vertices and is 2-connected. In a 2-connected graph, for every edge pair e, f, there is a cycle containing e and f.

b) If $e, f, g \in E(G)$, and G has a cycle through e and f and a cycle through f and g, then G also has a cycle through e and g. By part (a), e and f lie in the same block. By part (a), f and g lie in the same block. Since the blocks partition the edges, this implies that e and g lie in the same block. By part (a), this now implies that some cycle in G contains e and g.

4.2.20. *k-connectedness of the hypercube Q_k by explicit paths.* We use induction on k to show that for $x, y \in V(Q_k)$, there are k pairwise internally disjoint x, y-paths for each vertex pair $x, y \in V(Q_k)$. When $k = 0$, the claim holds vacuously.

For $k > 1$, consider vertex x and y as binary k-tuples. Suppose first that they agree in some coordinate. If they agree in coordinate j, then let Q be the copy of Q_{k-1} in Q_k whose vertices all have that value in coordinate j, and let Q' be the other copy of Q_{k-1}. By the induction hypothesis, Q

contains $k-1$ pairwise internally disjoint x, y-paths. Let x' and y' be the neighbors of x and y in Q'. Combining an x', y'-path in Q' with the edges xx' and yy' yields the kth path, since it has no internal vertices in Q.

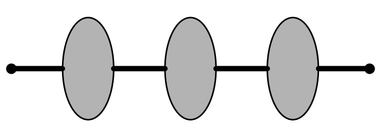
If x and y agree in no coordinate, then we define k paths explicitly as follows. The jth path begins from x by flipping the jth coordinate, then the $j+1$st, $j+2$nd, etc., cyclically (flipping the first coordinate after the kth). After k steps, the path reaches y. The vertices on the jth path agree with y for a segment of positions starting with coordinate j and agree with x for a segment ending with coordinate $j-1$, so the paths share no internal vertices.

4.2.21. *If G is $2k$-edge-connected and has at most two vertices of odd degree, then G has a k-edge-connected orientation.* It suffices to orient the edges so that at least k edges leave each nonempty proper subset of the vertices. When $k = 0$, the statement is trivial, so we may assume that $k > 0$.

Since G has at most two vertices of odd degree, G has an Eulerian trail. Choose an Eulerian trail T. Let D be the orientation obtained by orienting each edge of G in the direction in which T traverses it. Let $[S, \overline{S}]$ be an edge cut of G. When crossing the cut, the trail alternately goes from one side and then from the other, so it alternately orients edges leaving or entering S. Since G is $2k$-connected, $\left|[S, \overline{S}]\right| \ge 2k$, and the alternation means that at least k edges leave each side in the orientation.

4.2.22. *If $\kappa(G) = k$ and $\operatorname{diam} G = d$, then $n(G) \ge k(d-1)+2$ and $\alpha(G) \ge \lceil (1+d)/2 \rceil$, and these bounds are best possible.* Let G be a k-connected graph with diameter d, in which $d(x, y) = d$. Since G is k-connected, Menger's Theorem guarantees k pairwise internally disjoint x, y-paths in G. With x and y, these paths form a set of $k(d-1)+2$ vertices in G. The vertices consisting of all vertices having even distance from x along a shortest x, y-path form an independent set of size $\lceil (1+d)/2 \rceil$.

For optimality of the bounds, let $V_0, \ldots, V_d$ be "level sets" of size k, except that $|V_0| = |V_d| = 1$. Form G on these $k(d-1)+2$ vertices by making each vertex adjacent to the vertices in its own level and the two neighboring levels. The graph G has order $k(d-1)+2$ and diameter d. Also it is k-connected; if fewer than k vertices are deleted, then each internal set still has an element, so paths remain from each remaining vertex to each neighboring layer. The vertex set is covered by $\lceil (1+d)/2 \rceil$ cliques (each consisting of two consecutive levels), so $\alpha(G) \le \lceil (1+d)/2 \rceil$.



4.2.23. *König-Egerváry from Menger.* Let G be an X, Y-bigraph. Form a digraph G' by adding a vertex x with edges to X and a vertex y with edges from Y, and direct the edges of G from X to Y in G'. The idea is that internally disjoint x, y-paths in G' correspond to edges of a matching in G. Menger's Theorem states that the condition for having a set of k internally disjoint x, y-paths in G' (and hence a matching of size k in G) is that every x, y-separating set R has size at least k.

If we delete the endpoints from a set of internally disjoint x, y-paths in G', we obtain a set of edges in G with no common endpoints. Hence $\alpha'(G) \ge \lambda_{G'}(x, y)$.

An x, y-separating set R in G' consists of some vertices in X and some vertices in Y. In order to break all x, y-paths in G', such a set must contain an endpoint of every edge in G. Hence R is a vertex cover in G. Applying this to a smallest x, y-separating set yields $\kappa_{G'}(x, y) \ge \beta(G)$.

By Menger's Theorem, we now have $\alpha'(G) \ge \lambda_{G'}(x, y) = \kappa_{G'}(x, y) \ge \beta(G)$. Since weak duality yields $\alpha'(G) \le \beta(G)$ for every graph G, we have $\alpha'(G) = \beta(G)$ (König-Egerváry Theorem).

4.2.24. *If G is k-connected, and S, T are disjoint subsets of $V(G)$ with size at least k, then there exist k pairwise disjoint S, T-paths.* By the Expansion Lemma, we can add a vertex x adjacent to each vertex of S and a vertex y adjacent to each vertex of T, and the resulting graph will also be k-connected. Menger's Theorem then yields k disjoint x, y-paths, and since x is adjacent to all X and y to all Y we may assume each path has only one vertex of X and only one vertex of Y. If we delete x and y from these paths, we obtain k pairwise disjoint S, T-paths in G.

4.2.25. *Dirac's Theorem that every k vertices in a k-connected graph lie on a cycle is best possible.* $K_{k,k+1}$ is a k-connected graph where the $k+1$ vertices of the larger partite set do not lie on a cycle.

4.2.26. *For $k \ge 2$, a graph G with at least $k+1$ vertices is k-connected if and only if for every $T \subseteq S \subseteq V(G)$ with $|S| = k$ and $|T| = 2$, there is a cycle in G that contains T and avoids $S - T$.*

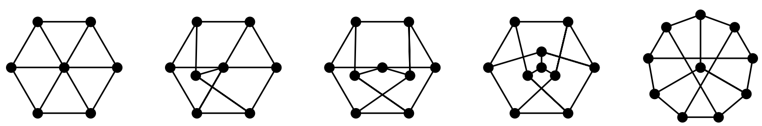
Necessity. If G is k-connected, then $G - (S - T)$ is 2-connected, since $|S - T| = k - 2$. In a 2-connected graph, every pair of vertices (such as T) lies on a cycle. Since $S - T$ has been discarded, the cycle avoids it.

Sufficiency. We prove the contrapositive. If G is not k-connected, then G has a separating set U of size $k-1$. Let T consist of one vertex from each of two components of $G - U$. Let $S = T \cup U$. The condition now fails, since deleting $S - T$ leaves no cycle through both vertices of T.

4.2.27. A *vertex k-split* of a graph G is a graph H obtained from G by replacing one vertex $x \in V(G)$ by two adjacent vertices x_1, x_2 such that $d_H(x_i) \ge k$ and that $N_H(x_1) \cup N_H(x_2) = N_G(x) \cup \{x_1, x_2\}$.

a) If G is a k-connected graph, and G' is a graph obtained from G by replacing one vertex $x \in V(G)$ with two adjacent vertices x_1, x_2 such that $N_H(x_1) \cup N_H(x_2) = N_G(x) \cup \{x_1, x_2\}$ and $d_H(x_i) \ge k$, then G is k-connected. Suppose S is a separating j-set of H, where $j < k$, and let $X = \{x_1, x_2\}$. Note that $H - S$ cannot have x_1, x_2, or X as a component, because $d_H(x_i) \ge k$ and X has edges to at least k distinct vertices of $H - X$. If $|S \cap X| = 2$, then $H - S = G - (S - X \cup x)$, and $S - X \cup x$ is a separating $j - 1$-set of G. If $|S \cap X| = 1$, then $S \cup X$ separates H, since $X - S$ is not a component of $H - S$. Hence $S - X \cup x$ is a separating j-set of G, which requires $j \ge k$. Finally, suppose $S \cap X = \varnothing$. Now $\{x_1, x_2\}$ must belong to the same component of $H - S$. Contracting an edge of a component in a disconnected graph leaves a disconnected graph, so in this case S separates G.

b) Every graph obtained from a "wheel" $W_n = K_1 \vee C_{n-1}$ by a sequence of edge additions and vertex 3-splits on vertices of degree at least 4 is 3-connected. Since wheels are 3-connected, part (a) implies that every graph arising from wheels by 3-splits and edge additions is also 3-connected. The Petersen graph arises by successively splitting off vertices from the central vertex of the wheel $K_1 \vee C_6$. Each newly-split vertex acquires two neighbors on the outside and remains adjacent to the central vertex.



4.2.28. *If X and Y are disjoint vertex sets in a k-connected graph G and are assigned nonnegative integer weights with $\sum_{x \in X} u(x) = \sum_{y \in Y} w(y) = k$, then G has k pairwise internally disjoint X, Y-paths from X to Y such that $u(x)$ of them start at x and $w(y)$ of them end at y.* We may assume that all weights are positive, since otherwise we delete vertices of weight 0 from X and Y and apply the argument to the sets that remain.

We construct a related G' and apply Menger's Theorem. Add copies of vertices in X and Y, with each new vertex having the same neighborhood as the vertex it copies. Since G is k-connected, these neighborhoods have size at least k, and by the Expansion Lemma the new graph is k-connected. We do this until there are $u(x)$ copies of each x and $w(y)$ copies of each y.

Next add two additional vertices s and t joined to the copies of all $x \in X$ and the copies of all $y \in Y$, respectively. Note that s and t each have degree k in this final graph G'. By the Expansion Lemma, G' is k-connected. By Menger's Theorem, there are k pairwise internally disjoint s, t-paths in G'.

These must depart s via its k distinct neighbors and reach t via its k distinct neighbors, so each path connects a copy of some $x \in X$ to a copy of some $y \in Y$, and no x or y appears in one of these paths except at endpoints. Collapsing G' to G by identifying the copies of each original vertex turns these into the desired paths, since there are $u(x)$ copies of each x and $w(y)$ copies of each y and one path at the original vertex arising from each copy of it in G'.

4.2.29. *Graph connectivity from connectivity in the corresponding symmetric digraph.* From a graph G, we form D be by replacing each edge with two oppositely-directed edges. Given two vertices a, b on a path P, let $P(a, b)$ denote the a, b-path along P.

If $\kappa'_D(x, y) = \lambda'_D(x, y)$, then $\kappa'_G(x, y) = \lambda'_G(x, y)$. It suffices to prove that $\lambda'_G(x, y) \ge \lambda'_D(x, y)$ and $\kappa'_G(x, y) \le \kappa'_D(x, y)$, since the weak duality $\lambda'_G(x, y) \le \kappa'_G(x, y)$ holds always.

Let **F** be a family of $\lambda'_D(x, y)$ pairwise edge-disjoint x, y-paths in D. If there is some vertex pair u, v such that uv appears in a path P in **F** and vu appears in another path Q in **F**, then we modify **F**. Let P' be path consisting of $P(x, u)$ followed by $Q(u, y)$, and let Q' be the path consisting of $Q(x, v)$ followed by $P(v, y)$. Replacing P and Q with P' and Q' in **F** reduces the number of edges that used in both directions. Repeating this replacement yields a family **F**$'$ with no such doubly-used pair. Now **F**$'$ becomes a family of $\lambda'_D(x, y)$ pairwise edge-disjoint x, y-paths in G using the same succesion of vertices, and hence $\lambda'_G(x, y) \ge \lambda'_D(x, y)$.

Let R be a set of $\kappa'_D(x, y)$ edges in D whose removal makes y unreachable from x. By the construction of D from G, every x, y-path in G must use an edge having a copy in R. Hence the corresponding edges in G form an x, y-disconnecting set, and $\kappa'_G(x, y) \le \kappa'_D(x, y)$.

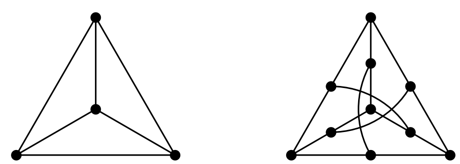
If $x \not\to y$ in D and $\kappa_D(x, y) = \lambda_D(x, y)$, then $\kappa_G(x, y) = \lambda_G(x, y)$. It suffices to prove that $\lambda_G(x, y) \ge \lambda_D(x, y)$ and $\kappa_G(x, y) \le \kappa_D(x, y)$, since the weak duality $\lambda_G(x, y) \le \kappa_G(x, y)$ holds always.

Let **F** be a family of $\lambda_D(x, y)$ pairwise internally-disjoint x, y-paths in D. Since these pairs pairwise share no vertices other than their endpoints, there is no pair u, v such that the edges uv and vu are both used. In particular, the paths (listed by vertices) in **F** also form a family of $\lambda_D(x, y)$ pairwise internally-disjoint x, y-paths in G, and $\lambda_G(x, y) \ge \lambda_D(x, y)$.

Let R be a set of $\kappa_D(x, y)$ vertices in D whose removal makes y unreachable from x. By the construction of D from G, every x, y-path in G uses a vertex of R. Hence R is an x, y-separating set in G, and $\kappa_G(x, y) \le \kappa_D(x, y)$.

4.2.30. *Expansion preserves 3-connectedness.* Suppose that G' is obtained from G by expansion (subdividing xy and wz and adding an edge st joining the two new vertices). It suffices to show that if G is 3-connected, then deleting a vertex from G' always leaves a 2-connected graph. If $v \in V(G)$, then we can obtain an ear decomposition of $G' - v$ from an ear decomposition of $G - v$ by making the ear a bit longer when the edge xy or wz is added and adding the edge st at the end. To obtain an ear decomposition of $G' - t$, observe that $G - wz$ is 2-connected (deleting an edge reduces connectivity by at most 1). Use an ear decomposition of $G - wz$, lengthening the ear when xy is added, and then add two ears through t. (There are many other proofs.)

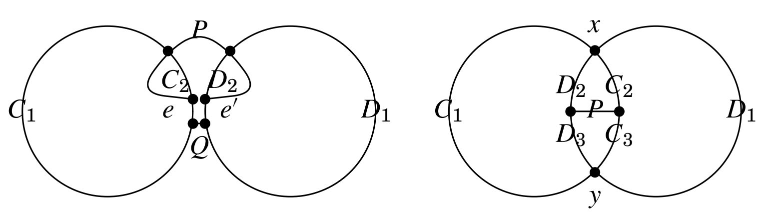
To obtain the Petersen graph from K_4 by expansions, perform expansion on the three pairs of nonincident edges in K_4, independently.



4.2.31. *Longest cycles in k-connected graphs.*

a) In a k-connected graph (for $k = 2, 3$), any two longest cycles have at least k vertices in common. (The claim is false for $k = 1$, as shown by two cycles joined by a single cut edge.) Let $l(H)$ denote the length of a cycle or path H, let C, D be two longest cycles, and let $S = V(C) \cap V(D)$. The proof is by contradiction; if $|S| < k$, it suffices to construct two other cycles C', D' such that $l(C') + l(D') > l(C) + l(D)$, because then C and D are not longest cycles in G.

Consider $k = 2$. Let e be an edge of C, and e' an edge of D, chosen to share the vertex of S if $|S| = 1$. Since G is 2-connected, there is a cycle R containing both e and e'. The two portions of R between e and e' contain paths P, Q that travel from $V(C)$ to $V(D)$ with no vertices of $V(C) \cup V(D)$ along the way. (If $|S| = 1$, then one of these paths is a single vertex and has length 0.) Note that since R is a cycle, P and Q are disjoint. The vertices where P and Q intersect C and D partition C and D into paths C_1, C_2 and D_1, D_2, respectively. Let $C' = C_1 \cup P \cup D_1 \cup Q$ and $D' = C_2 \cup P \cup D_2 \cup Q$; we have $l(C') + l(D') = l(C) + l(D) + 2l(P) + 2l(Q) > l(C) + l(D)$.



Consider $k = 3$. Since G is also 2-connected, we may assume by the

argument above that $|S| = 2$. Now $G - S$ is connected and has a shortest path P between $C - S$ and $D - S$. The vertices where P meets C and D, together with the vertices $S = \{x, y\}$, partition C and D into three paths C_1, C_2, C_3 and D_1, D_2, D_3, where C_1, D_1 are y, x-paths, C_2, D_2 are $x, V(P)$-paths, and C_3, D_3 are $y, V(P)$-paths. Let $C' = C_1 \cup C_2 \cup P \cup D_3$ and $D' = D_1 \cup D_2 \cup P \cup C_3$. Now $l(C') + l(D') = l(C) + l(D) + 2l(P) > l(C) + l(D)$.

b) For $k \ge 2$, one cannot guarantee more than k common vertices. The graph $K_{k,2k}$ is k-connected and has two cycles sharing only the smaller partite set.

4.2.32. *Given $k \ge 2$, let G_1 and G_2 be disjoint k-connected graphs, with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. If B is a bipartite graph with parts $N_{G_1}(v_1)$ and $N_{G_2}(v_2)$ that has no isolated vertex and has a matching of size at least k, then $(G_1 - v_1) \cup (G_2 - v_2) \cup B$ is k-connected.* Let $G = (G_1 - v_1) \cup (G_2 - v_2) \cup B$. It suffices to show that for distinct vertices $x, y \in V(G)$, there is a family of k independent x, y-paths.

If $x, y \in V(G_1)$, then there are k such paths from G_1, except that one of them may pass through v. If x' and y' are the neighbors of v along this path, then we replace $\langle x', v, y' \rangle$ with a path through G_2, using edges in B incident to x' and y'. The argument is symmetric when $x, y \in V(G_2)$.

If $x \in V(G_1)$ and $y \in V(G_2)$, then let $X \subseteq N_{G_1}(v_1)$ and $Y \subseteq N_{G_2}(v_2)$ be the partite sets of a matching M of size k in B. Deleting v_1 from k independent x, v_1-paths in G_1 leaves an x, X-fan. Similarly, deleting v_2 from k independent y, v_2-paths in G_2 leaves an y, Y-fan. Combining M with these two fans yields the desired x, y-paths.

The claim fails for $k = 1$. If G_1 and G_2 are stars, with centers v_1 and v_2, then the resulting graph G is simply the bipartite graph B. The only requirement on B is that it have no isolated vertices. In particular, it need not be connected.

4.2.33. *Ford-Fulkerson CSDR Theorem implies Hall's Theorem.* Given an X, Y-bigraph G with $X = \{x_1, \ldots, x_m\}$, let $A_i = B_i = N(x_i)$. If the systems $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ have a common CSDR, then $A_1, \ldots, A_m$ has an SDR, and thus G has a matching saturating X. Thus it suffices to show that Hall's Condition on G implies the Ford-Fulkerson condition for these systems.

Let $I, J \subseteq [m]$ be sets of indices. Since $\bigcup_{j \in J} B_j = \bigcup_{j \in J} A_j$, we have

$$\left|\bigcup_{i \in I} A_i \cap \bigcup_{j \in J} B_j\right| = \left|(\bigcup_{i \in I} A_i) \cap \bigcup_{j \in J} A_j)\right| \ge \left|\bigcup_{i \in I \cap J} A_i\right|.$$

By Hall's Condition, $\left|\bigcup_{i \in I \cap J} A_i\right| \ge |I \cap J| \ge |I| + |J| - m$. Thus the Ford-Fulkerson condition holds in G, as desired.

If $A_1, \dots, A_m$ and $B_1, \dots, B_m$ are partitions of a set E into sets of size s, then the two systems have a CSDR. It suffices to show that the systems satisfy the Ford-Fulkerson condition. By the defining condition, $\left|\bigcup_{i\in I} A_i\right| = |I|\,s$ and $\left|\bigcup_{j\in J} B_j\right| = |J|\,s$. Thus

$$\left|\bigcup_{i\in I} A_i \cap \bigcup_{j\in J} B_j\right| \ge \left|\bigcup_{i\in I} A_i\right| + \left|\bigcup_{j\in J} B_j\right| - ms = |I|\,s + |J|\,s - ms$$
$$= s(|I| + |J| - m) \ge |I| + |J| - m.$$

4.2.34. *Every minimally 2-connected graph has a vertex of degree 2.* Consider an ear decomposition of a minimally 2-connected graph G. If the last ear adds just one edge e, then $G - e$ also has an ear decomposition and is 2-connected. Hence the last ear added contains a vertex of degree 2.

A minimally 2-connected graph G with at least 4 vertices has at most $2n(G) - 4$ edges, with equality only for $K_{2,n-2}$. The graph $K_{2,n-2}$ is minimally 2-connected and has $2n - 4$ vertices. For the upper bound, we use induction on $n(G)$. When $n(G) = 4$, $K_{2,2}$ is the only minimally 2-connected graph. When $n(G) > 4$, consider an ear decomposition of G. If G is only a cycle, then the bound holds, with strict inequality. Otherwise, delete the last added ear from G to obtain G'. This deletes k vertices and $k + 1$ edges, where $k \ge 1$ as observed above.

The graph G' is also minimally 2-connected, since if $G' - e$ is 2-connected, then also $G - e$ is 2-connected. Hence $e(G') \le 2n(G') - 4$, by the induction hypothesis. In terms of G, this states that $e(G) - k - 1 \le 2n(G) - 2k - 4$, which simplifies to $e(G) \le 2n(G) - k - 3 \le 2n(G) - 4$. Equality requires $k = 1$, and by the induction hypothesis also $G' = K_{2,n-3}$. The only way to add an ear of length two to $K_{2,n-3}$ and obtain a minimally 2-connected graph is to add it connecting the two vertices of high degree.

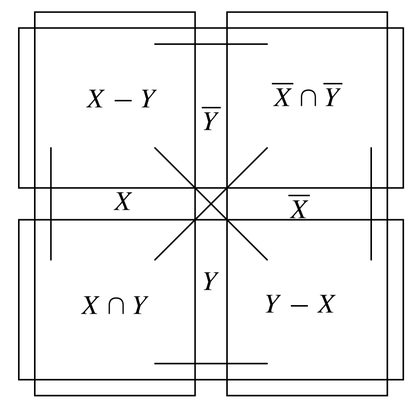
4.2.35. *A 2-connected graph is minimally 2-connected if and only if no cycle has a chord.* Suppose that G is 2-connected. We show that $G - xy$ is 2-connected if and only if x and y lie on a cycle in $G - xy$. If $G - xy$ is not 2-connected, then there is a vertex v whose deletion separates x and y, and thus all x,y-paths in $G - xy$ pass through v and $G - xy$ has no cycle containing x and y. Conversely, if $G - xy$ is 2-connected, then every pair of vertices (including x, y) lies on a cycle.

If a cycle in G has a chord x, y, then this argument shows that $G - xy$ is still 2-connected, and hence G is not minimally 2-connected. If no cycle has a chord, then for any edge xy, the graph $G - xy$ has no cycle containing x and y, and so $G - xy$ is not 2-connected.

4.2.36. If $X, Y \subseteq V(G)$, then $d(X \cap Y) + d(X \cup Y) \le d(X) + d(Y)$, where $d(S)$ is the number of edges leaving S. With respect to the sets X, Y, there

are four types of vertices, belonging to none, either, or both of the two sets. Between pairs of the four sets $X \cap Y$, $X - Y$, $Y - X$, $\overline{X} \cap \overline{Y}$, there are six types of edges. We list the contribution of each type to the counts on both sides of the desired inequality. Each edge contributes at least as much to the right side as to the left side of the inequality. This proves the inequality; note that equality holds if and only if $[X - Y, Y - X]$ is empty.

endpoints	$d(X \cap Y)$	$d(X \cup Y)$	$d(X)$	$d(Y)$
$X \cap Y,\ X - Y$	•			•
$X \cap Y,\ Y - X$	•		•	
$X \cap Y,\ \overline{X} \cap \overline{Y}$	•	•	•	•
$X - Y,\ \overline{X} \cap \overline{Y}$		•	•	
$Y - X,\ \overline{X} \cap \overline{Y}$		•		•
$X - Y,\ Y - X$			•	•



4.2.37. *Every minimally k-edge-connected graph G has a vertex of degree k.* Let $d(X) = \left|[S, \overline{S}]\right|$. If $d(X) > k$ whenever $\varnothing \ne X \subset V(G)$, then $G - e$ is k-edge-connected for each $e \in E(G)$, and G is not minimally k-edge-connected. Hence we may assume that $d(X) = k$ for some set X.

Suppose that $G[X]$ has an edge xy. Since $G - xy$ is not k-edge-connected, there is a nonempty $Z \subset V(G)$ (containing exactly one of $\{x, y\}$) such that $k - 1 \ge d_{G-xy}(Z) = d(Z) - 1$. Since G is k-edge-connected, $d(Z) \ge k$, so equality holds.

Now k-edge-connectedness of G and submodularity of d (the result of Exercise 4.2.36 yield

$$k + k \le d(X \cap Z) + d(X \cup Z) \le d(X) + d(Z) = k + k.$$

Since G is k-edge-connected, we obtain $d(X \cap Z) = k$. Since Z contains exactly one of $\{x, y\}$, the set $X \cap Z$ is smaller than X.

Hence a minimal set X such that $d(X) = k$ must be an independent set. Since each vertex of X has at least k incident edges leaving X, we have $|X| = 1$, and this is the desired vertex of degree k.

4.2.38. *Every $2k$-edge-connected graph has a k-edge-connected orientation.* To prove this theorem of Nash-Williams, we are given Mader's Shortcut Lemma: "If z is a vertex of a graph G such that $d_G(z) \notin \{0, 1, 3\}$ and z is incident to no cut-edge, then z has neighbors x and y such that $\kappa_{G-xz-yz+xy}(u, v) = \kappa_G(u, v)$ for all $u, v \in V(G) - \{z\}$."

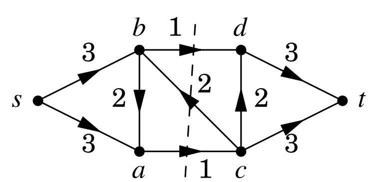
We use induction on $n(G)$. For the basis step, consider two vertices joined by at least $2k$ edges, and orient at least k in each direction.

For the induction step, let G be a $2k$-edge-connected graph with $n(G) > 2$. We discard edges to obtain a minimal $2k$-edge-connected graph; we may later orient the deleted edges arbitrarily. By Exercise 4.2.37, the resulting graph has a vertex z of degree $2k$, which is even. Mader's Shortcut Lemma iteratively finds shortcuts of z until we reduce the degree of z to 0. Throughout this process, we maintain $2k$-edge-connectedness for pairs of points not including z. At the end, we delete z to obtain a $2k$-edge-connected graph G' with $n(G) - 1$ vertices.

By the induction hypothesis, G' has a k-edge-connected orientation. Orient G by replacing each shortcut edge uv with the path u, z, v or v, z, u, oriented consistently with uv in G'. For $X \neq \{z\}$, lifting uv preserves $d(X) \geq k$ in the orientation; the only edge lost is uv, and if uv leaves X, then uz or zv is a new edge leaving X, depending on whether $z \in X$. The set $X = \{z\}$ itself reaches $d(X) = k$ after all k lifts.

4.3. NETWORK FLOW PROBLEMS

4.3.1. *Listing of feasible integer s,t-flows in a network.* This problem demonstrates the value of integer min-max relations in escaping exhaustive computation.



A feasible flow is an assignment of a flow value to each edge. It is not an assignment of flow paths. Every network has a feasible flow of value 0. In this network, there is a cycle ba, ac, cb with positive capacity, which makes it possible to "add" to a flow without adding

to the value of the flow. In particular, there are two feasible integer flows of value 0, eight of value 1, and four of value 2. We can specify each flow by the vector of values on the edges. We list these as $(f(sa), f(sb), f(ba), f(ac), f(bd), f(cb), f(cd), f(ct), f(dt))$, with each column of the matrix below corresponding to one flow.

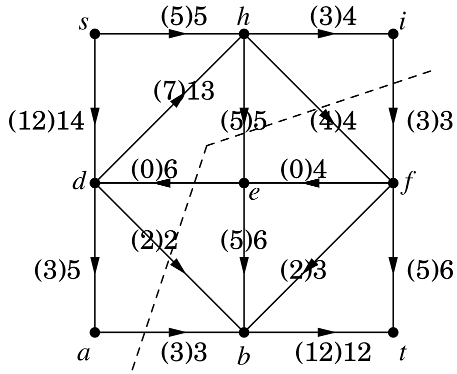
sa	0	0	1	1	1	0	0	0	0	0	1	0	1	0
sb	0	0	0	0	0	1	1	1	1	1	1	2	1	2
ba	0	1	0	0	0	1	1	1	0	1	0	1	0	1
ac	0	1	1	1	1	1	1	1	0	1	1	1	1	1
bd	0	0	0	0	1	0	0	1	1	1	1	1	1	1
cb	0	1	0	0	1	0	0	1	0	1	0	0	0	0
cd	0	0	0	1	0	0	1	0	0	0	0	0	1	1
ct	0	0	1	0	0	1	0	0	0	0	1	1	0	0
dt	0	0	0	1	1	0	1	1	1	1	1	1	2	2
value	0	0	1	1	1	1	1	1	1	1	2	2	2	2

Since the network has four internal vertices, there are $2^4 = 16$ ways to specify a source-sink cut $[S, T]$. In general, the resulting edge cuts might not be distinct as sets of edges, but for this network they are distinct. Incidence vectors for the cuts appear in the columns below; a 1 for edge e in column $[S, T]$ means that e belongs to the cut $[S, T]$. The cut $[sab, cdt]$ with capacity equal to the maximum flow value is suggested by dashed lines in the figure. Exhibiting a flow and a cut of equal value proves that the flow value is maximal and the cut capacity is minimal; this is a shorter and more reliable proof of flow optimality than listing all feasible flows.

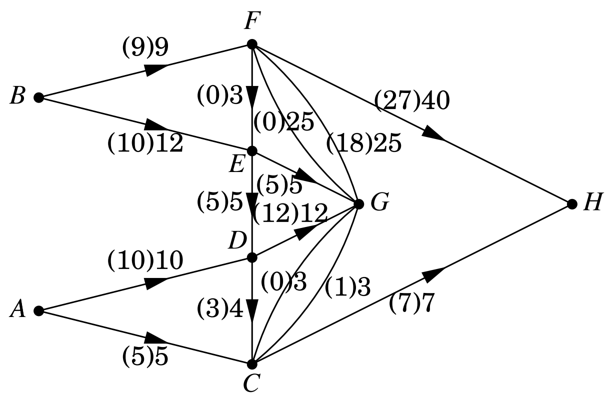
S	*s*	*sa*	*sb*	*sc*	*sd*	*sab*	*sac*	*sad*	*sbc*	*sbd*	*scd*	*sabc*	*sabd*	*sacd*	*sbcd*	*sabcd*
T	*abcdt*	*bcdt*	*acdt*	*abdt*	*abct*	*cdt*	*bdt*	*bct*	*adt*	*act*	*abt*	*dt*	*ct*	*bt*	*at*	*t*
sa	1	0	1	1	1	0	0	0	1	1	1	0	0	0	1	0
sb	1	1	0	1	1	0	1	1	0	0	1	0	0	1	0	0
ba	0	0	1	0	0	0	0	0	1	1	0	0	0	0	1	0
ac	0	1	0	0	0	1	0	1	0	0	0	0	1	0	0	0
bd	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0	0
cb	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0
cd	0	0	0	1	0	0	1	0	1	0	0	1	0	0	0	0
ct	0	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1
dt	0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1
capac	6	4	6	13	9	2*	10	7	11	8	14	6	4	11	11	6

4.3.2. *In the network with edge capacities as indicated below, the flow values listed in parentheses form a maximum feasible flow.*

By inspection, they satisfy the capacity constraints and the conservation constraints. The value of the flow is 17. To prove that the flow is optimal, we exhibit a solution to the dual (min cut) problem that has the same value. This proves optimality because the value of every feasible flow is at most the capacity of this cut. The cut has source set $\{s, h, i, d, a\}$ and sink set $\{e, f, b, t\}$. The edges of the cut are $\{if, hf, he, db, ab\}$ with total capacity 17; the edge ed does not belong to this cut.



4.3.3. *A maximum flow problem.* Add a source X with edges of infinite capacity to the true sources A and B. The optimal net flow from source to sink is 34. This is optimal because the cut $[XABE, CDFGH]$ has capacity 9+5+5+10+5=34, and no flow can have value larger than the capacity of any cut. The optimal flow values on the various edges are indicated in parentheses on the edges below. (The edges from $XABE$ to $CDFGH$ are saturated; those going back have zero flow.)



4.3.4. *Maximum flow in a network with lower bounds.* When the lower bounds equal 0, we have an ordinary network, and the ordinary Ford-Fulkerson labeling algorithm begins with the feasible 0 flow. In a network

with lower bound $l(uv)$ on the flow in each edge uv (and upper bound $c(uv)$), the labeling algorithm generalizes when we are given a feasible flow.

We seek an augmenting path to obtain a flow of larger value. In the statement of the algorithm, we still use the same definition of excess capacity when exploring vw from v. For an edge uv entering v, the requirement for reducing flow along uv to extend the potential augmenting path to u is "excess flow": $f(uv) > l(uv)$. Under either condition, we place the other endpoint of the edge involving v in the set R of "reached vertices". If we reach the sink, we have an augmenting path, and we let ε be the minimum value of the excess capacities $(c(uv) - f(uv))$ along the forward edges in the path and the excess flows $(f(vu) - l(vu))$ along the backward edges in the path. We adjust the values of f along the path by ε (up for forward edges, down for backward edges), again obtaining a feasible flow with value ε larger than f.

If we do not reach the sink after searching from all vertices of R, then the final searched set S provides a source/sink cut $[S, \overline{S}]$ that proves there is no larger feasible flow. Proving this needs a more general definition of cut capacity. The capacity of a source/sink partition $(S, \overline{S})$ is $\sum_{vw\in[S,\overline{S}]} c(vw) - \sum_{uv\in[\overline{S},S]} l(uv)$. This is an upper bound on the net flow from S to $\overline{S}$. Termination without reaching the sink in the algorithm above requires that the flow equals $c(vw)$ whenever $vw \in [S, \overline{S}]$ and equals $l(uv)$ whenever $uv \in [\overline{S}, S]$. Hence the net flow across this cut equals the generalized capacity of the cut.

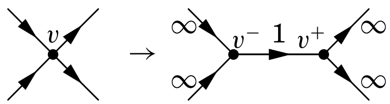
The conservation constraints force the value of a flow f to equal the net flow across any source/sink cut. Thus the value of the final flow equals the generalized capacity of the resulting cut. Since edge cut establishes an upper bound on the value of each feasible flow, this equality shows that both the flow and the cut are optimal.

4.3.5. *Menger for vertices in digraphs, from Ford-Fulkerson.* Consider a digraph G containing vertices x, y, with $xy \notin E(G)$. As usual, the definitions of $\kappa(x, y)$ and $\lambda(x, y)$ yield $\kappa(x, y) \ge \lambda(x, y)$ (weak duality), so the problem is to use the Ford-Fulkerson theorem to prove the opposite inequality. We want to design a suitable network G' so that

$$\lambda(x, y) \ge \max \operatorname{val} f = \min \operatorname{cap}(S, T) \ge \kappa(x, y).$$

In designing a suitable network G', we want to obtain pairwise internally-disjoint x, y-paths in G from units of flow in G'. Thus we have the problem of limiting the total flow through a vertex to 1. Since we can only limit flow via edge capacities, we expand vertex v into two vertices v^- and v^+ joined by an edge v^-v^+ of capacity 1 (for $v \notin \{x, y\}$). Call these the *intra-vertex* edges.

To complete the network, v^- inherits the edges entering v and v^+ inherits those leaving v. More precisely, an edge uv in G becomes an edge u^+v^- in G' (we think of the source x as x^+ and the sink y as y^-. The network also needs capacities on these edges. To simplify our later discussion of the cut, we assign huge capacities to these edges. We may view this as infinite capacity; any integer larger than $n(G)$ suffices.



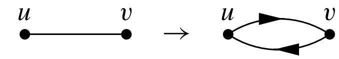
The equality is by the max-flow min-cut theorem. Let k be the common optimal value for the flow problem and the cut problem. For the first inequality, we convert a maximum flow of value k into k pairwise internally-disjoint x, y-paths in G; thus $\lambda(x, y) \geq k$. The integrality theorem breaks the x, y-flow into x, y-paths of unit flow, and these correspond to x, y-paths in G when we shrink the intra-vertex edges. Since each intra-vertex edge has capacity is 1, each vertex of G appears in at most one such path.

For the final inequality, we convert a minimum source/sink cut $[S, T]$ of capacity k into k vertices in G that break all x, y-paths; these yields $\kappa(x, y) \leq k$. If $S = x \cup \{v^-: v \neq x, y\}$ and $T = y \cup \{v^+: v \neq x, y\}$, then $cap(S, T) = n-2 < n(G)$. Thus no minimum capacity cut has an edge from S to T that is not an intra-vertex edge (this is the reason for assigning the other edges large capacity). As a result, the capacity of every minimum cut equals the number of vertices $v \in V(G)$ such that the intra-vertex edge for v belongs to $[S, T]$. Since deleting the edges of the cut leaves no capacity from S to T, these edges break all x, y-paths in G', and thus the corresponding k vertices form an x, y-separating set in G.

4.3.6. *Menger for edge-disjoint paths in graphs, from Ford-Fulkerson.* Consider a graph G containing vertices x, y. The definitions of $\kappa'(x, y)$ and $\lambda'(x, y)$ yield $\kappa'(x, y) \geq \lambda'(x, y)$ (weak duality), so the problem is to use the Ford-Fulkerson theorem to prove the opposite inequality. We design a suitable network G' so that

$$\lambda'(x, y) \geq \max \operatorname{val} f = \min \operatorname{cap}(S, T) \geq \kappa'(x, y).$$

In designing a suitable network G' with source x and sink y, we want to obtain pairwise edge-disjoint x, y-paths in G from units of flow in G'. An edge can be used in either direction. Thus we obtain G' from G by replacing each undirected edge uv with a pair of oppositely directed edges with endpoints u and v, as suggested below. We give each capacity 1 to each resulting edge.

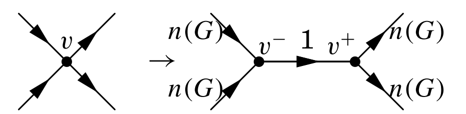


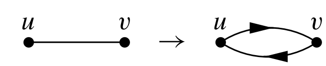
The Integrality Theorem guarantees a maximum flow from x to y in which all values are integers. If this assigns nonzero flow to two oppositely directed edges, then it assigns 1 to each. Replacing these values with 0 preserves the conservation conditions and does not change the value of the flow. Hence we may assume that in our maximum flow each edge from G is used in at most one direction. Now the Integrality Theorem breaks the flow into units of flow from x to y. These yield $\operatorname{val} f$ pairwise edge-disjoint x, y-paths in G, thereby proving the first part of the displayed inequality.

For any cut $[S, T]$ in G' each edge of G between S and T is counted exactly once, in the appropriate direction. Hence $\operatorname{cap}(S, T) = |[S, T]|$. Since $[S, T]$ is an edge cut, the last part of the displayed inequality also holds. We have proved the needed inequality $\lambda'(x, y) \geq \kappa'(x, y)$.

4.3.7. *Menger's Theorem for nonadjacent vertices in graphs: $\kappa(x, y) = \lambda(x, y)$.* Let x and y be vertices in a graph G. An x, y-separating set has a vertex of each path in a set of pairwise internally-disjoint x, y-paths, so $\kappa(x, y) \geq \lambda(x, y)$. It suffices to show that some x, y-separating set and some set of pairwise internally-disjoint x, y-paths have the same size.

Starting with G, first replace each edge uv with two directed edges uv and vu, as on the right below. Next, replace each vertex w outside $\{x, y\}$ with two vertices w^- and w^+ and an edge of unit capacity from w^- to w^+, as on the left below; call these *internal edges*. Every edge that had the form uv before this split now is replaced with the edge u^+v^-, having capacity $n(G)$. Let D be the resulting network, with source x and sink y. (We often write "∞" as a capacity to mean a sufficiently large capacity to keep those edges out of minimum cuts. Here $n(G)$ is enough.)

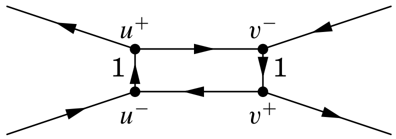




By the Max-Flow Min-Cut Theorem, the maximum value of a feasible flow in D equals the minimum value of a source/sink cut in D. Let k be the common value. We show that G has k pairwise internally disjoint x, y-paths and has an x, y-separating set of size k.

By the Integrality Theorem, there is a flow of value k that has integer flow on each edge. Since only the internal edge leaves v^-, with capacity 1, at most one edge into v^- has nonzero flow, and that flow would be 1. Since only the internal edge enters v^+, with capacity 1, at most one edge leaving

v^+ has nonzero flow, and that flow would be 1. Hence the k units of flow transform back into k x,y-paths in G, and the restriction of capacity 1 on v^-v^+ ensures that these paths are internally disjoint. (This includes the observation that we cannot have one path use the edge from u to v and another from v to u; one can see explicitly that the capacity of 1 on the internal edges directly prevents this, as illustrated below.)



Since the capacity of every edge of the form v^+w^- is $n(G)$, every source/sink cut $[S,T]$ that has some such edge has capacity at least $n(G)$. On the other hand, the cut that has x and all internal vertices of the form u^- in S and has y^- and all internal vertices of the form u^+ in T has capacity $n(G)-2$. Therefore, in every cut with minimum capacity the only edges from S to T are edges of the form u^-u^+. If such a set of edges $[S,T]$ breaks all x,y-paths in D, then $\{u \in V(G)\colon u^-u^+ \in [S,T]\}$ is a set of k vertices in G that breaks all x,y-paths in G.

4.3.8. *Networks to model vertex capacities.* Let G be a digraph with source x, sink y, and vertex capacities $l(v)$ for $v \in V(G)$. To find maximum feasible flow from x to y in G, we define an ordinary network N and use the maximum flow labeling algorithm. For each $v \in V(G)$, create two vertices v^-, v^+ in N, with an edge from v^- to v^+ having capacity $l(v)$. For each $uv \in E(G)$, create an edge $u^+v^- \in E(N)$ with infinite capacity.

Consider a maximum x^+,y^--flow in N, where x,y are the source and sink of G. Contracting all edges of the form v^-v^+ in N transforms any feasible flow in N into a vertex-feasible flow in G with the same value. Similarly, any feasible flow in the vertex-capacitated network G "expands" into a feasible flow in N with the same value. Therefore, the max flow algorithm in N solves the original problem.

4.3.9. *Use of Network Flow to characterize connected graphs.* Given a graph G, form a digraph D by replacing each edge uv of G with the directed edges uv and vu, and give each edge capacity 1. Then G has an x,y-path if and only if the network D with source x and sink y has a flow of value at least 1. By the Max Flow = Min Cut Theorem, this holds if and only every cut S,T with $x \in S$ and $y \in T$ has capacity at least 1, i.e. an edge from S to T. If all partitions S,T have such an edge in G, then for every choice of x and y there is an x,y-path. If for every pair x,y there is a path, the to explore the partition S,T we choose $x \in S$ and $y \in T$, and then the corresponding network problem guarantees that the desired edge exists.

4.3.10. *König-Egerváry from Ford-Fulkerson.* Let G be a bipartite graph with bipartition X,Y. Construct a network N by adding a source s and sink t, with edges of capacity 1 from s to each $x \in X$ and from each $y \in Y$ to t. Orient each edge of G from X to Y in N, with infinite capacity. By the integrality theorem, there is a maximum flow f with integer value at each edge. The edges of capacity one then force the edges between X and Y receiving nonzero flow in f to be a matching. Furthermore, $\mathrm{val}(f)$ is the number of these edges, since the conservation constraints require the flow along each such edge to extend by edges of capacity 1 from s and to t. We have constructed a matching of size $\mathrm{val}(f)$, so $\alpha'(G) \ge \mathrm{val}(f)$.

A minimum cut must have finite capacity, since $[s, V(N)-s]$ is a cut of finite capacity. Let $[S,T]$ be a minimum cut in N, and let $X' = S \cap X$ and $Y' = T \cap Y$. A cut of finite capacity has no edge of infinite capacity from S to T. Hence G has no edge from $S \cap X$ to $T \cap Y$. This means that $(X-S)\cup(Y-T)$ is a set of vertices in G covering every edge of G. Furthermore, the cut $[S,T]$ consist of the edges from s to $X \cap T = X - S$ and from $Y \cap S = Y - T$ to t. The capacity of the cut is the number of these edges, which equals $|(X-S)\cup(Y-T)|$. We have constructed a vertex cover of size $\cap(S,T)$, so $\beta(G) \le \cap(S,T)$.

By the Max flow-Min cut Theorem, we now have $\beta(G) \le \cap(S,T) = \mathrm{val}(f) \le \alpha'(G)$. But $\alpha'(G) \le \beta(G)$ in every graph, so equality holds throughout, and we have $\alpha'(G) = \beta(G)$ for every bipartite graph G.

4.3.11. *The Augmenting Path Algorithm for bipartite graphs (Algorithm 3.2.1) is a special case of the Ford-Fulkerson Labeling Algorithm.* Call these algorithms AP and FF, respectively.

Given an X,Y-bigraph G, construct a network N by directing each edge of G from X to Y, adding vertices s and t with edges sx for all $x \in X$ and yt for all $y \in Y$, and making all capacities 1. A matching M in G determines a flow f in N by letting $f(sx) = f(xy) = f(yt) = 1$ if $xy \in M$, and f be 0 on all other edges.

We run AP on a matching M and FF on the corresponding flow f. AP starts BFS from the unsaturated vertices $U \subseteq X$. Similarly, FF starts by adding U to the reached set $\{s\}$, since the sx arcs for $x \in U$ are exactly those leaving s with flow less than capacity.

Let W be the set of all $y \in Y$ adjacent to some $x \in U$. AP next reaches all of W. Similarly, FF adds W to R (since again the edges reaching them have flow less than capacity) and moves U to S. Next AP moves back to X along edges of M. FF will do so also when searching from W, because these are entering backward edges with flow equal to capacity. Note that FF cannot move forward to t since vertices of W are saturated by M, and thus the edges from W to t have flow equal to capacity.

Iterating this argument shows that AP and FF continue to search the same vertices and edges until one of two things happens. If AP terminates by reaching an unsaturated vertex $y \in Y$ and returns an M-augmenting path, then when FF searches y it finds that $f(yt) = 0$ and reaches t. It also stops and returns the corresponding f-augmenting path.

If AP terminates without finding an augmenting path and instead returns a minimal cover Q, then FF terminates at this time and returns an s,t-cut $[S,T]$ with capacity $|Q|$. It suffices to show that each $v \in G$ contributes 0 to both $|Q|$ and $\mathrm{cap}\,[S,T]$ or contributes 1 to both.

Let F be the search forest created by AP. By what we have shown, the corresponding tree F' of potential f-augmenting paths for FF is F with s attached as a root to the vertices of U. Consider $v = x \in X$. If $x \in V(F)$, then $x \notin Q$ and x contributes 0 to $|Q|$. Since $x \in V(F')$, also $x \in S$. There is no edge xy with $y \in Y \cap T$, since if there were then FF would have entered x from y. It follows that x contributes nothing to $\mathrm{cap}\,[S,T]$.

If on the other hand $x \notin V(F)$, then $x \in Q$, and x contributes 1 to $|Q|$. Also $x \notin V(F')$, so $x \in T$. Hence the only arc from S to T ending at x is sx, and x contributes 1 to $\mathrm{cap}\,[S,T]$. Similar considerations prove the claim when $v = y \in Y$.

4.3.12. Let $[S,\overline{S}]$ and $[T,\overline{T}]$ be source/sink cuts in a network N.

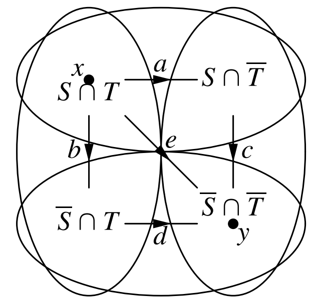
a) $\mathrm{cap}(S\cup T, \overline{S\cup T}) + \mathrm{cap}(S\cap T, \overline{S\cap T}) \le \mathrm{cap}([S,\overline{S}]) + \mathrm{cap}(T,\overline{T})$. Consider the contributions to the two new cuts, as suggested in the diagram below. Let a,b,c,d,e,f,g be the total capacities of the edges in $[S\cap T, S\cap\overline{T}]$, $[S\cap T, \overline{S}\cap T]$, $[S\cap\overline{T}, \overline{S}\cap\overline{T}]$, $[\overline{S}\cap T, \overline{S}\cap\overline{T}]$, $[S\cap T, \overline{S}\cap\overline{T}]$, $[S\cap\overline{T}, \overline{S}\cap T]$, and $[\overline{S}\cap T, S\cap\overline{T}]$, respectively. We have

$$\mathrm{cap}[S\cap T, \overline{S\cap T}] + \mathrm{cap}[S\cup T, \overline{S\cup T}] = (a+b+e) + (c+d+e),$$

$$\mathrm{cap}[S,\overline{S}] + \mathrm{cap}[T,\overline{T}] = (b+c+e+f) + (a+d+e+g).$$

Hence the desired inequality holds.

b) If $[S,\overline{S}]$ and $[T,\overline{T}]$ are minimum cuts, then $[S\cup T, \overline{S\cup T}]$ and $[S\cap T, \overline{S\cap T}]$ are also minimum cuts, and no edge between $S-T$ and $T-S$ has positive capacity. When $[S,\overline{S}]$ and $[T,\overline{T}]$ are minimum cuts, we obtain equality in the inequality of part (a). Neither summand on the left can be smaller, so both must equal the minimum. As shown in part (a), the difference between the two sides is $f+g$, which equals $\mathrm{cap}[S\cap\overline{T}, \overline{S}\cap T] + \mathrm{cap}[\overline{S}\cap T, S\cap\overline{T}]$. Equality requires that the difference be 0, so no edge between $S-T$ and $T-S$ has positive capacity.



4.3.13. *Modeling by network flows.* Several companies send delegates to a meeting; the ith company sends m_i delegates. The conference features simultaneous networking groups; the jth group can accommodate up to n_j delegates. The organizers want to schedule all delegates into groups, but delegates from the same company must be in different groups. The groups need not all be filled.

a) Use of network flow to test feasibility. Establish a network with a source s, sink t, vertex x_i for the ith company, and vertex y_j for the jth networking group. For each i, add an edge from s to x_i with capacity m_i. For each j, add an edge from y_j to t with capacity n_j. For each i, j, add $x_i y_j$ with capacity 1.

With integer capacities, the integrality theorem guarantees that some maximum flow breaks into paths of unit capacity. All s,t-paths have the form s, x_i, y_j, t and thus correspond to sending a delegate from company i to group j. The capacity on sx_i limits the ith company to m_i delegates. The capacity on $y_j t$ limits the jth group to n_j delegates. The capacity on $x_i y_j$ ensures that only one delegate from company i attends group j. The conditions of the problem are satisfiable if and only if this network has a flow of value $\sum m_i$. A flow of that value assigns, for each i, m_i delegates from company i to distinct groups.

b) A necessary and sufficient condition for successful construction is $k(q-l) + \sum_{j=1}^{l} n_j \ge \sum_{i=1}^{k} m_i$ for all $0 \le k \le p$ and $0 \le l \le q$, where $m_1 \ge \cdots \ge m_p$ and $n_1 \le \cdots \le n_q$.

Proof 1 (network flows). By the Max-flow/min-cut Theorem, there is a flow of value $\sum m_i$ if and only if there is no cut of capacity less than $\sum m_i$. Let $[S,T]$ be a source/sink cut, with $k = |S\cap\{x_1,\ldots,x_m\}|$ and $l = |S\cap\{y_1,\ldots,y_n\}|$. The capacity of the cut is $\sum_{i:\,x_i\in T} m_i + \sum_{j:\,y_j\in S} n_j + k(q-l)$. The network has a flow of value $\sum m_i$ if and only if this sum is at least $\sum m_i$ for each cut $[S,T]$. This will be true if and only if it is true when T has the $p-k$ companies with fewest participants and S has the l smallest groups. That is,

$$\sum_{i=k+1}^{p} m_i + \sum_{j=1}^{l} n_j + k(q-l) \ge \sum_{i=1}^{p} m_i,$$

which is equivalent to the specified inequality.

Proof 2 (bigraphic lists). The assignment of delegates to groups can be modeled by a bipartite graph. We may assume that $\sum n_j \ge \sum m_i$, since this is necessary to accommodate all the delegates. Let $t = \sum_j n_j - \sum_i m_i$. We add t phantom companies with one delegate each to absorb the excess capacity in the groups. Now there is a feasible assignment of delegates if and only if the pair (n, m) of lists is bigraphic, since each company sends at most one delegate to each group.

Note first that the given condition holds for all l if and only if it holds when $n_l \le k$ and $n_{l+1} \ge k$. The reason is that reducing l will cause terms smaller than k to contribute k and increasing l will cause contribution exceeding k from terms that contributed k.

By the Gale–Ryser Theorem, (n, m) is bigraphic if and only if $\sum_{i=1}^{q} \min\{n_i, k\} \ge \sum_{j=1}^{k} m_j$ for all $0 \le k \le p + t$. For $k > p$, we gain 1 with each increase in k on the right and at least 1 on the left unless we already have everything, so the inequality holds for all k if and only if it holds for $0 \le k \le p$. Since we have indexed $n_1, \ldots, n_q$ in increasing order, the left side equals $k(q-l) + \sum_{j=1}^{l} n_j$ when $n_l \le k$ and $n_{l+1} \ge k$. Thus the specified condition is equivalent to the condition in the Gale-Ryser Theorem and is necessary and sufficient for the existence of the bipartite graph and the assignment of delegates.

4.3.14. *A network flow solution to choosing $k/3$ assistant professors, $k/3$ associate professors, and $k/3$ full professors, one to represent each department.* We design a maximum flow problem with a node for each department, each professor, and each professorial rank. Let *unit edges* be edges of capacity 1. The source node s sends a unit edge to each departmental node. Each departmental node sends an edge to each of its professors' nodes; these may have infinite capacity. Each professorial node sends a unit edge to the node for that professor's rank. Finally, there is an edge of capacity $k/3$ from each rank to the sink t.

Each unit of flow selects a professor on the committee. The edges from the source to the departments ensure that each department is represented at most once. Since capacity one leaves each professor, the professor can represent only one department. The capacities on the three edges into the sink enforce balanced representation across ranks. The desired committee exists if and only if the network has a feasible flow of value k.

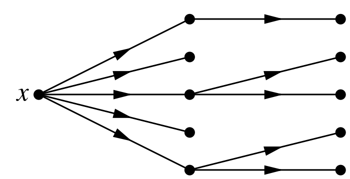
The network has a feasible flow of value k if and only if every source/sink cut has capacity at least k. Using edges with infinite capacity simplifies the analysis of finite cuts; such cuts $[S, T]$ cannot have an edge

of infinite capacity from S to T. Any capacity at least 1 on the edges from a department to its professors yields the same feasible flows.

4.3.15. *Spanning trees and cuts.* The *value* of a spanning tree T is the minimum weight of its edges, and the *cap* from an edge cut $[S, \overline{S}]$ is the maximum weight of its edges. Since every spanning tree contains an edge from every cut, the value of a tree T is at most the cap from $[S, \overline{S}]$.

Let m be the minimum cap from edge cuts in a connected graph G; thus every edge cut has an edge with weight at least m. Let H be the subgraph of G consisting of all edges with weight at least m. Since H has an edge from every edge cut in G, H is a spanning connected subgraph of G. Let T be a spanning tree of H. Since every edge in H has weight at least m, the minimum edge weight in T is at least m. Also T is a spanning tree of G. Hence equality holds between maximum value of a spanning tree and minimum cap from edge cuts.

4.3.16. *If x is a vertex of maximum outdegree in a tournament G, then G has a spanning directed tree rooted at x such that every vertex has distance at most 2 from x and every vertex other than x has outdegree at most 2, as sketched below.*



We create a network with source x. We keep the edges of G from x to $N^+(x)$ and edges from $N^+(x)$ to $N^-(x)$, and then we add a sink z and edges from $N^-(x)$ to z. The edges leaving x have capacity 2, the edges from $N^+(x)$ to $N^-(x)$ have infinite capacity, and the edges from $N^-(x)$ to z have capacity 1.

The Integrality Theorem yields an integer-valued maximum flow consisting of x, z-paths with unit flow, arriving at z from distinct vertices of $N^-(x)$. Since the capacity on edges out of x is 2, each successor of x is on at most two such paths. Hence a flow of value $d_G^-(x)$ yields the desired spanning tree, since successors of x belonging to none of the paths can be added freely as leaves of the tree.

The Max flow–Min cut Theorem guarantees that such a flow exists if we show that every cut has value at least $d_G^-(x)$. A cut of finite value has no edge from $N^+(x)$ to $N^-(x)$. Consider a source/sink cut $[S, T]$, and let $T' = T \cap N^-(x)$. Let Q be the set of vertices in $N^+(x)$ having successors in $T \cap N^-(x)$; such vertices must also be in T. Let $q = |Q|$ and $t = |T'|$. The capacity of this cut is at least $2q + d^-(x) - t$.

Every vertex in T' has as successors x and all of $N^+(x) - Q$; a total of $d^+(x)-q+1$ vertices. Also, some vertex of T' has outdegree at least $(t-1)/2$ in the subtournament induced by T'. Since x has maximum outdegree, we thus have $d^+(x) - q + 1 + (t-1)/2 \le d^+(x)$. This yields $2q - t \ge 1$ when T' is nonempty. Hence every cut other than the trivial cut $[S, T]$ that isolates z has capacity strictly greater than $d^-(x)$, and the desired flow exists.

Comment: Because the nontrivial cuts have capacity strictly greater than $d^-(x)$ in this argument, we still obtain a spanning tree of the desired form even under the additional restricted that any one desired successor of x be required to have outdegree at most 1.

4.3.17. *There is no simple bipartite graph for which the vertices in each partite set have degrees (5,4,4,2,1).* In any bipartite graph, the ith vertex on the side with degrees $\{p_i\}$ has at most $\min\{p_i, k\}$ neighbors among any set of k vertices on the other side. If we take the k largest degrees on the other side, their incident edges must come from somewhere, so $\sum \min\{p_i, k\} \ge \sum_{j=1}^k q_j$. The example given here violates this necessary condition when $k = 3$, because $3+3+3+2+1 = 12 < 13 = 5+4+4$.

4.3.18. *Given lists $r_1, \dots, r_n$ and $s_1, \dots, s_n$, there is a digraph D with vertices $v_1, \dots, v_n$ such that each ordered pair occurs at most once as an edge and $d^+(v_i) = r_i$ and $d^-(v_i) = s_i$ for all i if and only if $\Sigma r_i = \Sigma s_j$ and, for $1 \le k \le n$, the sum $\sum_{i=1}^n \min\{r_i, k\}$ is at least the sum of the largest k values in $s_1, \dots, s_n$.*

We transform this question into that of realization of degree lists by a simple bipartite graph. Splitting each vertex v into two vertices v^- and v^+ such that v^- inherits edges leaving v and v^+ inherits edges entering v turns such a digraph into a simple bipartite graph with degree lists $r_1, \dots, r_n$ and $s_1, \dots, s_n$ for the partite sets.

Conversely, a simple bipartite graph with vertices $u_1, \dots, u_n$ and $w_1, \dots, w_n$ such that $d(u_i) = r_i$ and $d(w_j) = s_j$ becomes a digraph as described if we orient each edge $u_i w_j$ from u_i to w_j and then merge u_i and w_i into v_i for each i.

Thus the desired condition is the necessary and sufficient condition for $r_1, \dots, r_n$ and $s_1, \dots, s_n$ to be bigraphic. The condition is the Gale–Ryser condition, found in Theorem 4.3.18.

4.3.19. *A consistent rounding of the data in the matrix A below appears in matrix B.* Every row permutation of B is a consistent rounding of A, as are some matrices with larger total sum, so the answer is far from unique.

$$A = \begin{pmatrix} .55 & .6 & .6 \\ .55 & .65 & .7 \\ .6 & .65 & .7 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

4.3.20. *Every 2-by-2 matrix can be consistently rounded.* We need only consider the fractional part of each entry. That is, we may assume that entries are at least 0 and are less than 1. A consistent rounding is now obtained by rounding each entry other than .5 to the nearest integer and, for each entry equal to .5, rounding down when the sum of the indices is odd and up when the sum of the indices is even. The resulting change in a column or row total is strictly less than 1. Hence it cannot be moved far enough to make the resulting total not be a rounding of the original total.

4.3.21. *If every entry in an n-by-n matrix is strictly between $1/n$ and $1/(n-1)$, then the possible consistent roundings are the $0,1$-matrices of order n with one or two 1s in each row and column.* Each entry in the rounding must be 0 or 1. Each row and column sum in the original is larger than 1 and less than $n/(n-1)$.

4.3.22. *A network D with conservation constraints at every node has a feasible circulation if and only if $\sum_{e\in[S,\overline{S}]} l(e) \le \sum_{e\in[\overline{S},S]} u(e)$ for every $S \subseteq V(D)$.* We have lower and upper bounds $l(e)$ and $u(e)$ for the flow $f(e)$ on each edges e. The conservation constraints require that the net flow out of each vertex is 0, and hence the net flow across any cut is 0. Thus

$$\sum_{e\in[S,\overline{S}]} l(e) \le \sum_{e\in[S,\overline{S}]} f(e) = \sum_{e\in[\overline{S},S]} f(e) \le \sum_{e\in[\overline{S},S]} u(e),$$

and hence the condition is necessary.

For sufficiency, we convert the circulation problem into a transportation network as in Solution 4.3.20. Let $b(v) = l^-(v) - l^+(v)$, where $l^-(v)$ and $l^+(v)$ are the totals of the lower bounds on edges entering and departing v, respectively; note that $\sum_v b(v) = 0$. We require $l(e) \le f(e) \le u(e)$ on each edge e; we transform this to $0 \le f'(e) \le c(e)$, where $c(e) = u(e) - l(e)$. Since the flow on each edge is adjusted in passing between f and f', the difference between the net out of v under these two functions is $b(v)$. If $b(v)$ is positive, then f sends $b(v)$ more into v than f' does when we alter f' by adding $l(e)$ on each edge e. Therefore, we want f' to produce net outflow $b(v)$ from v.

If $b(v) \ge 0$, then we make v a source and set $\sigma(v) = b(v)$; otherwise, we make v a sink and set $\partial(v) = -b(v)$. Since $\sum b(v) = 0$, the only way to satisfy all the demands at the sinks is to use up all the supply at the sources. If f' solves this transportation problem, then adding $l(e)$ to $f'(e)$ to obtain $f(e)$ on each edge e will solve the circulation problem with net outflow 0 at each node.

By Theorem 4.3.17, a transportation network with source set X and sink set Y is feasible if for every set T of vertices, the capacity of edges entering T is at least $\partial(Y\cap T)-\sigma(X\cap T)$; that is, the demand of T minus the supply in T.

Given a set T of vertices, the total capacity on entering edges is

$$c(\overline{T},T)=\sum_{e\in[\overline{T},T]}c(e)=\sum_{e\in[\overline{T},T]}[u(e)-l(e)].$$

For the supplies and demands, we compute

$$\begin{aligned}\partial(Y\cap T)-\sigma(X\cap T)&=\sum_{v\in Y\cap T}[l^+(v)-l^-(v)]-\sum_{v\in X\cap T}[l^-(v)-l^+(v)]\\&=\sum_{v\in T}[l^+(v)-l^-(v)]=\sum_{e\in[T,\overline{T}]}l(e)-\sum_{e\in[\overline{T},T]}l(e)\end{aligned}$$

The given condition $\sum_{e\in[S,\overline{S}]}l(e)\le\sum_{e\in[\overline{S},S]}u(e)$ for every $S\subseteq V(D)$ now implies that the condition for feasibility of the transportation problem holds. Hence there is a feasible solution to the transportation problem, and we showed earlier that such a solution produces a circulation in the original problem.

4.3.23. *A $(k+l)$-regular graph is (k,l)-orientable (it has an orientation in which each in-degree is k or l) if and only if there is a partition X,Y of $V(G)$ such that for every $S\subseteq V(G)$,*

$$(k-l)(|X\cap S|-|Y\cap S|)\le\left|[S,\overline{S}]\right|.$$

Note first that the characterization implies that every (k,l)-orientable with $k>l$ is also $(k-1,l+1)$-orientable, since $(k-1)-(l+1)\le k-l$; that is, the condition becomes easier to fulfill.

Note also that when $k=l$ the condition is always satisfied, and a consistent orientation of an Eulerian circuit is an orientation with the desired property. Hence we may assume that $k>l$.

Necessity. Given that G is (k,l)-orientable, let D be a suitable orientation of G. Let $X=\{v\in V(G)\colon d_D^-(v)=l\}$ and $Y=\{v\in V(G)\colon d_D^-(v)=k\}$. For a given set $S\subseteq V(D)$, the total indegree of the vertices in S is $l\,|X\cap S|+k\,|Y\cap S|$. The total outdegree in S is $k\,|X\cap S|+l\,|Y\cap S|$. Of the total outdegree, the amount generated by edges within S is at most the total outdegree minus the total indegree in S. Thus the left side of the displayed inequality is a lower bound on the number of edges in D that depart from S. The right side is an upper bound on that quantity.

Sufficiency. We are given a partition X,Y of $V(G)$ such that the displayed inequality holds for every $S\subseteq V(G)$. We create a transportation problem in which a feasible flow will provide the desired orientation. Replace each edge of G with a pair of opposing edges, each with unit capacity. Let X be the set of sources, and let Y be the set of sink. Let each supply and

demand value be $k-l$. The given condition is now precisely the necessary and sufficient condition in Theorem 4.3.17 for the existence of a feasible solution to the transportation problem.

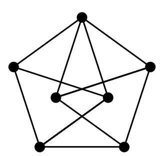
Since such a feasible solution is found by network flow methods, we may assume from the Integrality Theorem that there is a feasible solution in which the flow on each edge is 0 or 1. Also, we can cancel flows in opposing edges if they both equal 1. The edges that now have flow 1 specify an orientation of a subgraph of G. In this subgraph, $d^+(v)-d^-(v)=k-l$ for $v\in X$, and $d^+(v)-d^-(v)=l-k$ for $v\in Y$.

The degree remaining at a vertex v is $k+l-(d^+(v)+d^-(v))$. Always this value is even. A consistent orientation of an Eulerian circuit of each component of the remaining graph completes the desired orientation of G.

5.COLORING OF GRAPHS

5.1. VERTEX COLORING & UPPER BOUNDS

5.1.1. *Clique number, independence number, and chromatic number of the graph G below.* We have $\omega(G) = 3$ (no triangle extends to a 4-clique), $\alpha(G) = 2$ (every nonadjacent pair dominates all other vertices), and $\chi(G) = 4$ (a proper 3-coloring would give the top vertex the same color as the bottom two, but they are adjacent). Since there are seven vertices $\chi(G) \ge n(G)/\alpha(G)$ yields $\chi(G) \ge 4$. The graph is color-critical; checking each edge e shows that every $\chi(G-e)$ has a proper 3-coloring. By symmetry, there are only four types of edges to check.



5.1.2. *The chromatic number of a graph equals the maximum of the chromatic numbers of its components.* Since there are no edges between components, giving each component a proper coloring produces a proper coloring of the full graph. On the other hand, every proper coloring of the full graph must restrict to a proper coloring on each component.

5.1.3. *The chromatic number of a graph is the maximum of the chromatic number of its blocks.* We use induction on the number of blocks in G. If G has only one block, then the claim is immediate. Otherwise G is disconnected or has a cut-vertex v. In either case, we have subgraphs H_1, H_2 whose union is G, such that H_1, H_2 are disjoint (if G is disconnected) or share only the vertex v (if v is a cut-vertex).

The blocks of G are precisely the blocks of H_1 and H_2. Each has fewer blocks than G. Thus the induction hypothesis implies that $\chi(H_i)$ is the maximum of the chromatic numbers of the blocks in H_i. To complete the proof, it suffices to show that $\chi(G) = \max\{\chi(H_1), \chi(H_2)\}$.

The lower bound holds because both H_1 and H_2 are subgraphs of G. For the upper bound, assume by symmetry that $\chi(H_1) \ge \chi(H_2)$. Starting with an optimal coloring of H_1, we can incorporate an optimal coloring of H_2 by switching a pair of color names to make the coloring agree at v (if G is connected). This produces a proper coloring of G.

5.1.4. *The 5-cycle is a graph G with a vertex v so that $\chi(G-v) < \chi(G)$ and $\chi(\overline{G}-v) < \chi(\overline{G})$.* The 5-cycle is self-complementary and 3-chromatic, but deleting any vertex from C_5 (or $\overline{C}_5$) yields P_4, which is 2-colorable.

5.1.5. *Always $\chi(G+H) = \max\{\chi(G), \chi(H)\}$ and $\chi(G \vee H) = \chi(G) + \chi(H)$.* A coloring is a proper coloring of $G+H$ if and only if it restricts to a proper coloring on each of $\{G, H\}$, so the number of distinct colors needed is the maximum of $\chi(G)$ and $\chi(H)$.

In a proper coloring of $G \vee H$, the set of colors used on $V(G)$ must be disjoint from the set of colors used on $V(H)$. On the other hand, proper colorings of G and H that use disjoint sets of colors combine to form a proper coloring of $G \vee H$, so the number of colors needed is the sum of the numbers needed on G and H.

5.1.6. *If $\chi(G) = \omega(G) + 1$, and $H_1 = G$ and $H_k = H_{k-1} \vee G$ for $k > 1$, then $\chi(H_k) = \omega(H_k) + k$.* The union of a clique in F and a clique in H is a clique in $F \vee H$; hence $\omega(F \vee H) = \omega(F) + \omega(H)$. Since distinct colors must be used on $V(F)$ and $V(H)$ in a proper coloring of $F \vee H$, also $\chi(F \vee H) = \chi(F) + \chi(H)$.

Now we can prove the claim by induction on k. For $k = 1$, we are given $\chi(H_1) = \chi(G) = \omega(G) + 1 = \omega(H_1) + 1$. For $k > 1$, we compute

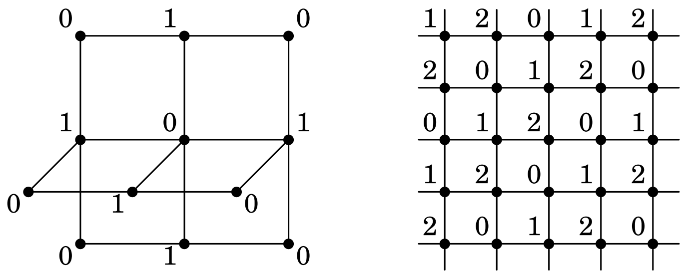
$$\begin{aligned}\chi(H_k) = \chi(H_{k-1} \vee G) = \chi(H_{k-1} + \chi(G) &= \omega(H_{k-1} + (k-1) + \omega(G) + 1 \\ &= \omega(H_{k-1} + \omega(G) + k = \omega(H_k) + k\end{aligned}$$

5.1.7. *The graph P_4 is neither a complete graph nor an odd cycle but has a vertex ordering relative to which greedy coloring uses $\Delta(P_4) + 1$ colors.* Although P_4 is bipartite, with maximum degree 2, coloring the endpoints first greedily with color 1 forces us to use colors 2 and 3 on the center.

5.1.8. *Comparison of $\chi(G) \le 1 + \Delta(G)$ and $\chi(G) \le 1 + \max_{H \subseteq G} \delta(H)$.* Let H' be a subgraph of G for which the minimum degree attains its maximum value. We have $\max_{H \subseteq G} \delta(H) = \delta(H') \le \Delta(H') \le \Delta(G)$. Hence the second bound is always at least as good as the first bound.

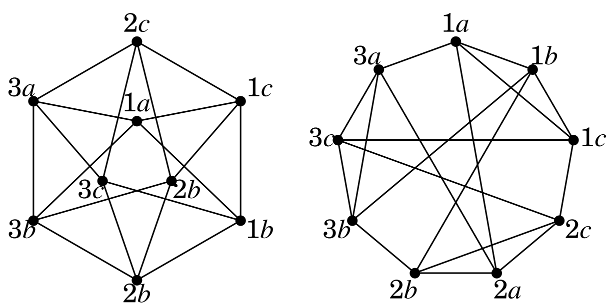
In order for equality to hold, we must have $\delta(H') = \Delta(H') = \Delta(G)$. Hence H' is k-regular, where $k = \Delta(G)$. This requires that no vertex of H' has a neighbor outside H'. Thus equality holds if and only if G has a component that is $\Delta(G)$-regular.

5.1.9. *Optimal (equitable) colorings of $K_{1,3} \square P_3$ and $C_5 \square C_5$.* The edges in the second figure wrap around to complete the 5-cycles in $C_5 \square C_5$.

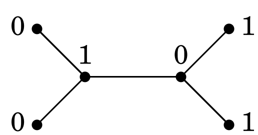


5.1.10. *The cartesian product graph $G \square H$ decomposes into a copies of H and b copies of G, where $n(G) = a$ and $n(H) = b$.* By the definition of cartesian product, $G \square H$ has two types of edges: those whose vertices have the same first coordinate, and those whose vertices have the same second coordinate. The edges joining vertices with a given value of the first coordinate form a copy of H, so the edges of the first type form aH. Similarly, the edges of the second type form bG, and the union is $G \square H$.

5.1.11. *Each graph below is isomorphic to $C_3 \square C_3$.* We label the vertices with $\{1, 2, 3\} \times \{a, b, c\}$ so that vertices are adjacent if and only if their labels agree in one coordinate and differ in the other.



5.1.12. *Every k-chromatic graph G has a proper k-coloring in which some color class has $\alpha(G)$ vertices—FALSE.* In the bipartite graph G below, every proper 2-coloring has three vertices in each color class, but $\alpha(G) = 4$.



5.1.13. *If $G = F \cup H$, then $\chi(G) \le \chi(F) + \chi(H)$—FALSE.* The complement of $K_{3,3}$ is $2K_3$. Hence we can express K_6 as the union of $K_{3,3}$ and $2K_3$. However, $\chi(K_6) = 6 > 5 = \chi(K_{3,3}) + \chi(2K_3)$.

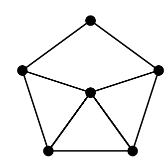
5.1.14. *For every graph G, $\chi(G) \le n(G) - \alpha(G) + 1$—TRUE.* We can produce a proper coloring by giving color 1 to a maximum independent set and giving distinct colors other than 1 to the remaining $n(G) - \alpha(G)$ vertices.

5.1.15. *It need not hold that $\chi(G) \le 1 + d$, where $d = 2e(G)/n(G)$ and G is a connected graph.* Form G by adding one edge joining a vertex of K_r to an endpoint of P_s. The graph G is connected, and $\chi(G) = r$. If $s > r$, then the average vertex degree is less than $(r+1)/2$. If also $r > 2$, then $r \ge (r+3)/2 > 1 + d$.

5.1.16. *Every tournament has a spanning path.* A n-vertex tournament D is an orientation of $G = K_n$, which has chromatic number n. By the Gallai–Roy Theorem, D has a path of length at least $\chi(G) - 1$, which equals $n - 1$. This is a spanning path.

5.1.17. *Chromatic number by critical subgraphs.* A graph with chromatic number at least 5 has a 5-critical subgraph, which has minimum degree at least 4. Since the graph below has only one vertex of degree at least 4, it has no subgraph with minimum degree at least 4.

A graph with chromatic number at least 4 has a 4-critical subgraph, which has minimum degree at least 3. Such a graph has at least 4 vertices. Deleting the one vertex with degree less than 2 from the graph below leaves only three vertices of degree at least 3. Hence there is no 4-critical subgraph, and $\chi(G) \le 3$.



5.1.18. *the number of colors needed to label $V(K_n)$ such that each color class induces a subgraph with maximum degree at most k is $\lceil n/k \rceil$.* With this many classes, we can partition the vertices into sets of size at most k.

5.1.19. *A false argument for Brooks' Theorem.* "We use induction on $n(G)$; the statement holds when $n(G) = 1$. For the induction step, suppose that G is not a complete graph or an odd cycle. Since $\kappa(G) \le \delta(G)$, the graph G has a separating set S of size at most $\Delta(G)$. Let $G_1, \dots, G_m$ be the components of $G - S$, and let $H_i = G[V(G_i) \cup S]$. By the induction hypothesis, each H_i is

$\Delta(G)$-colorable. Permute the names of the colors used on these subgraphs to agree on S. This yields a proper $\Delta(G)$-coloring of G."

Since $G[S]$ need not be a complete graph, it may not be possible to make the colorings of $H_1, \ldots, H_m$ agree on S. When x, y are nonadjacent vertices in S, they may have the same color in all proper $\Delta(G)$-colorings of H_i but have different colors in all proper $\Delta(G)$-colorings of H_j.

5.1.20. *If the odd cycles in G are pairwise intersecting, then $\chi(G) \le 5$.*

Proof 1 (direct). If G has no odd cycle, then $\chi(G) \le 2$, so we may assume that G has an odd cycle. Let C be a shortest odd cycle in G. If $\chi(G - V(C)) \ge 3$, then we have an odd cycle disjoint from C. Hence $\chi(G - V(C)) \le 2$. Since C is a shortest odd cycle, it has no chords, and the subgraph induced by C is 3-colorable. Thus we can combine a 2-coloring of $G - V(C)$ with a 3-coloring of C to obtain a 5-coloring of G.

Proof 2 (contrapositive). If $\chi(G) \ge 6$, consider an optimal coloring. The subgraph induced by vertices colored 1,2,3 coloring must have an odd cycle, else it would be bipartite and we could replace these three colors by two. Similarly, the subgraph induced by vertices colored 4,5,6 in the optimal coloring has an odd cycle, and these two odd cycles are disjoint.

5.1.21. *If every edge of a graph G appears in at most one cycle, then every block of G is an edge, a cycle, or an isolated vertex.* A block B with at least three vertices is 2-connected and has a cycle C. We show that $B = C$.

Proof 1. If B has an edge e not in C, then the properties of 2-connected graphs imply that e and an edge e' of C lie in a common cycle (Theorem 4.2.4). Now e' lies in more than one cycle.

Proof 2. Every 2-connected graph has an ear decomposition. If B is not a cycle, then adding the next ear completes two cycles sharing a path.

Proof 3. If B has a vertex x of degree at least 3, then consider $u, v, w \in N(x)$. Since $G - x$ is connected, it has a u, v-path and a v, w-path. These complete two cycles containing the edge uv.

For such a graph G, $\chi(G) \le 3$.

Proof 1 (structural property). By Exercise 5.1.3, $\chi(G)$ equal the largest chromatic number of its blocks. Here the blocks are edges or cycles and have chromatic number at most 3.

Proof 2 (induction on the number of blocks). If G has one block, then $\chi(G) \le 3$ since G is a vertex, an edge, or a cycle. Otherwise, we decompose G into G_1 and G_2 sharing a cut-vertex x of G. The blocks of G_1 and G_2 are the blocks of G. Using 3-colorings of G_1 and G_2 given by the induction hypothesis, we can permute colors in G_2 so the colorings agree at x.

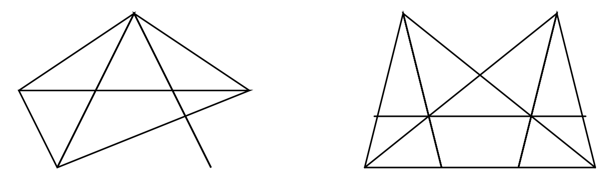
Proof 3 (subdivisions). Theorem 5.2.20 states that if G is not 3-colorable, then G contains a subdivision of K_4. Edges in such a subgraph appear in more than one cycle.

5.1.22. *The segment graph of a collection of lines in the plane with no three intersecting at a point is 3-colorable.* The vertices of G are the points of intersection of a family of lines; the edges are the segments on the lines joining two points of intersection.

Proof 1. By tilting the plane, we can ensure that no two vertices have the same x-coordinate. On each line, a vertex v has at most one neighbor with smaller x-coordinate. Thus each vertex has at most two earlier neighbors when $V(G)$ is indexed in increasing order of x-coordinates. Applying the greedy algorithm to this ordering uses at most three colors.

Proof 2. If $H \subseteq G$, the vertex of H with largest x-coordinate has degree at most 2 in H, for the same reason as above; on each line through that vertex it has at most one neighbor with smaller x-coordinate and none with larger x-coordinate. By the Szekeres–Wilf Theorem, $\chi(G) \le 1 + \max_{H \subseteq G} \delta(H) \le 3$.

The configurations below illustrate that the bound does not hold when more than two lines are allowed to meet at a point. The configuration on the left has seven lines, of which four meet at a point. The configuration on the right has eight lines, without four meeting at a point. In each case the resulting graph is 4-chromatic.



5.1.23. *The chromatic number of the graph $G_{n,k}$ obtained by joining each of n points on a circle to the $2k$ points nearest to it is $k+1$ if $k+1$ divides n and $k+2$ otherwise, if $n \ge k(k+1)$.* Every set of $k+1$ consecutive points forms a clique, so $\chi(G_{n,k}) \ge k+1$. If there is a $(k+1)$-coloring, each string of $k+1$ points must get distinct colors. Hence the coloring without loss of generality reads $123\cdots k(k+1)123\cdots k(k+1)123\cdots$ in order around the circle, since the new point must have the same color as the point just dropped from the most recent clique to avoid introducing a new color. The coloring will be proper if and only if the last vertices have colors $123\cdots k(k+1)$ before starting over, so $\chi(G_{n,k}) = k+1$ if and only if $k+1$ divides n.

If not, then one more color suffices if $n \ge k(k+1)$. Suppose $n = q(k+1) + r$, where $1 \le r < k+1$. After q complete stretches of $123\cdots k(k+1)$ in the scheme suggested above there are r vertices remaining to be colored. If $q \ge r$, then inserting color $k+2$ after $k+1r$ times will swell the sequence to fill up all the vertices with a proper coloring. In other words, expressing n as $r(k+2) + (q-r)(k+1)$, we can use $123\cdots(k+1)$ in order $q-r$ times and then $123\cdots(k+2)$ in order r times. If $n \ge k(k+1)$, then $q \ge k \ge r$, so the construction works.

If $n = k(k+1)-1$, then $\chi(G_{n,k}) > k+2$. If only $k+2$ colors are available, then some color must be used k times, since $(k-1)(k+2) = k(k+1)-2$. Following the n steps around the circle, the minimum separation between consecutive appearances among the k appearances of this color is less than $k+1$, since the total distance is $k(k+1)-1$. Since vertices at most k apart are adjacent, this prohibits a proper $(k+2)$-coloring.

5.1.24. *If G is a 20-regular graph with 360 vertices spaced evenly around a circle so that vertices separated by 1 or 2 angular degrees are nonadjacent and vertices separated by 3, 4, 5 or 6 degrees are adjacent, then $\chi(G) \le 19$.* We number the vertices 1 through 360 consecutively around the circle, and show that the greedy coloring algorithm uses at most 19 colors with respect to that order. Vertices $1, 2, 3$ receive color 1, and vertices $4, 5, 6$ receive color 2. Each of vertices 1 through 356 has at most 18 of its predecessors among its 20 neighbors, so color 20 will not have been assigned to any of the first 356 vertices.

Vertex 357 has 19 of its predecessors among its neighbors, but among those, vertices $1, 2, 3$ have the same color. Hence vertex 357 is assigned a color 19 or lower, having at most 17 differently colored predecessors. Similarly, vertex 358 has at most 18 differently colored predecessors ($1, 2, 3$ have the same color), 359 has at most 18 ($2, 3$ and $4, 5$ are pairs with the same color), and 360 has at most 18 ($4, 5, 6$ have the same color), so their assigned colors are 19 or lower.

5.1.25. *The unit-distance graph in the plane has chromatic number greater than 3 and at most 7.* For the lower bound, suppose the graph has a proper 3-coloring. Consider two equilateral triangles of side-length one that share an edge. The corners not on the shared edge must have the same color. The distance between these two points is $\sqrt{3}$. Hence in a proper 3-coloring, any two points $\sqrt{3}$ apart must have the same color. If C is a circle of radius $\sqrt{3}$, every point on C must have the same color as the center. This cannot be a proper coloring, since C contains two points that are distance 1 apart.

A 7-coloring can be obtained using regions in a tiling of the plane. Consider a tiling by hexagons of diameter 1, where each hexagon has two parallel horizontal edges and the hexagons lie in vertical columns. The interior of each hexagon receives a single color, along with the top half of the boundary (including the top two corners but not the middle two corners).

The rest of the boundary is colored as part of the top half of neighboring hexagons. In a single region, the distance between any pair of points is less than 1; we need only assign colors to regions so that no pair of regions with the same color contain pairs of points at distance 1. This we achieve

by using colors 1,2,3,4,5,6,7 cyclically in order on the regions in a column, with the region labeled 1 in a given column nestled between regions labeled 3 and 4 in the column to its left.

The closest points in two regions with the same color are opposite endpoints of a zig-zag of three edges in the tiling; the distance between these is greater than one. (An 8-coloring can be obtained using a grid of squares of diameter 1, with colors 1,2,3,4 on the odd columns and colors 5,6,7,8 on the even columns, cyclically in order, where the pattern in the odd rows repeats 1,5,3,7 and the pattern in the even rows repeats 2,6,4,8.)

5.1.26. *Chromatic number of a special graph.* Given finite sets $S_1, \dots, S_m$, let $V(G) = S_1 \times \cdots \times S_m$, and define $E(G)$ by putting $u \leftrightarrow v$ if and only if u and v differ in every coordinate.

The chromatic number is $\min_i |S_i|$. Let $k = \min_i |S_i|$. We may assume that $S_i = \{1, \dots, k\}$. We obtain a clique of size k by letting $v_i = (i, \dots, i)$ for $1 \le i \le k$ (when $i \ne j$, v_i and v_j differ in every coordinate). Hence $\chi(G) \ge k$.

To obtain a proper k-coloring, we color each vertex v with its value in a coordinate j such that $|S_j| = k$. The vertices having value i in coordinate j form an independent set, so this is a proper k-coloring, and $\chi(G) \le k$.

5.1.27. *The complement of the graph in Exercise 5.1.26 has chromatic number $\prod_{i=1}^m |S_i| / \min_i |S_i|$.* Nonadjacency means differing in every coordinate.

Let j be a coordinate such that $|S_j| = \min_i\{|S_i|\}$. The vertices with a fixed value in coordinate j form a clique of the specified size.

Let $k = |S_j|$. To obtain the desired coloring, we partition the vertices into independent sets of size k. Each must have a vertex with each value in coordinate j. The vertices with value 1 in coordinate j lie in different independent sets; use the remainder of the name of each such vertex as the name for its independent set. The ith vertex in this independent set has value i in coordinate j. Its value in coordinate r is obtained by adding $i-1$ (modulo $|S_r|$) to the value in coordinate r of the naming vertex.

To find the name of the independent set containing a vertex v, we let i be the value it has in coordinate j and subtract $i-1$ (modulo $|S_r|$) from the value in the rth coordinate, for each r.

5.1.28. *The traffic signal controlled by two switches is really controlled by one of the switches.* Each switch can be set in n positions. For each setting of the switches, the traffic signal shows one of n possible colors. Whenever the setting of *both* switches changes, the color changes.

Since the color changes when both coordinates change, assigning the color that shows to the vector of positions yields a proper n-coloring of the graph defined in Exercise 5.1.26, where $m = 2$ and both sets have size n. Since $\{(i, i)\colon 1 \le i \le n\}$ is a clique of size n, this is an optimal coloring.

Proof 1 (characterization of maximum independent sets). The vertices having value i in one coordinate form an independent set. This defines a proper n-coloring. We claim that every proper n-coloring has this form, and hence the color is controlled by the value in one coordinate. Every independent set has size at most n, since $n+1$ vertices cannot have distinct values among the n possible values in the first coordinate. In order to obtain an n-coloring of the n^2 vertices when each independent set has size at most n, we must use n independent sets of size n.

We claim that every independent set of size n is fixed in one coordinate. Let S be an independent set in which distinct values r and s appear in the first coordinate. Since these vertices in S are nonadjacent, they must agree in the second coordinate, so we now have $(r,t),(s,t) \in S$. If S has some vertex not having value t in the second coordinate, then its value in the first coordinate must equal both r and s, since it is nonadjacent to these two vertices. This is impossible, so the vertices S must agree in one coordinate.

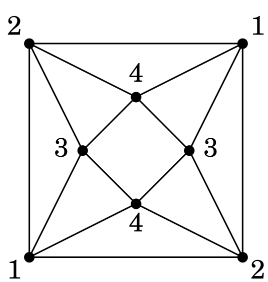
To partition $V(G)$ into n independent sets of size n, the sets must be parwise disjoint. Hence we cannot used one set fixed in the first coordinate and another set fixed in the second coordinate. Hence all the sets used in the coloring are constant in the same coordinate, and the color is controlled by the position in that coordinate.

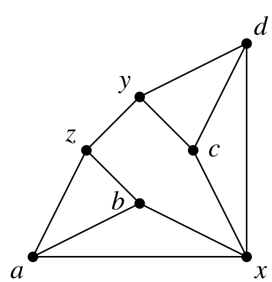
Proof 2 (induction on n). The claim is trivial for $n=1$. Let G_n be the product graph. For the induction step, note that $N_{G_n}((n,n))$ induces G_{n-1}. Also, the color n used on (n,n) cannot be used on that subgraph, so the coloring of G_n restricts to a proper $(n-1)$-coloring on that subgraph. By the induction hypothesis, it is determined by one coordinate. By symmetry, we may assume that (i,j) has color i when $i,j \in [n-1]$.

Now (n,j) for $1 \le j \le n-1$ is adjacent to vertices of the first $n-1$ colors, so it has color n. Now (i,n) is adjacent to vertices of all n colors except color i, so it has color i.

5.1.29. *A 4-critical subgraph in a 4-chromatic graph.*

The figure on the left illustrates a proper 4-coloring. On the right we show a 4-critical subgraph. Verifying that this is 4-critical also proves the lower bound to show that the full graph is 4-chromatic.





In any proper 3-coloring of $K_4 - e$, the nonadjacent vertices have the same color. Thus in a proper 3-coloring of the graph F on the right, y and z have the same color as x, which is forbidden because they are adjacent. 4-criticality is easy to verify using symmetry; there are only four "types" of edges. Thus the solution is completed by exhibiting proper 3-colorings of $F - yz$, $F - zb$, $F - ab$, and $F - ax$. If we remove any edge of the outer 5-cycle, then we can 2-color its vertices and use the third color on the two inner vertices. The analogous argument works for the inner 5-cycle. This leaves only $F - ab$ to consider, where we can obtain a proper 3-coloring by giving all of $\{a,b,x,y\}$ the same color.

5.1.30. *The chromatic number of the shift graph G_n is $\lceil \lg n \rceil$.* Here $V(G_n) = \binom{[n]}{2}$ and $E(G_n) = \{(ij, jk)\colon i < j < k\}$. It suffices to show that G_n is r-colorable if and only if $[r]$ has at least n distinct subsets.

Given a map $f\colon V(G) \to [r]$, define $T_j = \{f(jk) : k > j\}$. The labeling f is a proper coloring if and only if $f(ij) \notin T_j$ for all $i < j$. In particular, if f is proper, then $T_i \ne T_j$ for all $i < j$, and thus r must be large enough so that $[r]$ has n distinct subsets.

Conversely, if $[r]$ has n distinct subsets, we index n such subsets so that $A_j \not\subseteq A_i$ for $j > i$ (start with n and work back, always choosing a minimal set in the collection of subsets that remain). Now f can be defined by naming $f(ij)$ for each $i < j$ to be an element of $A_i - A_j$. This ensures that ij and jk receive distinct colors when $i < j < k$.

5.1.31. *A graph G is m-colorable if and only if $\alpha(G \square K_m) \ge n(G)$.*

Proof 1 (direct construction). Let $V(G) = \{v_i\}$ and $V(K_m) = \{1,\dots,m\}$, so the vertices of $G \square K_m$ are $\{(v_i, j)\colon 1 \le i \le n,\ 1 \le j \le m\}$. If G is m-colorable, let $C_1,\dots,C_m$ be the independent sets in a proper m-coloring of G. Then $\{(v,j)\colon v \in C_j\}$ is an independent set in $G \square K_m$ of size $n(G)$ (it contains one copy of each vertex of G).

Conversely, if $G \square K_m$ has an independent set S of size $n(G)$, then S can only contain one copy of each vertex of G (since (v,i) and (v,j) are adjacent), and the elements of S whose pairs use a single vertex of K_m must be an independent set in G. Hence from S we obtain a covering of $V(G)$ by m independent sets.

Proof 2 (applying known inequalities). If G is m-colorable, then $\chi(G \square K_m) = \max\{\chi(G), \chi(K_m)\} = m$. Because $\chi(H) \ge n(H)/\alpha(H)$ for every graph H, and $n(G \square K_m) = n(G)m$, we obtain $\alpha(G \square K_m) \ge n(G)m/m = n(G)$.

Conversely, $\alpha(G \square K_m) \ge n(G)$ yields $\alpha(G \square K_m) = n(G)$, since an independent set has at most one vertex in each copy of K_m. The vertices of a maximum independent set S have the form (v,i), where $v \in V(G)$ and $i \in [m]$. By the definition of cartesian product, adding 1 (modulo m) to the second coordinate in each vertex of S yields another independent set of size $n(G)$. Doing this m times yields m pairwise disjoint independent sets covering all the vertices of $G \,\square\, K_m$. Therefore, $G \,\square\, K_m$ is m-colorable. Since G is a subgraph of $G \,\square\, K_m$, also G is m-colorable.

5.1.32. *A graph G is 2^k-colorable if and only if G is the union of k bipartite graphs.* View the colors as binary k-tuples. If G has a proper 2^k-coloring f, let X_i be the set of all $v \in V(G)$ such that the binary expansion of $f(v)$ has a 0 in the ith coordinate, and let $Y_i = V(G) - X_i$. Define a bipartite subgraph B_i of G with bipartition X_i, Y_i and edge set $[X_i, Y_i]$. By construction, each such graph is bipartite. For every edge e in G, the endpoints of e have different colors in f, so their binary expansions differ in some coordinate, and thus e appears in one of these subgraphs.

Conversely, suppose that G is the union of k bipartite graphs, with X_i, Y_i being the bipartition of the ith subgraph. We use binary k-tuples as colors. Assign v the k-tuple that is 0 in the ith coordinate if $v \in X_i$, or 1 if $v \in Y_i$. Since each edge is in one of the bipartite graphs, the k-tuples assigned to its endpoints are distinct, and this is a proper 2^k-coloring.

5.1.33. *For each graph G, there is an ordering of $V(G)$ where the greedy algorithm uses only $\chi(G)$ colors.* Consider an optimal coloring f. Number the vertices of G as $v_1, \ldots, v_n$ as follows: start with the vertices of color 1 in f, then those of color 2, and so on. By induction on i, we prove that the greedy algorithm assigns v_i a color at most $f(v_i)$.

Certainly v_1 gets color 1. For $i > 1$, the induction hypothesis says that v_j has received color at most $f(v_j)$, for every $j < i$. Furthermore, the only such vertices v_j with $f(v_j) = f(v_i)$ are those in the same color class with v_i in the optimal coloring, and these are not adjacent to v_i. Hence the colors used on earlier neighbors of v_i are in the set $\{1, \ldots, f(v_i) - 1\}$, and the algorithm assigns color at most $f(v_i)$ to v_i.

5.1.34. *There is a tree T_k with maximum degree k having a vertex ordering such that the greedy algorithm uses $k+1$ colors.* There are at least three ways to construct the same tree T_k and essentially the same ordering. In each, we construct T_k by induction on k along with a vertex ordering such that the last vertex has degree k and receives color $k+1$ under the greedy algorithm. In each, the tree K_1 works as T_0 when $k = 0$.

Construction 1. For $k > 0$, T_k consists of copies of $T_0, \ldots, T_{k-1}$, with one additional vertex x joined to the vertex of maximum degree in each T_i.

By the induction hypothesis, each old vertex has degree at most $k-1$, and the only one that attains degree k, along with x, is the vertex of maximum degree in T_{k-1}. The vertex ordering uses $V(T_i), \ldots, V(T_{k-1})$ in order and puts x last. The ordering within $V(T_i)$ is the ordering guaranteed for it by the induction hypothesis. The coloring of each T_i happens independently

according to the order for that subtree, because the only edge leaving the copy of T_i goes to x. By the induction hypothesis, the neighbor of x in T_i, which is the last vertex of T_i, gets color $i+1$. Thus x has earlier neighbors of colors $1, \ldots, k$ and receives color $k+1$.

Construction 2. Build two copies of T_{k-1} (T' and then T''), with the vertex orderings given by the induction hypothesis. Include in the induction hypothesis the statement that the last vertex has degree $k-1$ and receives color k under the greedy coloring. When the last vertex x of T'' is created, make it also adjacent to the last vertex y of T'. Hence x and y have degree k in the resulting tree T.

When y is reached in the ordering, it receives color k, by the induction hypothesis. For the same reason, x is adjacent in T to vertices that have received colors $1, \ldots, k-1$ (in T'') and also to y. Hence x receives color $k+1$, as desired.

Construction 3. Given T_{k-1} and its ordering, form T_k by appending a leaf to each vertex. In the ordering, place all these leaves first. These form an independent set and receive color 1. After this independent set, use the ordering for T_{k-1} on the vertices in the copy of T_{k-1}. Since each of these vertices already has a neighbor with color 1, the colors assigned are 1 higher than the colors assigned under the ordering of T_{k-1}. Also the degree of each vertex is larger by 1. Hence this T_k has maximum degree k, and the given ordering assigns color $k+1$ to the last vertex.

Explicit Construction. The tree T_k can be described by letting the vertices be $\{0, \ldots, 2^k - 1\}$. Make i and j adjacent whenever $i \ge 2^k - 2^{r+1}$ and $j = i + 2^r$ for some r. This produces the same tree as above (it can be proved by induction on k that it is the same tree as in Construction 3), and the vertex ordering that puts color $k+1$ on $2^k - 1$ is $0 \ldots, 2^k - 1$.

5.1.35. *In a P_4-free graph, every greedy coloring is optimal.* Consider an ordering $v_1, \ldots, v_n$, and suppose that greedy coloring with respect to this ordering uses k colors. Let i be the smallest integer such that G has a clique consisting of vertices assigned colors i through k in this coloring. Proving that $i = 1$ yields a k-clique in G, which proves that the coloring is optimal.

Let $Q = \{u_i, \ldots, u_k\}$ be such a clique. If $i > 1$, then by the greedy procedure each element of Q has an earlier neighbor with color $i-1$. If some such vertex were adjacent to all of Q, then we could reduce i. Let x be a vertex with color $i-1$ that is adjacent to the most vertices of Q. Let z be a nonneighbor of x in Q. Let w be a neighbor of z with color $i-1$. By the choice of x, w is not adjacent to all neighbors of x in Q; choose $y \in (N(x) \cap Q) - N(w)$. Since x and w have the same color, they are nonadjacent. Now x, y, z, w induces P_4. The contradiction implies that $i = 1$.

5.1.36. *The ordering σ that minimizes the greedy coloring bound $f(\sigma) = 1+\max_i d_{G_i}(x_i)$ is the "smallest-last" ordering σ^* in which, for i from n down to 1, x_i is chosen to be a vertex of minimum degree in G_i.* Furthermore, $f(\sigma^*) = 1 + \max_{H\subseteq G} \delta(H)$. Let H^* be a subgraph of G maximizing $\delta(H)$. For a vertex ordering σ, let i be the position in σ where the last vertex of H^* is included. We have $d_{G_i}(x_i) \ge \delta(H^*)$, and thus $f(\sigma) \ge 1 + \delta(H^*) = 1 + \max_{H\subseteq G} \delta(H)$.

When greedy coloring is run with respect to σ^*, each v_i is a vertex of minimum degree in G_i. Thus $f(\sigma^*) = 1 + \max_i \delta(G_i) \le 1 + \max_{H\subseteq G} \delta(H)$. By the first paragraph, equality holds.

5.1.37. *The vertices of a simple graph G can be partitioned into $1 + \max_{H\subseteq G} \delta(H)/r$ classes such that the subgraph induced by the each class has a vertex of degree less than r.* Let v_n be a vertex of minimum degree in G, and for $i < n$ let v_i be a vertex of minimum degree in $G - \{v_{i+1}, \dots, v_n\}$. Place the vertices $v_1, \dots, v_n$ in order into the partition. Place v_i into the least-indexed set in which it has fewer than r neighbors already placed. This produces a partition of the desired form. Let $k = \max_{H\subseteq G} \delta(H)$. Since the degree of v_i in the subgraph induced by $v_1, \dots, v_i$ is at most k, v_i has r neighbors each in at most k/r classes, and therefore $1 + k$ classes suffice.

5.1.38. *If H is bipartite, then $\chi(\overline{H}) = \omega(\overline{H})$.* If H has isolated vertices, then in $\overline{H}$ they increase the clique number, and we may color them with extra colors. Hence we may assume that H has no isolated vertices.

Proof 1 (min-max relations). Because every independent set induces a clique in the complement and vice versa, we have $\omega(\overline{H}) = \alpha(H)$. Also $\chi(\overline{H})$ is the number of cliques in H needed to cover $V(H)$. If H is bipartite, then these cliques must be edges. Hence for a bipartite graph H with no isolated vertices, we have $\chi(\overline{H}) = \beta'(H) = \alpha(H) = \omega(\overline{H})$, using König's Theorem that $\beta'(H) = \alpha(H)$ in a bipartite graph with no isolated vertices.

We could also argue that each color in a proper coloring of $\overline{H}$ is used once or twice, since $\alpha(\overline{H}) = 2$. If k colors are used twice, then $k + (n - 2k) = n - k$ colors are used. The colors used twice color the edges of a matching in H, so $\chi(\overline{H}) = n - \alpha'(H) = \beta'(H)$ as before.

Proof 2 (construction). Let T be a maximum independent set in H, and let $A = X \cap T$ and $B = Y \cap T$, where H is an X, Y-bigraph. It suffices to find a matching of $Y - B$ into A and a matching of $X - A$ into B, because the edges of the matching disappear in $\overline{H}$, and this yields a covering of $V(\overline{H})$ using $|T|$ independent sets of sizes 1 and 2. To show that the matching exists, consider any $S \subseteq Y - B$ (the same argument works for $X - A$). Because $A - N(S) \cup S \cup B$ is a independent set and $A \cup B$ is a maximum independent set, we have $|N(S)| \ge |S|$, which by Hall's Theorem guarantees the desired matching.

5.1.39. *Every k-chromatic graph has at least $\binom{k}{2}$ edges.*

Proof 1. Consider a k-coloring of a k-chromatic G. If $e(G) < \binom{k}{2}$, then for some pair i, j of colors, no edge has colors i and j. Thus the vertices with colors i and j form a single independent set, and $V(G)$ is covered by $k - 1$ independent sets.

Proof 2. A k-chromatic graph G contains a k-critical subgraph G'. A k-critical graph has minimum degree at least $k - 1$, and since G' requires k colors it has at least k vertices. Hence

$$e(G) \ge e(G') \ge n(G')\delta(G')/2 \ge k(k-1)/2 = \tbinom{k}{2}.$$

If G is the union of m cliques of order m, then $\chi(G) \le 1 + m\sqrt{m-1}$. The construction of G implies $e(G) \le m\binom{m}{2}$. If $\chi(G) = k$, then $\binom{k}{2} \le m\binom{m}{2}$, or $k^2 - k - m^2(m-1) \le 0$. Using the quadratic formula and $\sqrt{x+1} < \sqrt{x} + 1$ (for $x > 0$), we have

$$k \le \tfrac{1}{2}(1 + \sqrt{1 + 4m^2(m-1)}) < \tfrac{1}{2}(2 + \sqrt{4m^2(m-1)} = 1 + m\sqrt{(m-1)}.$$

5.1.40. $\chi(G) \cdot \chi(\overline{G}) \ge n(G)$. If a proper coloring partitions n vertices into k color classes, there must be at least n/k vertices in some class, by the pigeonhole principle. These vertices form a clique in the complement, which forces $\chi(\overline{G}) \ge n/k$. Hence $\chi(\overline{G}) \ge n/\chi(G)$, or $\chi(G) \cdot \chi(\overline{G}) \ge n$.

$\chi(G) + \chi(\overline{G}) \ge 2\sqrt{n(G)}$. Two numbers with a fixed product x have smallest sum when they are equal; then their sum is $2\sqrt{x}$. Hence the first inequality implies this bound.

For $n = m^2$, the bound is achieved by $G = mK_m$, a disjoint union of $\sqrt{n}$ cliques of size $\sqrt{n}$. Since the complement is a complete $\sqrt{n}$-partite graph, both graphs have chromatic number $\sqrt{n}$.

5.1.41. $\chi(G) + \chi(\overline{G}) \le n(G) + 1$ *for every graph G.*

Proof 1 (induction on $n(G)$). The inequality holds (with equality) if $n = 1$. For $n > 1$, choose $v \in V(G)$, and let $G' = G - v$. By the induction hypothesis, $\chi(G') + \chi(\overline{G'}) \le n$. When we replace v to obtain G and $\overline{G}$, each chromatic number increases by at most 1. We have the desired bound unless they both increase.

If both increase, then v must have at least $\chi(G')$ neighbors in G (else we could augment a proper coloring of G' to include v) and similarly at least $\chi(\overline{G'})$ neighbors in $\overline{G}$. Since v has altogether $n - 1$ neighbors in G and $\overline{G}$, we conclude that in this case $\chi(\overline{G'}) + \chi(G') \le n - 1$, and adding 2 again yields the desired bound $\chi(\overline{G}) + \chi(G) \le n + 1$.

Proof 2 (greedy coloring bound). When the vertices are colored greedily in nonincreasing order of degree, the color used on the ith vertex is at most $\min\{d_i + 1, i\}$. Let k be the index where the maximum of $\min\{d_i + 1, i\}$

is achieved, so that $d_i + 1 \le k$ for $i \ge k$ and $d_i + 1 > k$ for $i < k$. Greedy coloring yields $\chi(G) \le k$.

Let d'_j denote the jth largest vertex degree in $\overline{G}$. Since $d'_j = n-1-d_{n-j}$, we have $d'_j < n-k$ for $n-j < k$ and $d'_j \ge n-k$ for $n-j \ge k$. This becomes $d'_j \le n-k$ for $j \ge n-k+1$ and $d'_j > n-k$ for $j < n-k+1$. Therefore $\max_j \min\{d'_j+1, j\} = n-k+1$, so $\chi(G) + \chi(\overline{G}) \le k + (n-k+1) = n+1$.

Proof 3 (degeneracy). By the Szekeres–Wilf Theorem, it suffices to show that $\max_{H\subseteq G} \delta(H) + \max_{H\subseteq \overline{G}} \delta(H) \le n-1$. Let H_1 and H_2 be subgraphs of G and $\overline{G}$ achieving the maximums. Let $k_i = \delta(H_i)$. Note that $n(H_i) \ge k_i + 1$. If $k_1 + k_2 \ge n$, then H_1 and H_2 have a common vertex v. Now v must have at least k_i neighbors in H_i, for each i, but only $n-1$ neighbors are available in total.

5.1.42. *Analysis of the ratio of $\chi(G)$ to $n(G)/\alpha(G)$.*

a) $\chi(G)\cdot\chi(\overline{G}) \le (n(G)+1)^2/4$, and the ratio of $\chi(G)$ to $(n+1)/\alpha(G)$ is at most $(n+1)/4$. Two numbers with a fixed sum x have largest product when they are equal, in which case their product is $x^2/4$. Hence the previous exercise implies this bound. The ratio of $\chi(G)$ to $(n+1)/\alpha(G)$ equals $\chi(G)\alpha(G)/(n+1)$. Since $\alpha(G) = \omega(\overline{G})$, we have $\chi(\overline{G}) \ge \alpha(G)$. Hence $\chi(G)\alpha(G)/(n+1) \le \chi(G)\cdot\chi(\overline{G})/(n+1) \le (n+1)/4$.

b) Construction for equality when n is odd. Let G be the join of a clique of order $(n-1)/2$ and a independent set of order $(n+1)/2$. Since the independent set can receive a single color and G has cliques of order $(n+1)/2$, $\chi(G) = (n+1)/2$. Also $\alpha(G) = (n+1)/2$, and equality holds in the bound $\chi(G)\chi(\overline{G}) \le (n+1)^2/4$.

5.1.43. *Paths and chromatic number in digraphs.*

a) $\chi(F\cup H) \le \chi(F)\chi(H)$. Let $G = F \cup H$. We may assume that $V(F) = V(H)$, because otherwise we can add vertices in exactly one of these digraphs as isolated vertices in the other without affecting any of the chromatic numbers. The chromatic number of a digraph is taken to be the same as the chromatic number of the underlying undirected graph.

Proof 1 (producing a coloring)

Assign to each vertex of G the "color" that is the pair of colors it gets in optimal colorings of F and H. Since every edge of G comes from F or H, no pair of adjacent vertices in G get the same color pair. Since there are $\chi(F)\chi(H)$ possible pairs, we have a proper $\chi(F)\chi(H)$-coloring of G.

Proof 2 (covering by independent sets) Let $U_1, \ldots, U_r$ be the color classes in an optimal coloring of F, and let $W_1, \ldots, W_s$ be the color classes in an optimal coloring of H. Each vertex belongs to exactly one class in each graph, so it belongs to $U_i \cap W_j$ for exactly one pair (i, j). Furthermore, $U_i \cap W_j$ is an independent set in G, since it is independent in both F and H.

Now the sets of the form $U_i \cap W_j$ for $1 \le i \le r = \chi(F)$ and $1 \le j \le s = \chi(H)$ partition $V(G)$ into the desired number of independent sets.

b) If D is an orientation of G, and $\chi(G) > rs$, and each $v \in V(G)$ is assigned a real number $f(v)$, then D has a path $u_0, \ldots, u_r$ with $f(u_0) \le \cdots \le f(u_r)$ or a path $v_0, \ldots, v_s$ with $f(v_0) > \cdots > f(v_s)$. Obtain from D two digraphs F and H defined as follows. Given the edge xy in D, put xy in F if $f(x) \le f(y)$, and put xy in H if $f(x) > f(y)$. If D has no nondecreasing path of length r and no decreasing path of length s, then F has no path of length r and H has no path of length s. By the Gallai–Roy Theorem, this implies $\chi(F) \le r$ and $\chi(H) \le s$. By part (a), we have $\chi(G) \le rs$, where $G = F \cup H$, but this contradicts the hypothesis on G. Hence one of the specified long paths exists.

c) Every sequence of $rs+1$ distinct real numbers has an increasing subsequence of size $r+1$ or a decreasing subsequence of size $s+1$. Let D be the tournament with vertices $v_1, \ldots, v_{rs+1}$ and $v_i \to v_j$ if $i > j$, and let $f(v_i)$ be the ith value in the sequence σ. Every path in D corresponds to a subsequence of σ, where the vertex labels are the values in σ. Because $\chi(D) = rs+1$, part (b) guarantees an increasing path with $r+1$ vertices or a decreasing path with $s+1$ vertices.

5.1.44. *Minty's Theorem.* Given an acyclic orientation D of a connected graph G, let $r(D) = \max_C \lceil a/b \rceil$, where a counts the edges of C that are forward in D and b counts those that are backward in D. Fix a vertex $x \in V(G)$, and let W be a walk from x in G. Let $g(W) = a - b\cdot r(D)$, where a counts the steps along W followed forward in D and b counts those followed backward in D. For $y \in V(G)$, let $g(y) = \max\{g(W)\colon W \text{ is an } x, y\text{-walk}\}$.

a) $g(y)$ is finite and thus well-defined, and G is $1 + r(D)$-colorable. By the definition of r, every cycle with a forward edges has at least ra backward edges. Hence traversing a cycle makes no positive contribution to $g(W)$, and $g(y) = g(W)$ for some x, y-path W. Thus there are only finitely many paths to consider, and $g(y)$ is well-defined.

To obtain a proper coloring of G, let the color of y be the congruence class of $g(y)$ modulo $1 + r(D)$. If $u \to v$ in D, then $g(v) \ge g(u) + 1$, since uv can be appended to an x, u-walk. On the other hand $g(u) \ge g(v) - r(D)$, since vu can be appended to an x, v-walk. Thus $g(u)+1 \le g(v) \le g(u)+r(D)$ when u and v are adjacent in G, which means that $g(u)$ and $g(v)$ do not lie in the same congruence class modulo $r(D)+1$.

b) $\chi(G) = \min_{D\in\mathbf{D}} 1 + r(D)$, where $\mathbf{D}$ is the set of acyclic orientations of G. The upper bound follows immediately from part (a). For the lower bound, we present an acyclic orientation D such that $r(D) \le \chi(G) - 1$. Given an optimal coloring f with colors $1, \ldots, \chi(G)$, orient each edge xy in the direction of the vertex with the larger color. Since colors increase strictly along every path, the orientation is acyclic and has maximum path length at most $\chi(G) - 1$.

5.1.45. *Gallai-Roy Theorem from Minty's Theorem.* We first prove that $1 + l(D)$ is minimized by an acyclic orientation, to which we can then apply Minty's Theorem. If D is an arbitrary orientation, let D' be a maximal acyclic subgraph of D. Let xy be an edge of $D - D'$. Since adding xy to D' creates a cycle, D' contains a y, x-path.

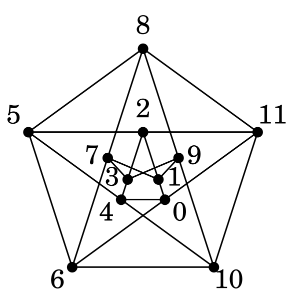
Let D^* be the orientation of G obtained from D by reversing the orientation on each edge of $D - D'$. If D^* contains a cycle C, then for each reversed edge yx on C corresponding to an edge xy of $D - D'$, we replace yx with a y, x-path that exists in D'. The result is a closed (directed) walk in D'. This yields a cycle in D', because a shortest closed walk in a digraph that has a closed walk is a cycle. Since by construction D' is acyclic, we conclude that D^* is acyclic.

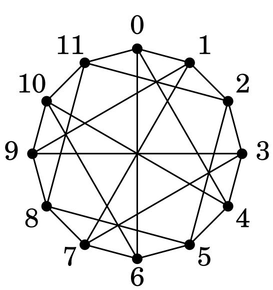
We also claim that $l(D^*) \le l(D')$. Let P be a u, v-path in D^*; some edges of P may have opposite orientation in D and D^*. For such an edge $yx \in E(P)$, there is a y, x-path in D'. When we replace all such edges of $D^* - D$ in P by paths in D', we obtain a u, v-walk in D'. This must in fact be a u, v-path in D', because D' is acyclic. Finally, the path we have found in D' is at least as long as P, because we replaced each edge of P not in D' with a nontrivial path in D'.

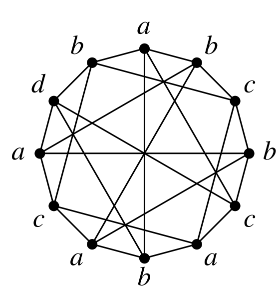
Since $D' \subseteq D$, also $l(D') \le l(D)$, so $l(D^*) \le l(D)$, and maximum path length is minimized over all orientations by an acyclic orientation.

With D^* an acyclic orientation minimizing the maximum path length, Minty's Theorem yields $\chi(G) \le 1 + r(D^*)$, where $r(D^*)$ is the floor of the maximum ratio of forward edges of D^* to backward edges of D^* when traversing a cycle of G. If a cycle of G achieving the maximum has k backward edges, then it has at least $kr(D^*)$ forward edges, and by the pigeonhole principle it has $r(D^*)$ consecutive forward edges. Hence $l(D^*) \ge r(D^*)$, and we obtain the desired inequality $\chi(G) \le 1 + r(D^*) \le 1 + l(D^*) \le 1 + l(D)$, where D is any orientation.

5.1.46. *4-regular triangle-free 4-chromatic graphs.* The graph on the left is isomorphic to the graph in the middle and properly 4-colored on the right.







We show that there is no proper 3-coloring. In a proper 3-coloring, the largest independent set has size at least 4, and the remaing vertices induce a bipartite subgraph. Thus it suffices to show that deleting a maximal independent set of size at least 4 always leaves a 3-cycle.

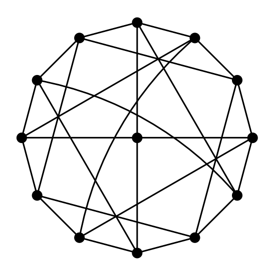
Rotational and reflectional symmetries partition the vertices into two orbits. Using the labeling on the left, class 1 is $\{2, 5, 8, 11\}$, and class 2 is the rest. The subgraph G_1 induced by class 1 is a 4-cycle, with the maximal independent sets consisting of opposite vertices. The subgraph G_2 induced by class 2 is an 8-cycle plus chords joining opposite vertices. The maximal independent sets have size 3. By symmetries and flips, all such sets are equivalent by isomorphisms to $\{0, 7, 9\}$. This set is adjacent to all vertices of Class 1 except 2 and 5, and we can add just one of those two.

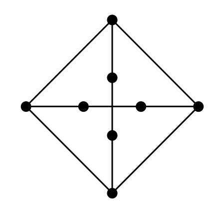
Therefore, when we use two vertices of Class 1, we can add only two from class 2. By symmetry, we may assume that 2 and 8 are used from Class 1. This eliminates all vertices of Class 1 except $\{4, 6, 10, 0\}$. The maximal additions are $\{4, 6\}$ and $\{10, 0\}$, equivalent by symmetry.

Below we list in the first column a representative for each type of maximal independent set of size at least 4. The second column gives an odd cycle among the remaining vertices. Hence there is no proper 3-coloring.

$\{0, 7, 9, 2\}$	$(10, 11, 8, 5, 6)$
$\{0, 7, 9, 5\}$	$(10, 11, 2, 3, 4)$
$\{0, 2, 8, 10\}$	$(3, 4, 5, 6, 7)$

Another 4-chromatic graph. This graph is obtained from the graph above by deleting the 0-6 and 3-9 edges and replacing them with a new vertex z adjacent to 0, 3, 6, 9. Hence we obtain a proper 4-coloring from the coloring of the previous graph by using color d on z.





To show that there is no proper 3-coloring, consider the previous graph. Vertices 0 and 6 together dominate all but 2, 3, 8, 9, which now induce $2K_2$. By rotational symmetry, 3 and 9 also belong to no independent 5-set of old vertices. Since the new graph has 13 vertices, 3-coloring requires a color class of size 5, and this must include the new vertex z.

The subgraph induced by the nonneighbors of z is shown on the right above. The four vertices of degree 2 are not independent, so a vertex of degree 3 is needed to form an independent 4-set. However, deleting a vertex of degree 3 leaves $2K_2$, and only two additional vertices can be chosen.

We have shown that the 13-vertex graph has no independent 5-set, so its chromatic number is 4.

5.1.47. *Brooks' Theorem and the following statement (*) are equivalent: every $(k-1)$-regular k-critical graph is a complete graph or an odd cycle.* Suppose first that Brooks' Theorem is true. Let G be a $(k-1)$-regular k-critical graph. Thus $\Delta(G) = k-1$ and $\chi(G) = k$. By Brooks' Theorem, G must be a complete graph or an odd cycle. Hence (*) follows.

Conversely, assume (*): every $(k-1)$-regular k-critical graph is a complete graph or an odd cycle. Let G be a connected graph with chromatic number k. In order to prove Brooks' Theorem, we must show that $\Delta(G) \ge k$ unless G is a complete graph or an odd cycle.

Let H be a k-critical subgraph of G. Since H is k-critical, $\delta(H) \ge k-1$. If $\Delta(G) < k$, then $k-1 \le \delta(H) \le \Delta(H) \le \Delta(G) < k$, which requires H to be $(k-1)$-regular. By (*), H is a complete graph or an odd cycle. If also $\Delta(G) = k-1$, then no vertex of H has an additional incident edge in G. This means that H is a component of G, so it is all of G, since G is connected. We have shown that if $\Delta(G) < \chi(G)$, then G is a complete graph or an odd cycle. Hence (*) implies Brooks' Theorem.

5.1.48. *A simple graph G with maximum degree at most 3 and no component isomorphic to K_4 has a bipartite subgraph with at least $e(G) - n(G)/3$ edges.* By Brooks' Theorem, G is 3-colorable. In a proper 3-coloring, let red be the smallest color class, with green and blue being the other two. By the pigeonhole principle, there are at most $n(G)/3$ red vertices.

Each red vertex v has three neighbors. By the pigeonhole principle, blue or green appears at most once in $N(v)$. If blue appears at most once, then we delete the edge from v to its blue neighbor (if it has one) and change the color of v to blue. If green appears at most once, then we delete the edge to the green neighbor (if it has one) and make v green.

This alteration deletes at most $n(G)/3$ edges and eliminates the color red. Thus it produces the desired bipartite subgraph.

5.1.49. *The Petersen graph can be 2-colored so that each color class induces only isolated edges and vertices.* Such a coloring appears on the front cover of the text. One color class is an independent set of size 4. Deleting an independent 4-set from the Petersen graph leaves $3K_2$.

5.1.50. *Improvement of Brooks' Theorem.*

a) For any graph G and parameters $D_1,\dots,D_t$ such that $\sum D_i \ge$

$\Delta(G) - t + 1$, $V(G)$ can be partitioned into t classes such that the subgraph G_i induced by the ith class has $\Delta(G_i) \le D_i$.

Proof 1 (extremality). Given a partition $V_1,\dots,V_t$ of $V(G)$, let $e_i = e(G[V_i])$ for each i, and let $d_i(x) = |N(x) \cap V_i|$. We claim that a partition minimizing $f = \sum e_i/D_i$ has the desired property. If $d_i(x) > D_i$ for some $x \in V_i$, then $|N(x) - V_i| \le d(x) - D_i \le \Delta(G) - D_i \le \sum_{j\ne i}(D_j+1)$. Thus for some j other than i, we have $d_j(x) \le D_j$. Moving x from V_i to V_j reduces f by $d_i(x)/D_i - d_j(x)/D_j$, which is positive. Thus when f is minimized each induced subgraph meets its degree bound.

Proof 2 (induction on t). For $t = 2$, we claim that the partition minimizing $D_1 e(G_2) + D_2 e(G_1)$ satisfies the desired bounds. If not, then there is a vertex x, say $x \in V(G_1)$, such that $d_{G_1}(x) > D_1$; x has at most $D_2 - 1$ neighbors in $V(G_2)$. Moving x to the other part gains less than $D_1 D_2$ and loses more than $D_2 D_1$, which contradicts the optimality of the original partition.

For $t > 2$, let $D = D_1 + \cdots D_{t-1} + (t-2)$. We have $D + D_t \ge \Delta(G) - 1$, and the hypothesis for 2 parameters guarantees a vertex partition where the induced subgraphs have maximum degrees bounded by D and D_t. Since $D_1 + \cdots + D_{t-1} = D - t + 2$, the hypothesis for $t-1$ parameters yields the rest of the desired partition.

b) (general case). If G contains no r-clique, where $4 \le r \le \Delta(G)+1$, then $\chi(G) \le \left\lceil \frac{r-1}{r}(\Delta(G)+1) \right\rceil$. Let $D_1 = \cdots = D_{t-1} = r-1$, and require $D_t \ge r-1$ and $\sum D_i = \Delta(G) - t + 1$. Thus $t = \lfloor (\Delta(G)+1)/r \rfloor$. By (a), $V(G)$ partitions into t classes such that $\Delta(G_i) \le D_i$. Since G has no r-clique, Brooks' Theorem implies $\chi(G_i) \le D_i$. Coloring the subgraphs with disjoint color sets, we have $\chi(G) \le \sum \chi(G_i) \le \sum D_i = \Delta(G) + 1 - t$.

b) (special case). If G contains no K_4 and $\Delta(G) = 7$, then $\chi(G) \le 6$. Letting $t = 2$ and $D_1 = D_2 = 3$, we have $D_1 + D_2 = \Delta(G) - t + 1$ Applying part (a), we are guaranteed a partition of $V(G)$ into two sets such that the subgraph induced by each set has maximum degree at most 3. Since G has no K_4, Brooks' Theorem guarantees that both subgraphs are 3-colorable. Coloring the two subgraphs with disjoint color sets, we have $\chi(G) \le 6$.

5.1.51. *If G is a k-colorable graph, and P is a set of vertices in G such that $d(x,y) \ge 4$ whenever $x, y \in P$, then every coloring of P with colors from $[k+1]$ extends to a proper $k+1$ coloring of G.* Let $c\colon P \to [k+1]$ be the coloring on the precolored set P, and let $f\colon V(G) \to [k]$ be a proper k-coloring of G. We define an $(k+1)$-coloring g of G by

$$g(u) = \begin{cases} c(u) & u \in P \\ k+1 & u \in N(v),\ v \in P, \text{ and } f(u) = c(v) \\ f(u) & \text{otherwise} \end{cases}$$

Since the neighbors of $v \in P$ having color $c(v)$ under f must form an independent set and vertices of P are separated by distance at least 4,

$\{u\colon g(u) = r + 1\}$ is independent. A color among the first r is used only on P or on vertices receiving that color under f. The latter type form an independent set, and when a color is used on a vertex $v \in P$, all neighbors of P explicitly receive a color different from that. Hence each color class in g is an independent set, and g is a proper coloring that extends c.

5.1.52. *Every graph G can be $\lceil(\Delta(G) + 1)/j\rceil$-colored so that each color class induces a subgraph having no j-edge-connected subgraph.* A j-edge-connected subgraph has minimum degree at least j. Hence it suffices to color $V(G)$ so that when each vertex is assigned a color, it has fewer than j neighbors among vertices already colored. In this way, no j-edge-connected subgraph is ever completed within a color class.

Color vertices according to some order σ. When a vertex is reached, it has at most $\Delta(G)$ earlier neighbors. Since $\lceil(\Delta(G) + 1)/j\rceil$ colors are available, by the pigeonhole principle some color has been used on fewer than j earlier neighbors. We assign such a color to the new vertex.

No smaller number of classes suffices if G is an j-regular j-edge-connected graph or an n-clique with $n \equiv 1 \pmod j$ (or an odd cycle when $j = 1$). If G is a j-regular j-edge-connected graph, then two colors are needed. If $G = K_n$ with $n \equiv 1 \pmod j$, then we can give each color to at most j vertices, and thus $\lceil n/j\rceil$ colors are needed. If $j = 1$ and G is an odd cycle, then 3 colors are needed. In each case, the needed number of colors equals $\lceil(\Delta(G) + 1)/j\rceil$.

5.1.53. *Relaxed colorings of the $2k$-regular graph $G_{n,k}$ of Exercise 5.1.23.* For $k \le 4$, we seek n such that there is a 2-coloring in which each color class induces a subgraph with maximum degree at most k.

Of the $2k$ neighbors of a vertex v, at most half can have the same color as v. When n is even, alternating colors works, and when n is odd we can insert one additional vertex with either color. This solution is trivial because the problem was improperly stated: "at most k" should be "less than k". Say that a 2-coloring is *good* if each vertex has at most $k - 1$ neighbors of its own color. We solve the intended problem.

If n is even and k is odd, then alternating the colors around C gives an good 2-coloring of $G_{n,k}$, since each vertex has exactly $(k - 1)/2$ neighbors with its own color in each direction. More generally, let n be a multiple of $2j$, and 2-color $G_{n,k}$ using runs of j consecutive vertices with the same color. Suppose that $k = q \cdot 2j + r$, where $j \le r < 2j$. Consider the ith vertex in a run. Following it, this vertex has $(j - i) + qj + [r - j - (j - i)]$ neighbors of its own color; preceding it, this vertex has $(i - 1) + qj + [r - j - (i - 1)]$ neighbors of its own color. Altogether, then, every vertex has fewer than k neighbors of its own color, since $q(2j) + 2r - 2j = k + r - 2j < k$. Thus there is a good 2-coloring when n is a multiple of $2j$ and j is a positive integer such that the k is congruent modulo $2j$ to a value in $\{j, \ldots, 2j - 1\}$. For $k \in \{1, 3\}$, this permits all even n. For $k = 2$, it permits multiples of 4. For $k = 4$, it permits multiples of 6 or 8. Let T denote this set of values of n.

When $k \le 4$, we show that the set T consists of all values of n that permit good 2-colorings. Consider a good 2-coloring of $G_{n,k}$ for n not in this set. Obviously there is no run of length at least $k + 1$ in the same color; each vertex neighbors all others in a run of length $k + 1$. If there is a run of length k, then since n is not a multiplie of $2k$ there is a run of length k followed by a shorter run. Now the last vertex of the k-run has $k - 1$ neighbors of its own color in that run and another neighbor after the subsequent run, which is forbidden.

Hence we may assume that all runs have length less than k. If they all have the same length j, then n is a multiple of $2j$. If $k = q \cdot 2j + r$ with $0 \le r < j$, then the last vertex of a run has qj neighbors of its own color following it and $qj - 1 + r + 1$ neighbors of its own color preceding it. These sum to k, so such a coloring is not good.

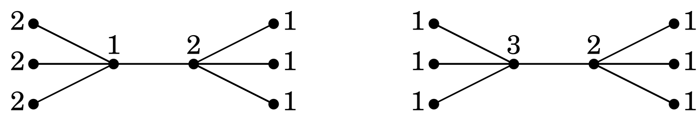
Hence we may assume that the coloring has adjacent runs of different sizes, each each less than k. For $k \le 2$, this is impossible. If the coloring has a $(k - 1)$-run A, then the absence of k-runs implies that the first vertex has of A has an earlier neighbor in its own color. Since it also has $k - 2$ neighbors in A, the run following A must have length at least 2. Thus runs of length $k - 1$ are surrounded by runs of length at least 2. For $k = 3$, this forbids runs of distinct length and therefore completes the proof.

Now consider $k = 4$. If the coloring has a 3-run, then the preceding argument and the requirement of having runs of different lengths allows us to assume colors *abbbaab* in order on $v_0, \ldots, v_6$. Now successively examining the neighborhoods of v_3 and v_4 allows us to conclude that $f(v_7) = a$ and then $f(v_8) = b$. Next the neighborhood of v_6 leads us to $f(v_9) = f(v_{10}) = a$, but now v_7 has four neighbors of its own color.

The only remaining possibility is that the largest run has size 2. With runs of different lengths, we may assume collors *abbab* on $v_0, \ldots, v_4$. Since v_5 has a later neighbor of its own color, $f(v_5) = a$. Since there is no 3-run, v_2 has two earlier neighbors of its own color b, and thus $f(v_6) = a$. Since there is no 3-run, $f(v_7) = a$. Now v_4 has three neighbors in its own color, which forces $f(v_8) = a$. We have produced the pattern of $f(v_0), \ldots, f(v_4)$ on vertices $v_4, \ldots, v_8$, with the colors exchanged. Hence the pattern *abba baab abba baab*, continues. This is a good coloring, but it requires n to be a multiple of 8, and these values of n are already in T.

Comment. West–Weaver [1994] conjectured that T contains all n for which $G_{n,k}$ has a good coloring is not valid for larger k. This was disproved by Brad Friedman, who discovered other values of n with good colorings for all larger k.

5.1.54. Let f be a proper coloring of a graph G in which the colors are natural numbers. The *color sum* is $\sum_{v\in V(G)} f(v)$. Minimizing the color sum may require using more than $\chi(G)$ colors. In the tree below, for example, the best proper 2-coloring has color sum 12, while there is a proper 3-coloring with color sum 11. Construct a sequence of trees in which the kth tree T_k use k colors in a proper coloring that minimizes the color sum. (Kubicka–Schwenk [1989])



5.1.55. *Chromatic number is bounded by one plus longest odd cycle length.*

a) If G is a 2-connected non-bipartite graph containing an even cycle C, then there exist vertices x, y on C and an x, y-path P internally disjoint from C such that $d_C(x, y) \not\equiv d_P(x, y)$ (mod 2). Let C' be an odd cycle in G. Since G is 2-connected, G has a cycle containing an edge of C and an edge of C'. Using edges of this and C', we can form an odd cycle D containing at least two vertices of C. Let $x_1, \ldots, x_t$ be the common vertices of C and D, indexed in order of their appearance on D. Letting $x_{t+1} = x_1$, we have $\sum_{i=1}^{t} d_D(x_i, x_{i+1}) \equiv 1 \pmod 2$, since D is an odd cycle. On the other hand, since C is an even cycle, it has a bipartition, and $d_C(x_i, x_{i+1})$ is even if x_i and x_{i+1} are on the same side of the bipartition of C, odd if they are on opposite sides. Hence $\sum_{i=1}^{t} d_C(x_i, x_{i+1}) \equiv 0 \pmod 2$. Therefore, for some value of i we have $d_D(x_i, x_{i+1}) \not\equiv d_C(x_i, x_{i+1})$, and we use this portion of D as P.

b) If $\delta(G) \ge 2k$ and G has no odd cycle longer than $2k-1$, then G has a cycle of length at least $4k$. Let $P = x_1, \ldots, x_t$ be a maximal path in G, so $N(x_1) \subseteq V(P)$. Let x_r be the neighbor of x_1 farthest along P; $d(x_1) \ge 2k$ implies $r \ge 2k+1$. By the odd cycle condition, r is even, and neither x_{2i+1} nor x_{r-2i+1} can belong to $N(x_1)$ if $i \ge k$. If $\{2i+1\colon i \ge k\}$ and $\{x_{r-2i+1}\colon 2 \ge k\}$ are disjoint, then together with $N(x_1)$ we have at least $r - 2k + 2k = r$ vertices with indices from 2 to r. This is impossible, so we must have $2k+1 \le r - 2k + 1$, implying $r \ge 4k$.

c) If G is a 2-connected graph having no odd cycle longer than $2k-1$, then $\chi(G) \le 2k$. We use induction on $n(G)$. For $n(G) = 2$, the claim holds using $k = 1$. For the induction step, suppose $n(G) = n > 2$ and the claim holds for graphs with fewer than n vertices. Since $\chi(G)$ is the maximum chromatic number of its blocks, we may assume G is 2-connected. Suppose the longest odd cycle in G has length $2k-1$, but $\chi(G) > 2k$. For any $x \in V(G)$, the induction hypothesis implies $\chi(G - x) \le 2k$. Hence G is vertex-$(2k+1)$-critical, which implies $\delta(G) \ge 2k$. By part (b), G has a cycle C of length at least $4k$. By part (a), G has a path P joining two vertices x, y of C such that P together with either x, y-path along C forms an odd cycle. The sum of the lengths of these two odd cycles is at least $4k+2$. Hence one of them has length at least $2k+1$, contradicting the hypothesis. The contradiction yields $\chi(G) \le 2k$.

5.2. STRUCTURE OF k CHROMATIC GRAPHS

5.2.1. *If $\chi(G - x - y) = \chi(G) - 2$ for all pairs x, y of distinct vertices, then G is a complete graph.* If $x \not\leftrightarrow y$, then a proper coloring of $G - x - y$ can be augmented with one new color on x and y to obtain a proper coloring of G. This yields $\chi(G) \le \chi(G - x - y) + 1$, so the given condition forces $x \leftrightarrow y$ for all $x, y \in V(G)$.

5.2.2. *A simple graph is a complete multipartite graph if and only if it has no induced three-vertex subgraph with one edge.* If a connected graph is not a clique, then the shortest of all paths between nonadjacent pairs of vertices has length two, and the three vertices of this path induce a subgraph with exactly two edges. Hence each successive pair of the following statements are equivalent: (1) G has no induced 3-vertex subgraph with one edge. (2) $\overline{G}$ has no induced 3-vertex subgraph with two edges. (3) Every component of $\overline{G}$ is a clique. (4) G is a complete multipartite graph.

5.2.3. *The smallest k-critical graphs.*

a) If x, y are vertices in a color-critical graph G, then $N(x) \subseteq N(y)$ is impossible, and hence there is no k-critical graph with $k+1$ vertices. If G is k-critical, then $G - x$ is $(k-1)$-colorable, but $N(x) \subseteq N(y)$ would allow us to return x with the same color as y to obtain a $(k-1)$-coloring of G. If $n(G) = k+1$, then we have $\delta(G) < k$ since K_{k+1} is not k-critical, and we have $\delta(G) \ge k-1$ by the properties of k-critical graphs. Hence $\delta(G) = k-1$, which implies that nonadjacent vertices x, y have the same set of neighbors (the remaining $k-1$ vertices), which contradicts the statement just proved. Hence there is no k-critical graph with $k+1$ vertices.

b) $\chi(G \vee H) = \chi(G) + \chi(H)$, and $G \vee H$ is color-critical if and only if both G and H are color-critical, and hence there is a k-critical graph with $k+2$ vertices. Coloring G and H optimally from disjoint sets yields a proper coloring of $G \vee H$, so $\chi(G \vee H) \le \chi(G) + \chi(H)$. The colors used on the subgraph of $G \vee H$ arising from G must be disjoint from the colors on the copy of H, since each vertex of the former is adjacent to each of the latter; hence $\chi(G \vee H) \ge \chi(G) + \chi(H)$.

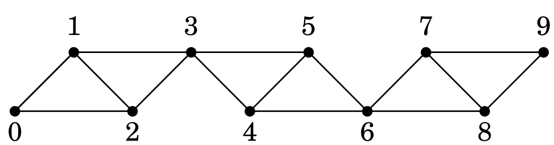
For criticality, consider an arbitrary edge $xy \in E(G \vee H)$. If $xy \in E(G)$, then $(G \vee H) - xy = (G - xy) \vee H$, and hence $\chi(G \vee H) - xy =$

$\chi(G \vee H) - 1$ if and only if $\chi(G - xy) = \chi(G) - 1$. Similarly for $xy \in E(H)$. Hence $G \vee H$ being color-critical implies that G and H are color-critical. For the converse, assume that G and H are color-critical. We have already considered $G \vee H - xy$ for $xy \in E(G) \cup E(H)$; we must also consider $xy \in E(G \vee H)$ with $x \in V(G)$ and $y \in V(H)$. By the properties of color-critical graphs, we know that G and H have optimal colorings in which x and y, respectively, are the only vertices in their color classes. In $G \vee H - xy$, we use these colorings but change the color of y to agree with x. This uses $\chi(G) + \chi(H) - 1$ colors.

Since C_5 is 3-critical and K_{k-3} is $(k-3)$-critical, we conclude that $C_5 \vee K_{k-3}$ is a k-critical graph with $k+2$ vertices.

5.2.4. *Blocks and coloring in a special graph.* Let G be the graph with vertex set $\{v_0, \ldots, v_{3n}\}$ defined by $v_i \leftrightarrow v_j$ if and only if $|i-j| \le 2$ and $i+j$ is not divisible by 6.

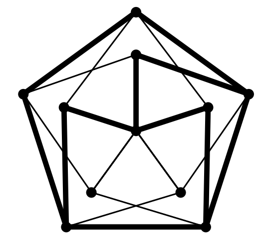
a) The blocks of G. Because consecutive integers sum to a number that is odd and hence not divisible by 6, the vertices $v_0, \ldots, v_{3n}$ form a path in order. Edges of the form $\{v_i, v_{i+2}\}$ are added when i is congruent to one of $\{0, 1, 3, 4\}$ modulo 6, but not when i is congrent to 2 or 5 modulo 6. Thus G is the graph below, and there are n blocks. The blocks are the subgraphs induced by $\{v_{3i-3}, v_{3i-2}, v_{3i-1}, v_{3i}\}$ for $1 \le i \le n$.



b) Adding the edge v_0v_{3n} to G creates a 4-critical graph. In a proper 3-coloring of G, the induced kites force successive vertices whose indices are multiples of 3 to have the same color. When the edge v_0v_{3n} is added to form G', the graph is no longer 3-colorable.

If an edge in the ith kite is deleted, then giving its endpoints the same color permits properly 3-coloring the remainder of the subgraph induced by $\{v_{3i-3}, v_{3i-2}, v_{3i-1}, v_{3i}\}$ so that v_{3i-3} and v_{3i} have different colors. Continuing the proper 3-coloring in both directions gives v_0 the color of v_{3i-3} and gives v_{3n} the color of v_{3i}. Thus the edge v_0v_{3n} is also properly colored. We have shown that for each edge e, the graph $G' - e$ is 3-colorable, so G' is 4-critical.

5.2.5. *A subdivision of K_4 in the Grötzsch graph.* The subgraph in bold below is a subdivision of K_4.



5.2.6. *The minimum number of edges in a connected n-vertex graph with chromatic number k is $\binom{k}{2} + n - k$.* Equality holds for the graph obtained by identifying a vertex of K_k with an endpoint of P_{n-k+1}. The desired lower bound on $e(G)$ when $k = 2$ is $n - 1$ and holds trivially for connected graphs, so we may assume that $k \ge 3$.

Proof 1 (critical subgraph). Let G be a connected k-chromatic n-vertex graph. Let H be a k-critical subgraph of G. If H has t vertices, then $e(H) \ge (k-1)t/2$, since $\delta(H) \ge k-1$. With H and the remaining $n-t$ vertices of G as $n-t+1$ components, we must add at least $n-t$ additional edges to reduce the number of components to 1. Hence

$$e(G) \ge (k-1)t/2 + n - t = (k-3)t/2 + n.$$

Since $n \ge t \ge k$, this is minimized when $t = k$, yielding the desired value.

Proof 2 (induction on n). For $n = k$, the bound again is trivial. For $n > k$, let G be a minimal connected k-chromatic n-vertex graph. By the choice of G, deletion of any edge disconnects G or reduces k.

If $G - e$ is disconnected for some $e \in E(G)$, then it has two components. At least one of these must be k-chromatic, else we can recolor G with fewer than k colors. Letting l be the number of vertices in a k-chromatic component of $G - e$, the induction hypothesis yields

$$e(G) \ge \left[\binom{k}{2} + l - k\right] + 1 + (n - l - 1) = \binom{k}{2} + n - k,$$

where the additional terms count e itself and the edges of a spanning tree of the other component.

In the remaining case, $\chi(G - e) < k$ for all $e \in E(G)$. Hence G is k-critical, which requires $\delta(G) \ge k - 1$. Hence

$$e(G) \ge n(k-1)/2 = n + n(k-3)/2 > n + k(k-3)/2 = n - k + \binom{k}{2}.$$

5.2.7. *In an optimal coloring of a graph, for each color there is a vertex of that color that is adjacent to vertices of all other colors.* Let C be the set of vertices of color i, and consider $v \in C$. If v has no neighbor of color j, then we can switch the color of v to j. Since we are changing colors only for vertices in C, moving several of them to color j in this way creates no conflicts, since C is an independent set. After relabeling all vertices of C, we have obtained a proper coloring without using color i. Hence C must have some "unmovable" vertex, adjacent to vertices of every other color.

5.2.8. *Critical subgraph approach to $\chi(G) \le \max_i \min\{d_i+1, i\}$.* If $\chi(G) = k$, then G has a k-critical subgraph, which has at least k vertices of degree at least $k-1$. These vertices also have degree at least k in G, so $d_k \ge k-1$. Hence $\chi(G) = k = \min\{d_k+1, k\} \le \max_i \min\{d_i+1, i\}$.

5.2.9. *If G is a color-critical graph, then the graph G' generated from it by applying Mycielski's construction is also color-critical.* We use properties of a k-critical graph G obtained in Proposition 5.2.13a: a) For $v \in V(G)$, there is a proper k-coloring of G in which color k appears only at v, and b) For $e \in E(G)$, every proper $(k-1)$-coloring of $G-e$ uses the same color on the endpoints of e.

Given $V(G) = \{v_1, \dots, v_n\}$, let G' be as in Mycielski's construction, with $V(G') = V(G) \cup \{u_1, \dots, u_n\} \cup \{w\}$. Suppose that G is k-critical. The proof of Theorem 5.2.2 yields $\chi(G') = k+1$; thus it suffices to show that $\chi(G'-e) = k$ for $e \in E(G')$.

For $e = wu_j$, let f be a proper k-coloring of G with color k appearing only on v_j. Extend f to $G'-e$ by setting $f(u_i) = f(v_i)$ for $1 \le i \le n$ and $f(w) = f(v_j)$.

For $e = v_r v_s$, let f be a proper $(k-1)$-coloring of $G-e$, which exists because G is k-critical. Extend f to $G'-e$ by letting $f(u_i) = k$ for $1 \le i \le n$ and $f(w) = 1$.

For $e = v_r u_s$, let f be a proper $(k-1)$-coloring of $G - v_r v_s$. By Proposition 5.2.13b, $f(v_r) = f(v_s)$. Extend f to $G'-e$ by letting $f(u_i) = f(v_i)$ for $1 \le i \le n$ and $f(w) = k$. This uses k colors and uses color k only on w, but this is not a proper coloring of $G'-e$, because the endpoints of the edges $v_r v_s$ and $u_r v_s$ have received the same color. We correct this to a proper coloring by changing the color of v_s to k.

5.2.10. *Given $H \subseteq G$ with $V(G) = \{v_1, \dots, v_n\}$, if G'' is obtained from G by applying Mycielski's construction and adding the edges $\{u_i u_j : v_i v_j \in E(H)\}$, then $\chi(G'') = \chi(G)+1$ and $\omega(G'') = \max\{\omega(G), \omega(H)+1\}$.* Since G'' is a supergraph of the result G' of Mycielski's construction, $\chi(G'') \ge \chi(G') = \chi(G)+1$. On the other hand, the proper coloring of G' that uses a proper $\chi(G)$-coloring on $v_1, \dots, v_n$, copies the color of v_i onto u_i for each i, and assigns a new color to w, is still a proper coloring of G'', so $\chi(G'') = \chi(G)+1$.

Since u_i and v_i remain nonadjacent for all i, every complete graph induced by $\{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\}$ is a copy of a complete subgraph of G, using at most one of $\{v_i, u_i\}$ for each i. Every complete graph involving w is an edge wu_i or consists of w together with a complete subgraph in H. Hence $\omega(G'') = \max\{\omega(G), \omega(H)+1\}$.

5.2.11. *If G has no induced $2K_2$, then $\chi(G) \le \binom{\omega(G)+1}{2}$.* To prove the upper bound, we define $k + \binom{k}{2}$ independent sets that together cover $V(G)$.

Let $Q = \{v_1, \dots, v_k\}$ be a maximum clique in G. Let S_i be the set of vertices in G that are adjacent to all of Q except v_i. This set is independent, since two adjacent vertices in S_i would form a $(k+1)$-clique with $Q - \{v_i\}$.

Let $T_{i,j}$ be the set of vertices in G that are adjacent to neither of $\{v_i, v_j\}$. This set is independent, since two adjacent vertices in $T_{i,j}$ would form an induced $2K_2$ with $\{v_i, v_j\}$.

Every vertex of G has at least one nonneighbor in Q, since Q is a maximum clique. Thus every vertex of G is in some S_i or in some $T_{i,j}$, and we have covered $V(G)$ with the desired number of independent sets.

5.2.12. *Zykov's construction.* Let $G_1 = K_1$. For $k > 1$, construct G_k from $G_1, \dots, G_{k-1}$ by taking the disjoint union $G_1 + \cdots + G_{k-1}$ and adding a set T of $\prod_{i=1}^{k-1} n(G_i)$ additional vertices, one for each way to choose exactly one vertex v_i from each G_i. Let the vertex of T corresponding to a particular choice of $v_1, \dots, v_{k-1}$ have those $k-1$ vertices as its neighborhood.

a) $\omega(G_k) = 2$ and $\chi(G_k) = k$. Giving all of T a single color k and using an i-coloring from $\{1, \dots, i\}$ on each copy of G_i yields a proper k-coloring of G_k. Since the neighbors of each vertex of T are in distinct components of $G_k - T$, the edges to T introduce no triangle.

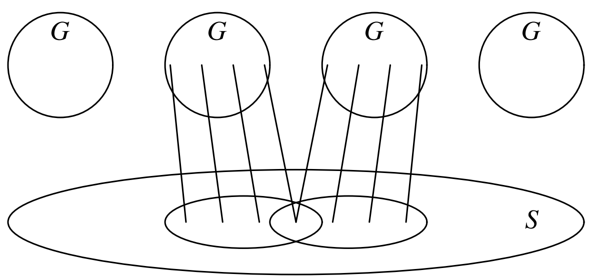
Suppose that G_k has a proper $(k-1)$-coloring. Because $\chi(G_i) = i$, some color is used on G_1, some other color is used on G_2, some third color used on G_3, and so on. Thus vertices can be selected from the subgraph $G_1 + \cdots + G_{k-1}$ having all $k-1$ colors. By the construction of G_k, some vertex of T has these as neighbors, and the proper coloring cannot be completed.

b) Zykov's construction produces color-critical graphs. We must show that, for any edge xy, $G_k - xy$ has a proper $(k-1)$-coloring. Suppose this has been shown for $G_1, \dots, G_{k-1}$, and consider an edge xy of G_k. If $x, y \notin T$, then xy is an edge of G_t for some $t < k$. Color each G_i with colors $1, \dots, i$ for $i \ne t$, but color $G_t - xy$ with colors $1, \dots, t-1$. Each vertex of T has $k-1$ neighbors, but it is not possible for each of the colors $1, \dots, k-1$ to appear among the neighbors of any vertex of T, because its neighbors in $\{G_1, \dots, G_t\}$ have received only $t-1$ colors. Hence there is a color in $\{1, \dots, k-1\}$ available for any vertex of T.

Finally, suppose we delete an edge xy with $x \in T$, $y \notin T$, and let $S = N(x)$. By the criticality of $G_1, \dots, G_{k-1}$, each G_i has a proper i-coloring with colors $1, \dots, i$ in which the only vertex of color i is the neighbor of x in G_i; use these colorings to color $G_k - T$. In order to choose one vertex from each G_i and obtain a set with colors $1, \dots, k-1$, we must choose the vertex with color i from G_i; the only way to do this is to choose S. Since x is the only vertex of T with these neighbors, for every other vertex of T there is a color in $\{1, \ldots, k-1\}$ available for it. Finally, if the other endpoint y of the deleted edge xy is the neighbor of x in G_i, we can give color i to x to complete the proper $(k-1)$-coloring of $G - xy$.

5.2.13. *Inductive construction of k-chromatic graphs of girth at least six.* Given G with girth at least 6 and $\chi(G) = k$, form G' by taking $\binom{kn(G)}{n(G)}$ copies of G and an independent set S with $kn(G)$ vertices, with each subset of n vertices in S joined by a matching to one copy of G (distinct subsets match to different copies of G). Since G is k-colorable, G' has a proper $(k+1)$-coloring where all of S has color $k+1$. If G' is k-colorable, then any proper k-coloring of G' gives the same color to at least $n(G)$ vertices of S with the same color, by the pigeonhole principle. This color is forbidden from the copy of G matched to this $n(G)$-subset of S. Now the coloring cannot be completed, since proper coloring of this copy of G require at least k colors.

Since G has girth at least 6, any cycle of length less than 6 must use at least two vertices of S. However, S is independent, and vertices of S have no common neighbors, so it must take at least 3 edges to go from one vertex of S to another.



5.2.14. *Chromatic number and cycle lengths.*

a) If v is a vertex in a graph G, and T is a spanning tree that maximizes $\sum_{u \in V(G)} d_T(u, v)$, then every edge of G joins vertices belonging to a path in T starting at v. View v as the root of T. If u is on the v, w-path in T, then w is a *descendant* of u. Suppose that xy is an edge in G such that neither of $\{x, y\}$ is a descendant of the other in T. We may assume that $d_T(v, x) \le d_T(v, y)$. Now cutting the edge reaching x on the v, x-path in T and replacing it with yx increases the distance from v to x and to all its descendants. This contradicts the choice of T, so there is no such edge.

b) If $\chi(G) > k$, then G has a cycle whose length is one more than a multiple of k. Define T as in part (a). Define a coloring of G by letting the color assigned to each vertex x be the congruence class modulo k of $d_T(v, x)$. This is a proper coloring unless G has an edge xy outside T that joins vertices of the same color. By part (a), x or y is a descendant of the other, and the length of the x, y-path in T is a multiple of k. If G has no cycle with length one more than a multiple of k, then there is no such edge, and the coloring is proper. We have proved the contrapositive of the claim.

5.2.15. *Every triangle-free n-vertex graph G has chromatic number at most $2\sqrt{n}$.* Since G is triangle-free, every vertex neighborhood is an independent set. Iteratively use a color on a largest remaining vertex neighborhood and delete those vertices. After $\lfloor \sqrt{n} \rfloor$ iterations, the maximum degree in the remaining subgraph is less than $\sqrt{n}$. Otherwise, we have deleted at least $\sqrt{n}$ vertices $\lfloor \sqrt{n} \rfloor$ times, and there are at most $\sqrt{n}$ vertices remaining. Since the maximum degree of the remaining subgraph is less than $\sqrt{n}$, we can use $\sqrt{n}$ additional colors to properly color what remains.

5.2.16. *A simple n-vertex graph with no K_{r+1} has at most $(1 - \frac{1}{r})\frac{n^2}{2}$ edges.*

Proof 1 (induction on r). Basis step: If $r = 1$, then G has no edges, as claimed. Induction step: For $r > 1$, let x be vertex of maximum degree, with $d(x) = k$. Since G has no $(r+1)$-clique, the subgraph G' induced by $N(x)$ has no r-clique. Hence G' has at most $\frac{r-2}{r-1}k^2/2$ edges, by the induction hypothesis. The remaining edges are incident to the remaining $n - k$ vertices; since each such vertex has degree at most k, there are at most $k(n-k)$ such edges. Summing the two types of contributions, we have $e(G) \le k(n - ak)$, where $a = r/(2r-2)$. The function $k(n - ak)$ is maximized by setting $k = \frac{n}{2a}$, where it equals $\frac{n^2}{4a}$. Hence $e(G) \le \frac{n^2}{4a} = (1 - \frac{1}{r})\frac{n^2}{2}$.

Proof 2 (by Turán's Theorem). By Turán's Theorem, the maximum number of edges in a graph with no $(r+1)$-clique is achieved by the complete r-partite graph with no two part-sizes differing by more than one. If the part-sizes are $\{n_i\}$, the degree-sum is $\sum_{i=1}^r n_i(n - n_i) = n^2 - \sum_{i=1}^r n_i^2$.

By the convexity of the squaring function, the sum of the squares of numbers summing to n is minimized when they all equal n/r. Hence if G has no $(r+1)$-clique, we have $2e(G) \le (1 - \frac{1}{r})n^2$.

5.2.17. *Lower bounds on $\omega(G)$ and $\alpha(G)$ for n-vertex graphs with m edges.*

a) $\omega(G) \ge \lceil n^2/(n^2 - 2m) \rceil$, and this is sometimes sharp. Let r be the number of vertices in the largest clique in G. By Exercise 5.2.16, $m \le (1 - 1/r)n^2/2$. This is equivalent by algebraic manipulation to $n^2/(n^2 - 2m) \le r$.

This bound is sometimes best possible. Let $r = \lceil n^2/(n^2 - 2m) \rceil$. Since $m \le \binom{n}{2}$, we have $r \le n$. For the bound to be sharp, it suffices to show that $T_{n,r}$ has at least m edges and that $m \ge \binom{r}{2}$. If these two statements are true, then we can discard edges from $T_{n,r}$ to obtain a graph G with n vertices and m edges such that $\chi(G) = \omega(G) = r$.

If r is an integer that divides n, then $e(T_{n,r}) = (1 - 1/r)n^2/2 = m$ and the desired properties hold. However, when $n = 12$ and $m = 63$, there are three edges in $\overline{G}$. We have $\lceil n^2/(n^2-2m)\rceil = 8$, but every 12-vertex simple graph with only three edges in the complement has clique number 9.

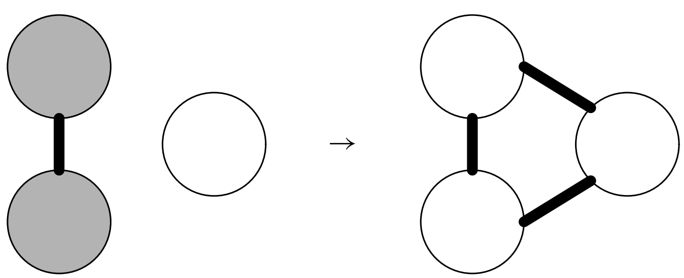
b) $\alpha(G) \ge \lceil n^2/(n+2m)\rceil$ *vertices, and this is sometimes sharp.* We can transform this question into an instance of part (a) by taking complements. Every clique in G becomes an independent set in $\overline{G}$, and vice versa. Let $H = \overline{G}$. Let $m' = \binom{n}{2} - m$ be the number of edges in H. If the largest independent set in G has s vertices, then the largest clique in H has s vertices. From part (b), we have $s \ge \lceil n^2/(n^2-2m')\rceil$. Substituting $m' = \binom{n}{2} - m$ yields $s \ge \lceil n^2/(n+2m)\rceil$. Since this lower bound for s is achieved for some m, n by letting H be the appropriate Turán graph, it is also achieved by letting G be the complement of that graph.

5.2.18. *Counting edges in the Turán graph.* Let $T_{n,r}$ denote the r-partite Turán graph on n vertices, and let $a = \lfloor n/r \rfloor$ and $b = n - ra$.

a) $e(T_{n,r}) = (1-1/r)n^2/2 - b(r-b)/(2r)$. By the degree-sum formula, we need only show that the vertex degrees sum to $(1-1/r)n^2 - b(r-b)/r$. Every vertex has degree $n-a$ or $n-a-1$, with $(r-b)a$ of the former and $b(a+1)$ of the latter. Hence the degree sum is $n(n-a) - b(a+1)$. Substituting $a = (n-b)/r$ yields $n^2 - n(n-b)/r - b(n-b+r)/r$, which equals the desired formula.

b) The least r where $e(T_{n,r})$ can differ from $\lfloor (1-1/r)n^2/2 \rfloor$ is $r = 8$, and $e(T_{n,8}) < \lfloor (1-1/r)n^2/2 \rfloor$ whenever $n \equiv 4 \pmod 8$. In general, $e(T_{n,r}) < \lfloor (1-1/r)n^2/2 \rfloor$ if and only if $b(r-b)/(2r) \ge 1$. Since $e(T_{n,r})$ is an integer, the formula for $e(T_{n,r})$ in part (a) differs from $\lfloor (1-1/r)n^2/2 \rfloor$ if and only if the difference between $(1-1/r)n^2/2$ and $e(T_{n,r})$ is at least 1. Hence the condition is $b(r-b)/(2r) \ge 1$. For fixed r, the left side is maximized by $b = r/2$, where it equals $r/8$. Hence the condition occurs if and only if $r \ge 8$, and when $r = 8$ it occurs if and only if $b = 4$.

5.2.19. *Comparison of the Turán graph $T_{n,r}$ with the graph $\overline{K}_a + K_{n-a}$ yields $e(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$.* Here $a = \lfloor n/r \rfloor$. The initial graph $\overline{K}_a + K_{n-a}$ has $\binom{n-a}{2}$ edges. We transform it into $T_{n,r}$ and study the change in the number of edges. Let A be the independent set of size a. We create $T_{n,r}$ by iteratively removing the edges within a set of size a or $a+1$ to make it one of the desired partite sets, replacing these edges by edges to A.



The number of edges from A to a new partite set B is $a\,|B|$. Whether $|B|$ is a or $a+1$, this numerically equals $\binom{|B|}{2} + \binom{a+1}{2}$. Thus replacing the edges of the clique on B with these edges gains $\binom{a+1}{2}$ edges. Repeating this $r-1$ times to create the other partite sets gains $(r-1)\binom{a+1}{2}$ edges, and thus $e(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$.

5.2.20. *For positive integers n and k, if $q = \lfloor n/k \rfloor$, $r = n - qk$, $s = \lfloor n/(k+1) \rfloor$, and $t = n - s(k+1)$, then $\binom{q}{2}k + rq \ge \binom{s}{2}(k+1) + ts$.* The Turán graph $T_{n,k}$ has partite sets of sizes q and $q+1$, with r of the latter. Hence its complement has $\binom{q}{2}k + rq$ edges. Similarly, $\overline{T}_{n,k+1}$ has $\binom{s}{2}(k+1) + ts$ edges. To prove the desired inequality, it thus suffices to show that $e(\overline{T}_{n,k}) \ge e(\overline{T}_{n,k+1})$, or $e(T_{n,k}) \le e(T_{n,k+1})$.

This follows from Turán's Theorem. Since $e(T_{n,k+1})$ is the maximum number of edges in an n-vertex graph not containing K_{k+2}, and $T_{n,k}$ is such a graph, we have $e(T_{n,k}) \le e(T_{n,k+1})$.

5.2.21. *$T_{n,r}$ is the unique n-vertex K_{r+1}-free graph of maximum size.* We use induction on r. The statement is immediate for $r = 1$. For the induction step, suppose $r > 1$. Let G be an n-vertex K_{r+1}-free graph, and let x be a vertex of maximum degree in x. Let $G' = G[N(x)]$. Let $H' = T_{d(x),r-1}$, and let $H = \overline{K}_{n-d(x)} \vee H'$. Since G' has no r-clique, the induction hypothesis yields $e(H') \ge e(G')$, with equality only if $G' = H'$. Let $S = V(G) - N(x)$. Since $e(G) - e(G') \ge \sum_{v\in S} d_G(v)$ and $e(H) - e(H') = (n-d(x))d(x) = |S|\,\Delta(G)$, we have $e(H) - e(H') \ge e(G) - e(G')$. Hence $e(H) \ge e(G)$, with equality only if equality occurs in both transformations.

We have seen (by the induction hypothesis) that equality in the first transformation requires $G' = H'$. Equality in the second transformation requires each edge of $E(G) - E(G')$ to have exactly one endpoint in S and requires each vertex of S to have degree $d(x)$. Thus every vertex of S is adjacent to every vertex of $N(x)$ and to no other vertex of S, which means that G is the join of G' with an independent set. Since G' is a complete $(r-1)$-partite graph, this makes G a complete r-partite graph. Finally, we know by shifting vertices between partite sets that $T_{n,r}$ is the only n-vertex complete r-partite graph that has the maximum number of edges.

5.2.22. *Vertices of high degree.* We have 18 vertices in a region of diameter 4, with $E(G)$ consisting of the pairs at most 3 units apart. Since $3 > 4/\sqrt{2}$, Application 5.2.11 (in particular the absence of independent 4-sets) guarantees that G lacks at most 108 edges of its 153 possible edges and thus has at least 45 edges. If at most one vertex has degree at least five, then the degree-sum is at most $(17)4 + (1)17 = 85$, which only permits 42 edges.

The result can be strengthened by a more detailed argument (communicated by Fred Galvin). Let S be the set of vertices with degree less than 5. Because there cannot be four vertices that are pairwise separated by at least 3 units, the subgraph induced by S has no independent set of size 4. Thus $|S| \le 15$, since the edges incident to the vertices of a maximal independent set in S must cover all the vertices in S. This shows among any 17 vertices there must be two with degree at least 5.

Furthermore, consider a set with 16 vertices. If $|S| < 15$, then again we have two vertices with degree at least 5. If $|S| = 15$, let T be an independent 3-set in S, and let z be the vertex outside S. Since the vertices of $S - T$ have degree at most 4, they have degree-sum at most 48. However, Theorem 1.3.23 or the absence of independent 4-sets guarantees that $S-T$ has at least 18 edges. Adding the 12 edges to S and at least five edges to z yields degree-sum at least 53. The contradiction implies that at least two vertices have degree at least five.

The result for 16 is sharp, because 15 vertices can be placed in three clumps forming cliques, and then all vertices have degree four.

5.2.23. *Turán's proof of Turán's Theorem.*

a) Every maximal simple graph with no $(r+1)$-clique has an r-clique. If making x and y adjacent creates an $(r+1)$-clique, then the graph must already have a clique of $r-1$ vertices all adjacent to both x and y. Thus x or y forms an r-clique with these vertices.

b) $e(T_{n,r}) = \binom{r}{2} + (n-r)(r-1) + e(T_{n-r,r})$. Since n and $n-r$ have the same remainder under division by r, the size of the ith largest partite set of $T_{n-r,r}$ is one less than the size of the ith largest partite set of $T_{n,r}$, for each i. Hence deleting one vertex from each partite set of $T_{n,r}$ leaves a copy of $T_{n-r,r}$ as an induced subgraph. The deleted edges form a complete subgraph on the vertices removed plus an edge from each of the $n-r$ remaining vertices to all but one of the deleted vertices. The terms in the claimed equation directly count these contributions.

c) The Turán graph $T_{n,r}$ is the unique simple graph with the most edges among n-vertex graphs without K_{r+1}. We use induction on n. Basis step: $n \le r$. We can include all edges without forming K_{r+1}. Thus the maximum graph is K_n, and this is $T_{n,r}$.

Induction step: $n > r$. Let G be a largest simple n-vertex graph avoiding K_{r+1}. By part (a), G contains K_r; let S be an r-vertex clique in G. Since G avoids K_{r+1}, every vertex not in S has at most $r-1$ neighbors in S. Therefore, deleting S loses at most $\binom{r}{2} + (n-r)(r-1)$ edges. The remaining graph G' avoids K_{r+1}. By the induction hypothesis, $e(G') \le e(T_{n-r,r})$, with equality only for $T_{n-r,r}$.

Since $e(G) \le \binom{r}{2} + (n-r)(r-1) + e(G')$, part (b) implies that $e(G) \le e(T_{n,r})$. To achieve equality, G' must be $T_{n-r,r}$, and each vertex of G' must have exactly $r-1$ neighbors in S. If some vertex of S has a neighbor in each partite set of G', then G contains K_{r+1}. Hence each vertex of S has neighbors at most $r-1$ partite sets of G'. Since each vertex of G' is adjacent to $r-1$ vertices in S, the vertices of S miss different partite sets in G'. Thus the vertices of S can be added to distinct partite sets in G' to form $T_{n,r}$.

5.2.24. *An n-vertex graph having $t_r(n) - k$ edges and an $(r+1)$-clique has at least $f_r(n) - k + 1$ such cliques, where $f_r(n) = n - r - \lceil n/r \rceil$ and $k \ge 0$.*

Let G be a graph with exactly one $(r+1)$-clique Q; we first use Turán's Theorem to bound $e(G)$. Note that $e(G-Q) \le t_r(n-r-1)$, and furthermore each $v \in V(G) - Q$ has at most $r-1$ neighbors in Q. Thus

$$e(G) \le t_r(n-r-1) + (r-1)(n-r-1) + \binom{r+1}{2}.$$

To express this in terms of $t_r(n)$, we compute $t_r(n) - t_r(n-r-1)$. First, deleting one vertex from each partite set in $T_{n,r}$ loses the edges among them plus an edge from each remaining vertex to $r-1$ deleted vertices. Hence $t_r(n) - t_r(n-r) = \binom{r}{2} + (r-1)(n-r)$. Also, $T_{n-r,r}$ becomes $T_{n-r-1,r}$ when we delete a vertex from a largest partite set, which has degree $n - r - \lceil (n-r)/r \rceil$. Thus $t_r(n-r) - t_r(n-r-1) = (n-r) - \lceil (n-r)/r \rceil$. Hence

$$t_r(n) - t_r(n-r-1) = \binom{r}{2} + r(n-r) - \lceil n/r \rceil + 1.$$

Together,

$$\begin{aligned} e(G) &\le t_r(n) - \binom{r}{2} - r(n-r) + \lceil n/r \rceil - 1 + (r-1)(n-r-1) + \binom{r+1}{2} \\ &= t_r(n) - (n - r - \lceil n/r \rceil) = t_r(n) - f_r(n). \end{aligned}$$

Suppose now that G has $t_r(n) - k$ edges and $s \ge 1$ copies of K_{r+1}. By iteratively deleting an edge that does not belong to every $(r+1)$-clique, we can delete fewer than s edges from G to obtain a graph G' with exactly one $(r+1)$-clique. By the preceding argument, $e(G') \le t_r(n) - f_r(n)$. Since $e(G) - e(G') \le s - 1$, we have $t_r(n) - k = e(G) \le t_r(n) - f_r(n) + s - 1$, or $s \ge f_r(n) - k + 1$.

5.2.25. *Bounds on* $\mathrm{ex}(n, K_{2,m})$.

a) If G is simple and $\sum_{v\in V} \binom{d(v)}{2} > (m-1)\binom{n}{2}$, then G contains $K_{2,m}$. If any pair of vertices has m common neighbors, then G contains $K_{2,m}$. Since there are $\binom{n}{2}$ pairs of vertices $\{x, y\}$, this means by the pigeonhole principle that a graph with no $K_{2,m}$ has at most $(m-1)\binom{n}{2}$ selections $(v, \{x, y\})$ such that v is a common neighbor of x and y. Counting such selections by v shows that there are exactly $\sum_{v\in V} \binom{d(v)}{2}$ of them, which completes the proof.

b) If G has e edges, then $\sum_{v\in V} \binom{d(v)}{2} \ge e(2e/n - 1)$. Because $\binom{x}{2}$ is a convex function of x, $\binom{x}{2} + \binom{y}{2} \ge 2\binom{(x+y)/2}{2}$. Hence $\sum_{v\in V} \binom{d(v)}{2}$ is minimized over fixed degree sum (number of edges) by setting all $d(v) = \sum d(v)/n = 2e/n$, in which case the sum is $e(2e/n - 1)$.

c) A graph with more than $\frac{1}{2}(m-1)^{1/2}n^{3/2} + \frac{n}{4}$ edges contains $K_{2,m}$. Since this edge bound implies $2e/n - 1 > (m-1)^{1/2}n^{1/2} - \frac{1}{2}$, we conclude

$$e\left(\frac{2e}{n} - 1\right) > \frac{1}{2}\left[(m-1)^{1/2}n^{3/2} + \frac{n}{2}\right]\left[(m-1)^{1/2}n^{1/2} - \frac{1}{2}\right]$$
$$= \frac{1}{2}(m-1)n^2 - \frac{n}{8} > (m-1)\binom{n}{2}.$$

By (b), this implies the hypothesis of (a) (if $m \ge 2$), and then (a) implies that G contains $K_{2,m}$.

d) Among n points in the plane, there are at most $\frac{1}{\sqrt{2}}n^{3/2} + \frac{n}{4}$ pairs with distance exactly one. Let $V(G)$ be the n points, with edges corresponding to the pairs at distance 1. If G has more than the specified number of edges, then (c) with $m = 3$ implies that G contains $K_{2,3}$. However, no two points in the plane have three points at distance exactly 1 from each of them.

5.2.26. *Every n-vertex graph G with more than $\frac{1}{2}n\sqrt{n-1}$ edges has girth at most 4.* The sum $\sum_{v \in V(G)} \binom{d(v)}{2}$ counts the triples u, v, w such that v is a common neighbor of u and w. If G has no 3-cycle and no 4-cycle, then we can bound the common neighbors of pairs u, w. If $u \leftrightarrow w$ in G, then they have no common neighbor. If $u \not\leftrightarrow w$ in G, then they have at most one common neighbor. Thus $\sum_{v \in V(G)} \binom{d(v)}{2} \le \binom{n}{2} - e(G)$.

Since the vertex degrees have the fixed sum $2e(G)$, we also have a lower bound on $\sum_{v \in V(G)} \binom{d(v)}{2}$ due to the convexity of $x(x-1)/2$. When $\sum d(v) = 2e(G)$, the sum $\sum \binom{d(v)}{2}$ is numerically minimized when $d(v) = 2e(G)/n$ for each v. Letting $m = e(G)$, we now have $n(m/n)(2m/n - 1) \le n(n-1)/2 - m$. Clearing fractions yields the quadratic inequality $2m(2m - n) \le n^2(n-1) - 2mn$, which simplifies to $m \le \frac{1}{2}n\sqrt{n-1}$.

5.2.27. *For $n \ge 6$, the maximum number of edges in a simple n-vertex graph not having two edge-disjoint cycles is $n+3$.* We argue first that $K_{3,3}$ does not have two edge-disjoint cycles. Deleting the edges of a 6-cycle leaves $3K_2$, and deleting the edges of a 4-cycle leaves a connected spanning subgraph, which must therefore use the remaining five edges. Thus every cycle other than the deleted one shares an edge with the deleted one.

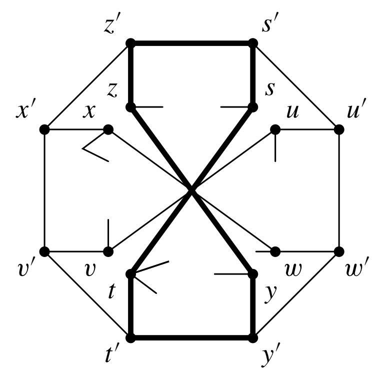
The number of edges in $K_{3,3}$ exceeds the number of vertices by 3. This is preserved by subdividing edges, and the property that every two cycles have a common edge is also preserved by subdividing edges. Hence every n-vertex subdivision of $K_{3,3}$ is a graph of the desired form with $n+3$ edges. This establishes the lower bound.

For the upper bound, consider an n-vertex graph G without two edge-disjoint cycles. We may assume that G is connected, since otherwise we can add an edge joining two components without adding any cycles. To prove that $e(G) \le n+3$, we may assume that G has a cycle C. Let H be a maximal unicyclic subgraph of G containing C. That is, we add edges from G to C without creating another cycle. Since G is connected, H is a spanning subgraph and has n edges. In addition to C, the rest of H forms a spanning forest H', with components rooted at the vertices of C.

Each edge of $G - E(H)$ joins two components of H', since otherwise it creates a cycle edge-disjoint from C using the path joining its endpoints in H'. Furthermore, for any two such edges xy and uv, with endpoints in components of H' having roots x', y', u', v' on C, it must be that C does not contain an x', y'-path and a u', v'-path that are disjoint, since these would combine with xy, uv, and the x, x'-, y, y'-, u, u'-, and v, v'-paths in H' to form edge-disjoint cycles. Therefore, the x', y'- and u', v'-paths on C must alternate endpoints (or share one endpoint).

Suppose that there are four such extra edges, say $\{st, uv, wx, yz\}$, such that the corresponding roots on C in order are $s', u', w', y', t', v', x', z'$ (consecutive vertices in the list may be identical, and any a may equal a'). Suppose that these vertices in order split into distinct pairs, such as $s' \ne u'$, $w' \ne y'$, $t' \ne v'$, and $x' \ne z'$. We now build edge-disjoint cycles by taking the cycle through $s', s, t, t', y', y, z, z'$ indicated in bold below and the analogous cycle through $u', u, v, v', x', x, w, w'$ (note that $t' = y'$, etc., is possible). If for example $s' = u'$, so that these two cycles are not edge-disjoint, then edge-disjoint cycles can be extracted in other ways (we omit the details).

We conclude that only three additional edges are possible, which limits $e(G)$ to $n+3$.



5.2.28. *For $n \ge 6$, the maximum number of edges in a simple n-vertex graph G not having two disjoint cycles is $3n-6$.* To construct such a graph, form a triangle on a set S of three vertices, and let S be the neighborhood of each remaining vertex. Each cycle uses at least two vertices from S, so there cannot be two disjoint cycles. The graph has $3 + 3(n-3) = 3n-6$ edges.

For the upper bound, we use induction on n. Basis step ($n = 6$): G has at most two missing edges. We find one triangle incident to all the missing edges, and then the remaining three vertices also form a triangle.

Induction step ($n > 6$): If G has a vertex v of degree at most 3, then the induction hypothesis applied to $G - v$ yields the claim. Thus we may assume that $\delta(G) \ge 4$. Since $e(G) \ge n$, there is a cycle in G. Let C be a shortest cycle in G, and let $H = G - V(C)$. We may assume that H is a forest, since otherwise we have a cycle disjoint from C.

Since $\delta(G) \ge 4$, every leaf or isolated vertex in H has at least three neighbors on C. This yields a shorter cycle than C unless C is a triangle. Hence we may assume that C is a triangle, and now $\delta(G) \ge 4$ implies that H has no isolated vertices.

Since every leaf of H is adjacent to all of $V(C)$, two leafs in a single component of H plus one additional leaf yield two disjoint cycles. Hence we may assume that H is a single path. Thus every internal vertex of H has at least two neighbors in C, and there is at least one such vertex since $n > 6$. We now have two disjoint triangles: the first two vertices of the path plus one vertex of C, and the last vertex of the path plus the other two vertices of C.

5.2.29. Let G be a claw-free graph (no induced $K_{1,3}$).

a) The subgraph induced by the union of any two color classes in a proper coloring of G consists of paths and even cycles. Let H be such a subgraph. Since H is 2-colorable, it is triangle-free. Hence a vertex of degree 3 in H is the center of a claw. Since G is claw-free, every induced subgraph of G is claw-free. Hence $\Delta(H) \le 2$. Every component of a graph with maximum degree at most 2 is a path or a cycle. Since H is 2-colorable, the cycle components have even order.

b) If G has a proper coloring using exactly k colors, then G has a proper k-coloring where the color classes differ in size by at most one. Consider a proper k-coloring of G. If some two color classes differ in size by more than 1, then we alter the coloring to reduce the size of a largest color class A and increase the size of a smallest color class B. Consider the subgraph H induced by $A \cup B$. By part (a), the components of H are paths and even cycles. The even cycles have the same number of vertices from A and B. Since $|A| \ge |B| + 2$, there must be a component of H that is a path P with one more vertex from A than from B. Switching the colors on P brings the two color classes closer together in size. Iterating this procedure leads to all pairs of classes differing in size by at most 1.

5.2.30. *If G has a proper coloring in which each color class has at least two vertices, then G has a $\chi(G)$-coloring in which each color class has at least two vertices.* (Note that C_5 doesn't have either type of coloring.)

Proof 1 (induction on $\chi(G)$; S. Rajagopalan). The statement is immediate if $\chi(G) = 1$. If $\chi(G) > 1$, let f be an optimal coloring of G, and let $g\colon V(G) \to \mathbb{N}$ be a coloring in which each class has at least two vertices. If f has a singleton color set $\{x\}$, let $S = \{v \in V(G)\colon g(v) = g(x)\}$, and let $G' = G - S$. Since f restricts to a $(\chi(G)-1)$-coloring of G' (because x is omitted) and g restricts to a coloring of G' in which every color is used at least twice (because only vertices with a single color under G were omitted), the induction hypothesis implies that G' has a $(\chi(G)-1)$-coloring in which every color is used at least twice. Replacing S as a single color class yields such a coloring for G.

Proof 2 (algorithmic version). Define f and g as above. if x is a singleton color in the current $\chi(G)$-coloring f, change all vertices in $\{v\colon g(v) = g(x)\}$ to color $f(x)$. The new coloring is proper, since $f(x)$ appeared only on x and since the set of vertices with color $g(x)$ in g is independent. No new colors are introduced, so the new coloring is optimal. Vertices that have been recolored are never recolored again, so the procedure terminates after at most $\chi(G)$ steps. It can only terminate with an optimal coloring in which each color is used at least twice.

5.2.31. *If G is a connected graph that is not a complete graph or a cycle whose length is an odd multiple of 3, then in every minimum proper coloring of G there are two vertices of the same color with a common neighbor.* For odd cycles, if every two vertices having the same color are at least three apart, then the coloring must be $1, 2, 3, 1, 2, 3, \cdots$, cyclically, so the length is an odd multiple of 3. For other graphs, Brooks' Theorem yields $\chi(G) \le \Delta(G)$. Since only $\Delta(G) - 1$ colors are available for the neighborhood of a vertex of maximum degree, the pigeonhole principle implies that a vertex of maximum degree has two neighbors of the same color in any optimal coloring.

5.2.32. *The Hajós construction.* Applied to graphs G and H sharing only vertex v, with $vu \in E(G)$ and $vw \in E(H)$, the Hajós construction produces the graph $F = (G - vu) \cup (H - vw) \cup uw$.

a) If G and H are k-critical, then F is k-critical. A proper $(k-1)$-coloring of F contains proper $(k-1)$-colorings of $G - vu$ and $H - vw$. Since G and H are k-critical, every $(k-1)$-coloring of F gives the same color to v and u and gives the same color to v and w. Since this gives the same color to u and w, there is no such coloring of F.

Thus $\chi(F) \ge k$, and equality holds because we can combine proper $(k-1)$-colorings of $G - vu$ and $H - vw$ and change w to a new color.

Finally, for $e \in E(F)$ we show that $F - e$ is $(k-1)$-colorable. For $F - uw$, the coloring described above is proper. Let xy be another edge of F; by symmetry, we may assume that $xy \in E(G)$. Since G is k-critical, we have a proper $(k-1)$-coloring f of $G - xy$. Since uv is an edge in $G - xy$, this coloring gives distinct colors to u and v. In a proper $(k-1)$-coloring of $H - vw$ that gives v and w the same color, we can permute labels so this color is $f(v)$. Combining these colorings now yields a proper $(k-1)$-coloring of $F - xy$.

b) For $k \geq 3$, a k-critical graph other than K_k. Apply the Hajós construction to the graph consisting of two edge-disjoint k-cliques sharing one vertex v. This deletes one edge incident to v from each block and then adds an edge joining the two other vertices that lost an incident edge. The resulting graph is $(k-1)$-regular except that v has degree $2k-4$.

c) Construction of 4-critical graphs with n vertices for all $n \geq 6$. Since the join of color-critical graphs is color-critical, we can use $C_{2k+1} \vee K1$, which yields 4-critical graphs for all even n. In particular, this works for $n \in \{4, 6, 8\}$, which has a member of each congruence class modulo 3.

If we apply the Hajós construction to a 4-critical graph G with $2l$ vertices and the 4-critical graph $H = K_4$, we obtain a 4-critical graph F with $2l+3$ vertices. Thus we obtain a 4-critical n-vertex graph whenever n exceeds one of $\{4, 6, 8\}$ by a multiple of 3. This yields all $n \geq 4$ except $n = 5$.

5.2.33. *a) If a k-critical graph G has a 2-cut $S = \{x, y\}$, then 1) $x \nleftrightarrow y$, 2) G has exactly two S-lobes, and 3) we may index them as G_1 and G_2 such that $G_1 + xy$ and $G_2 \cdot xy$ are k-critical.* Since no vertex cut of a k-critical graph induces a clique, we have $x \nleftrightarrow y$. By k-criticality, every S-lobe of G is $(k-1)$-colorable. If each S-lobe has a proper $(k-1)$-coloring where x, y have the same color, then colors can be permuted within S-lobes so they agree on $\{x, y\}$, so G is $(k-1)$-colorable.

The same can be done if each S-lobe has a proper $(k-1)$-coloring where x, y have different colors. Hence there must be an S-lobe G_1 such that u, v receive the same color in every proper $(k-1)$-coloring and an S-lobe G_2 such that u, v receive the different colors in every proper $(k-1)$-coloring. Deletion of any other S-lobe would therefore leave a graph that is not $(k-1)$-colorable, so criticality implies that there is no other S-lobe.

Since every proper $(k-1)$-coloring of G_1 gives x and y the same color, $G_1 + xy$ is not $(k-1)$-colorable. Since every proper $(k-1)$-coloring of G_2 gives x and y different colors, $G_2 \cdot xy$ is not $(k-1)$-colorable. To see that $G_1 + xy$ is k-critical, let $G' = G_1 + xy$ and consider edge deletions. First $G' - xy = G_1$, which is $(k-1)$-colorable. For any other edge e of G', $G - e$ has a proper $(k-1)$-coloring that contains a proper $(k-1)$-coloring of G_2, hence it gives distinct colors to x and y. Therefore the colors it uses on the vertices of G_1 form a proper $(k-1)$-coloring of $G' - e$. The analogous argument holds for $G_2 \cdot xy$.

b) Every 4-chromatic graph contains a K_4-subdivision. Part (a) can be used to shorten the proof of this. We use induction on $n(G)$, with the basis $n(G) = 4$ and K_4 itself. Given $n(G) > 4$, let G' be a 4-critical subgraph of G. We know G' has no cutvertex. If G' is not 3-connected, then we have a 2-cut $S\{x, y\}$. Part (a) guarantees an S-lobe G_1 such that $G_1 + xy$ is 4-critical. By the induction hypothesis, $G_1 + uv$ contains a subdivision of K_4; if this subdivision uses the edge uv, then this edge can be replaced by a path through G_2 to obtain a subdivision of K_4 in G. If G' is 3-connected, the proof is as in the text.

5.2.34. *In a 4-critical graph G with a separating set S of size 4, $e(G[S]) \leq 4$.* If $e(G[S]) = 6$, then S is a 4-clique, and G is not 4-critical. If $e(G[S]) = 5$, then $G[S]$ is a kite. Every proper 3-coloring of the S-lobes of G assigns one color to the vertices of degree 2 in the kite and two other colors to the vertices of degree 3 in the kite. Hence the names of colors in the proper 3-colorings of the S-lobes can be permuted so that the coloring agree on S. This yields a proper 3-coloring of G. The contradiction implies that $G[S]$ cannot have five edges.

5.2.35. *Alternative proof that k-critical graphs are $(k-1)$-edge-connected.*

a) If G is k-critical, with $k \geq 3$, then for any $e, f \in E(G)$ there is a $(k-1)$-critical subgraph of G containing e but not f. Any $(k-1)$-coloring ϕ of $G - e$ assigns the same color to both endpoints of e. The endpoints of f get distinct colors under ϕ; by renumbering colors, we may assume one of them gets color $k-1$. Let $S = \{v : \phi(v) = k-1\}$; note that $G - e - S$ is $(k-2)$-colored by ϕ. However, $G - S$ is $(k-1)$-chromatic, since S is an independent set, so any $(k-1)$-critical subgraph of $G - S$ must contain e and be the desired graph. (Toft [1974])

b) If G is k-critical, with $k \geq 3$, then G is $(k-1)$-edge-connected. Since the 3-critical graphs are the odd cycles, this is true for $k = 3$, and we proceed by induction. For $k > 3$, consider an edge cut with edge set F. If $|F| = 1$, we permute colors in one component of $G - F$ to obtain a $(k-1)$-coloring of G from a $(k-1)$-coloring of $G - F$, so we may assume $|F| \geq 2$. Choose $e, f \in F$. By part (a), there is a $(k-1)$-critical subgraph G' containing e but not f. Deleting $F - f$ from G' separates it, since it separates the endpoints of e. By the induction hypothesis, $|F - f| \geq k-2$, and thus $|F| \geq k-1$.

5.2.36. *If G is k-critical and every $(k-1)$-critical subgraph of G is isomorphic to K_{k-1}, then $G = K_k$ (if $k \geq 4$).* Since K_k is k-critical, a k-critical graph cannot properly contain K_k, so if we can find K_k in G, then $G = K_k$. Let G have the specified properties; since $k \geq 4$, G has a triangle x, y, z. Toft's critical graph lemma says that for any edges e, f, G contains a $(k-1)$-critical subgraph that contains e and avoids f.

Let G_1 be such a graph that contains xy but omits yz. Since every $(k-1)$-critical subgraph is a clique, by hypothesis, G_1 cannot contain z at all. Similarly, let G_2 be a $(k-1)$-critical graph that contains yz but omits x. Both G_1 and G_2 are $(k-1)$-cliques, so a proper $(k-2)$-coloring of $G_1 - xy$ must give x and y the same color, and a proper $(k-2)$-coloring of $G_2 - yz$ must give y and z the same color. This means that the graph $H = (G_1 - xy) \cup (G_2 - yz) \cup xz$ is not $(k-2)$-colorable, so it contains some $(k-1)$-critical subgraph H', which by hypothesis is a $(k-1)$-clique.

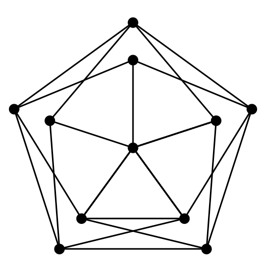
Furthermore, the set of vertices common to G_1 and G_2 induce a clique, which means that the $(k-2)$-colorings of G_1 and G_2 can be made to agree on their intersection. This means that $H - xz$ is $(k-2)$-colorable, which implies that $xz \in H'$. By construction, $N_H(x) = V(G_1) - y$ and $N_H(z) = V(G_2) - y$. Since H' is a clique containing x, z, this forces G_1, G_2 to have $k-3$ common vertices other than y. We add x, y, z to these to obtain a k-clique in G, which as noted earlier implies that $G = K_k$.

5.2.37. *Vertex-color-critical graphs.*

a) Every color-critical graph is vertex-color-critical. Every proper subgraph of a color-critical graph has smaller chromatic number, including those obtained by deleting a vertex, which is all that is needed for vertex-color-critical graphs.

b) Every 3-chromatic vertex-color-critical graph G is color-critical. Since it needs 3 colors, G is not bipartite, but $G - v$ is bipartite for every $v \in V(G)$. Hence every vertex of G belongs to every odd cycle of G; let C be a spanning cycle of G. If G has any edge e not on C, then e creates a shorter odd cycle with a portion of C, leaving out some vertices. Since G is vertex-color-critical, this cannot happen, and G is precisely an odd cycle.

c) the graph below is vertex-color-critical but not color-critical. This graph G is obtained from the Grötzsch graph by adding an edge, so $\chi(G) \ge 4$. An explicit coloring shows that $\chi(G) = 4$. Hence G is not color-critical. Explicit 3-colorings of the graphs obtained by deleting one vertex show that G is vertex-color-critical.



5.2.38. *Every nontrivial simple graph with at most one vertex of degree less than 3 contains a K_4-subdivision.* Call a vertex with degree less than 3 a *deficient* vertex. By considering the larger class of graphs that may have one deficient vertex, we obtain a stronger result than $\delta(G) \ge 3$ forcing a K_4-subdivision, but one that is easier to prove inductively.

We use induction on $n(G)$; the only graph with at most four vertices that satisfies the hypothesis is K_4 itself. For the induction step, we seek a graph G' having at most one deficient vertex and having $n(G') < n(G)$. If G contains G' or a G'-subdivision, we obtain a K_4-subdivision in G, because the K_4-subdivision in G' guaranteed by the induction hypthesis is a subgraph of G or yields a subgraph of G by subdividing additional edges.

If G is disconnected, let G' be a component of G. If G has a cut-vertex x, then some $\{x\}$-lobe of G has at most one deficient vertex; let this be G'. Hence we may assume G is 2-connected. If G is 3-connected, then as in the proof of Theorem 5.2.20 we find a cycle C in $G - x$ and an $x, V(C)$-fan in G to complete a subdivision of K_4.

Hence we may assume that $\kappa(G) = 2$, with S a separating 2-set. Only one S-lobe of G can have a vertex outside S that is deficient in G. Let H be an S-lobe of G containing no vertex outside S that is deficient in G.

Note that x and y each have degree at least 1 in H, since $\kappa(G) = 2$, and in fact they must have distinct neighbors in $V(H) - S$. If x or y has degree at least 3 in H, then let $G' = H$.

If $d_H(x) = d_H(y) = 1$, then x and y cannot be adjacent. Merge x and y to form G' from H; degrees of other vertices don't change, since x and y have no common neighbors in H. Also G contains a subdivision of G' by undoing the merging and adding an x, y-path through another S-lobe.

Hence we may assume that $d_H(x) = 2$. If $xy \notin E(G)$, then add the edge xy to H to form G'; only y can now be deficient. Also G contains a subdivision of G' by replacing the added edge with an x, y-path through another S-lobe. If $xy \in E(G)$, then y has a neighbor in H other that the neighbor of x (and it is the only neighbor of y other than x). Now we contract xy to obtain G', with the new vertex having degree 2. Now H is a subdivision of G' that is a subgraph of G.

5.2.39. *For $n \ge 3$, the maximum number of edges in a simple n-vertex graph G having no K_4-subdivision is $2n - 3$.* If G has at least $2n - 2$ edges, then $n \ge 4$; we prove by induction on n that G has a K_4-subdivision. For $n = 4$, G has (at least) 6 edges and must be K_4. For $n > 4$, if $\delta(G) \ge 3$, then Dirac's Theorem guarantees that G has a K_4-subdivision.

When $\delta(G) < 3$, let x be vertex of minimum degree. The graph $G - x$ has at least $2(n-1) - 2$ edges; by the induction hypothesis, $G - x$ has a K_4-subdivision, and this subgraph appears also in G.

To show this is the best bound, we observe that $K_2 \vee (n-2)K_1$ has $2n-3$ edges but no K_4-subdivision. It cannot have a K_4-subdivision because it has only two vertices with degree at least 3. Another example is $K_1 \vee P_{n-1}$, but it requires induction to show that this example with $2n-3$ edges has no K_4-subdivision.

5.2.40. *For $G_7 = C_5[K_3, K_2, K_3, K_2, K_3]$ and $G_8 = C_5[K_3, K_3, K_3, K_3, K_3]$, the graph G_k is k-chromatic but contains no K_k-subdivision.* In these constructions, the vertices substituted for two successive vertices of C_5 (call these *groups*) induce a clique. For G_7, we use colors 123, 45, 671, 23, 456 in the successive cliques. For G_8, we use 123, 456, 781, 234, 567.

In these graphs, one cannot take two vertices from the same group or adjacent groups in an independent set. Thus each graph has independence number 2. Thus $\chi(G_7) \ge n(G_7)/2 = 6.5$ and $\chi(G_8) \ge n(G_8)/2 = 7.5$. Since $\chi(G)$ is always an integer, we have $\chi(G_k) \ge k$.

If G_k has a K_k-subdivision H, then H must have two vertices u, v of degree $k-1$ in nonadjacent groups, since adjacent groups together have size at most $k-2$. Since there must be $k-1$ pairwise internally disjoint u, v-paths in H, this is impossible when G_k has a u, v-separating set of size $k-2$. In all cases except one, G_k has such a u, v-separating set consisting of two groups. The exception is u, v chosen from the groups of size 2 in G_7.

In this exceptional case, we have forbidden the high-degree vertices of H from the consecutive groups of size 3, since that would yield the case already discussed. Thus the seven high-degree vertices must consist of the two groups of size 2 and the triangle between them. Now the four needed paths connecting the two groups of size 2 must use the two consecutive groups of size 3, but only three paths can do this.

5.2.41. *If $m = k(k+1)/2$, then $K_{m,m-1}$ contains no subdivision of K_{2k}.* In $K_{m,m}$ there is such a subdivision: place k branch vertices in each partite set, and then there remain $\binom{k}{2}$ unused vertices in each partite set to subdivide edges joining the branch vertices in the other partite set. We prove that if an X, Y-bigraph G contains a subdivision of K_{2k}, then $n(G) \ge 2m$.

Proof 1 (counting argument). The paths representing edges of K_{2k} are pairwise internally-disjoint. When a partite set has a "branch vertices" (degree more than two in the subdivision), the other partite set has at least $\binom{a}{2}$ vertices that are not branch vertices. If the subdivision of K_{2k} has a branch vertices in X, we thus need at least $\binom{a}{2} + 2k - a + \binom{2k-a}{2} + a$ vertices. Using the identity $\binom{a}{2} + a(n-a) + \binom{n-a}{2} = \binom{n}{2}$, the formula becomes $\binom{2k}{2} + 2k - a(2k-a)$. Since $a(2k-a) \le k^2$, the number of required vertices is at least $\binom{2k+1}{2} - k^2$. This quantity is $k(k+1) = 2m$.

Proof 2 (extremal bipartite subgraphs). In a subdivision of K_{2k} within a graph G, there are $2k$ branch vertices. The maximum number of edges in a bipartite graph with $2k$ vertices is k^2. Hence if more than k^2 edges joining branch vertices are left unsubdivided, then the subgraph of G induced by these vertices will not be bipartite. Since we require the host graph ($G = K_{m,m-1}$) to be bipartite, at least $\binom{2k}{2} - k^2$ edges must be subdivided. This requires $k^2 - k$ additional vertices. Together with the branch vertices, a bipartite graph containing a subdivision of K_{2k} must have at least $k^2 + k$ vertices. (Comment: The uniqueness of the $2k$-vertex bipartite graph with k^2 edges leads to the uniqueness of $K_{m,m}$ as a graph with $k^2 + k$ vertices having a subdivision of K_{2k}.)

5.2.42. *If F is a forest with m edges, and G is a simple graph such that $\delta(G) \ge m$ and $n(G) \ge n(F)$, then $F \subseteq G$.* We may assume that F has no isolated vertices, since those could be added at the end.

Let F' be a subgraph of F obtained by deleting one leaf from each nontrivial component of F. Let R be the set of neighbors of the deleted vertices. Map R onto an m-set $X \subseteq V(G)$ that minimizes $e(G[X])$. Since $\delta(G) \ge m$ and $n(F') = m$, we can extend X to a copy of F' in G (each vertex has at least m neighbors, but fewer than m of its neighbors are used already in F' when we need to add a neighbor to it).

To extend this copy of F' to become a copy of F, we show that G contains a matching from X into the set Y of vertices not in this copy of F'. Let H be the maximal bipartite subgraph of G with bipartition X, Y. By Hall's Theorem, the desired matching exists unless there is a set $S \subseteq X$ such that $|N_H(S)| < |S|$. Consider $t \in S$ and $u \in Y - N_H(S)$. Outside S, t has at most $(n(F') - |X|) + |N_H(S)|$ neighbors in G. Since $\delta(G) \ge m$, we have $|N_G(t) \cap S| \ge |X| - |N_H(S)|$. On the other hand, since $u \notin N_H(S)$, we have $|N_G(u) \cap X| \le |X| - |S|$. Since $|N_H(S)| < |S|$, replacing t with u in X reduces the size of the subgraph induced by X. This contradicts the choice of X, and hence Hall's condition holds.

5.2.43. *Every proper k-coloring of a k-chromatic graph contains each labeled k-vertex tree as a labeled subgraph.* We use induction on k, with trivial basis $k = 1$. For $k > 1$, let f be the coloring, and let $C_i = \{v \in V(G) \colon f(v) = i\}$. Suppose that x is a leaf of T with neighbor y and that we seek label p for x and q for y. Let $S \subset C_q$ be the vertices in C_q adjacent to no vertex of C_p. We have $S \ne C_q$, else we can combine color classes in f.

The vertices of S cannot be used in the desired embedding of T, so we will discard them. Let $G' = G - (S \cup C_p)$. We have $\chi(G') \le k-1$ because we have discarded all vertices of color p in f, and we have $\chi(G') \ge k-1$ because $S \cup C_p$ is an independent set in G. By the induction hypothesis, G' has $T - x$ as a labeled subgraph H, and the image of y in H belongs to C_q. We have retained in $C_q - S$ only vertices having a neighbor with color p in

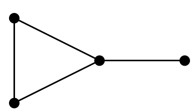
f (by part (a), this set is non-empty). Hence G has a vertex in C_p that we can use as the image of x to obtain T as a labeled subgraph.

5.2.44. *Every k-chromatic graph with girth at least 5 contains every k-vertex tree as an induced subgraph.* If $\chi(G) = k$ and $d(x) < k-1$ for some $x \in V(G)$, then $\chi(G-x) = k$, so it suffices to prove the claim for graphs in which the minimum degree is at least $k-1$. In fact, with this condition, we do not need the condition on the chromatic number. For $k \le 2$, the result is obvious; we proceed by induction for $k > 2$.

Suppose T is a k-vertex tree, x is a leaf of T with neighbor y, and $T' = T - x$. By the induction hypothesis, G has T as an induced subgraph $f(T')$; let $u = f(y)$. It suffices to show that $S = N(u) - f(T')$ contains a vertex adjacent to no vertex of $f(T')$ except u. Each vertex in $f(N(y))$ has no neighbor in S, because G has no triangles. Each vertex in $f(T - N[y])$ has at most one neighbor in S, else it would complete a 4-cycle in G with two such vertices and u. Hence S has at most $n(T') - 1 - d(y)$ unavailable vertices. Since $|S| \ge k - 1 - d(y)$, there remains an available vertex in S to assign to x.

5.3. ENUMERATIVE ASPECTS

5.3.1. *The chromatic polynomial of the graph below is $k(k-1)^2(k-2)$.* The graph is chordal, and the polynomial follows immediately from a simplicial elimination ordering. It can also be obtained from the recurrence, from the inclusion-exclusion formula, etc.



5.3.2. *The chromatic polynomial of an n-vertex tree is $k(k-1)^{n-1}$, by the chromatic recurrence.* We use induction on n. For $n = 1$, the polynomial is k, as desired. Contracting an edge of an n-vertex tree leaves a tree with $n-1$ vertices. Deleting the edge leaves a forest of two trees, with orders m and $n-m$ for some m between 1 and $n-1$. The polynomial for a disconnected graph is the porduct of the polynomials for the components. We use the induction hypothesis and the chromatic recurrence and extract the factors k and $(k-1)^{n-2}$ to obtain the polynomial

$$k(k-1)^{m-1}k(k-1)^{n-m-1} - k(k-1)^{n-2} = k(k-1)^{n-2}(k-1) = k(k-1)^{n-1}.$$

5.3.3. *$k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial.* In $\chi(G;k)$, the degree is $n(G)$, and the second coefficient is $-e(G)$. Hence we need a 4-vertex graph with four edges. The only such graphs are C_4 and the paw, which have chromatic polynomials $k(k-1)(k^2-3k+3)$ and $k(k-1)(k-2)(k-1)$, each with nonzero linear term. (Note: The linear term of the chromatic polynomial of a connected graph is nonzero; see Exercise 5.3.12.)

Alternatively, observe that the value at 2 is negative, so it cannot count the proper 2-colorings in any graph.

5.3.4. *a) The chromatic polynomial of the n-cycle is $(k-1)^n + (-1)^n(k-1)$.* **Proof 1** (induction on n). The chromatic polynomial of the loop (C_1) is 0, which the formula reduces to when $n = 1$. Those considering only simple graphs can start with $\chi(C_3;k) = k(k-1)(k-2) = (k-1)^3 - (k-1)$. For larger n, the chromatic recurrence yields $\chi(C_n;k) = \chi(P_n;k) - \chi(C_{n-1};k)$. By the induction hypothesis and the formula for trees, this equals $k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) = (k-1)^n + (-1)^n(k-1)$.

Proof 2 (Whitney's formula). We use $\chi(G;k) = \sum_{S\subseteq E(G)}(-1)^{|S|}k^{c(G(S))}$. For every set S of size j, the number of components of $G(S)$ is $n-j$, except that for $S = E(G)$ the number of components is 1, not 0. Since there are $\binom{n}{j}$ sets with j edges, we obtain $\chi(C_n;k) = (\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}k^{n-j}) + (-1)^n k$. By the binomial theorem, $(k-1)^n = (\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}k^{n-j}) + (-1)^n$. Thus we obtain $\chi(C_n;k)$ from $(k-1)^n$ by adding $(-1)^n k$ and subtracting $(-1)^n$.

b) If $H = G \vee K_1$, then $\chi(H;k) = k\chi(G;k-1)$. Let x be the vertex added to G to obtain H. In every proper coloring, the color used on x is forbidden from the rest of H. Each of the k ways to color x combines with each of the $\chi(G;k-1)$ ways to properly color the rest of H to form a proper coloring of H. Hence $\chi(H;k) = k\chi(G;k-1)$;in particular, $\chi(C_n \vee K_1;k) = k(k-2)^n + (-1)^n k(k-1)$.

5.3.5. *If $G_n = K_2 \square P_n$, then $\chi(G_n;k) = (k^2-3k+3)^{n-1}k(k-1)$.*

Proof 1 (induction on n). Since G_1 is a 2-vertex tree, $\chi(G_1;k) = k(k-1)$. For $n > 1$, let u_n, v_n be the two rightmost vertices of G_n. The proper colorings of G_n are obtained from proper colorings of G_{n-1} by assigning colors also to u_n and v_n. Each proper coloring f of G_{n-1} satisfies $f(u_{n-1}) \ne f(v_{n-1})$. Thus each such f extends to the same number of colorings of G_n.

There are $(k-1)^2$ ways to specify $f(u_n)$ and $f(v_n)$ so that $f(u_n) \ne f(u_{n-1})$ and $f(v_n) \ne f(v_{n-1})$. Of these extensions, $k-2$ give u_n and v_n the same color, and we delete them. Since $(k-1)^2 - (k-2) = k^2 - 3k + 3$, the induction hypothesis yields

$$\chi(G_n;k) = (k^2-3k+3)\chi(G_{n-1};k) = (k^2-3k+3)^{n-1}k(k-1).$$

Proof 2 (induction plus chromatic recurrence). Again $\chi(G_1;k)=k(k-1)$. Let $e=u_nv_n$. For $n>1$, observe that $\chi(G_n-e;k)=\chi(G_{n-1};k)(k-1)^2$ and $\chi(G_n\cdot e;k)=\chi(G_{n-1};k)(k-2)$, by counting the ways to extend each coloring of G_{n-1} to the last column. Thus

$$\begin{aligned}\chi(G_n;k)&=\chi(G_n-u_nv_n;k)-\chi(G_n\cdot u_nv_n;k)\\&=\chi(G_{n-1};k)[(k-1)^2-(k-2)]=(k^2-3k+3)^{n-1}k(k-2).\end{aligned}$$

5.3.6. *Non-inductive proof that the coefficient of $k^{n(G)-1}$ in $\chi(G;k)$ is $-e(G)$.* Let n be the number of vertices in G. By Proposition 5.3.4, $\chi(G;k)=\sum_{r=1}^n p_r k_{(r)}$, where p_r is the number of partitions of G into exactly r nonempty independent sets. Since $k_{(r)}$ is a polynomial in k of degree r, contributiong to the coefficient of k^{n-1} in $\chi(G;k)$ can arise only from the terms for $r=n$ and $r=n-1$.

The only partition of $V(G)$ into n independent sets is the one with each vertex in a set by itself, so $p_n=1$. When partitioning $V(G)$ into $n-1$ independent sets, there must be one set of size 2 and $n-2$ sets of size 1. Thus each such partition is determined by choosing two nonadjacent vertices. There are $\binom{n}{2}-e(G)$ such pairs (G is simple), so $p_{n-1}=\binom{n}{2}-e(G)$.

The term involving k^{n-1} in $k_{(n-1)}$ arises only by choosing the term k from each factor when expanding the product. Thus the coefficient of k^{n-1} in $k_{(n-1)}$ is 1. Contributions to the coefficient of k^{n-1} in $k_{(n)}$ arise by choosing the term k from $n-1$ factors and the constant from the remaining term. Thus the contributions are $-1,-2,\ldots,-(n-1)$, and the coefficient is $-\sum_{i=0}^{n-1} i$, which equals $-\binom{n}{2}$.

Combining these computations yields the coefficient of k^{n-1} in $\chi(G;k)$ as $1\cdot[-\binom{n}{2}]+[\binom{n}{2}-e(G)]\cdot 1$.

5.3.7. *Roots of chromatic polynomials.*

a) The chromatic polynomial $\chi(G;k)$ of an arbitrary graph G is a nonnegative linear combination of chromatic polynomials of cliques with at most $n(G)$ vertices. This holds trivially when G itself is a clique, which is the situation where $e(\overline{G})=0$. This is the basis step for a proof by induction on $e(\overline{G})$. For $e(\overline{G})>0$, let G' be the graph obtained by adding the edge $e=uv$ and contracting it; we have $\chi(G;k)=\chi(G+uv;k)+\chi(G';k)$ by the chromatic recurrence. To apply the induction hypothesis, note that $e(\overline{G+uv})=e(\overline{G})-1$ and $e(\overline{G'})=e(\overline{G})-1-|\overline{N}(u)\cap\overline{N}(v)|$, where $e=uv$. Hence we can express $\chi(G';k)$ and $\chi(G'\cdot e;k)$ as nonnegative linear combinations of the polynomials $\chi(K_j;k)$ for $j\le n$.

b) The chromatic polynomial of a graph on n vertices has no real root larger than $n-1$. The combinatorial definition of the chromatic polynomial as the function of k that counts the proper colorings of G using at most k colors guarantees that the value cannot be 0 for $k\ge n$, because we can arbitrarily assign the vertices distinct colors to obtain at least $k(k-1)\cdots(k-n+1)>0$ proper colorings. However, this argument applies only to integers. To forbid all real roots exceeding n, we use part (a). Observe that $\chi(K_j;x)$ is the product of positive real numbers whenever $x>j-1$; hence these polynomials have no real roots larger than $j-1$. Since any chromatic polynomial is a nonnegative linear combination of these for $j\le n$, its value at any $x>n-1$ is the sum of at most n positive numbers and therefore is also positive.

5.3.8. *The number of proper k-colorings of a connected graph G is less than $k(k-1)^{n-1}$ if $k\ge 3$ and G is not a tree.* If G is connected but not a tree, let T be a spanning tree contained in G, and choose $e\in E(G)-E(T)$. Every proper coloring of G must be a proper coloring of the subgraph T, and there are exactly $k(k-1)^{n-1}$ proper k-colorings of T. It suffices to show that at least one of these is not a proper k-coloring of G. Since T is bipartite and $k\ge 3$, we can construct such a coloring by using a 2-coloring of T and then changing the endpoints of e to a third color. This is still a proper k-coloring of T, but it is not a proper k-coloring of G.

If $k=2$, then T has exactly two proper k-colorings, and these are both proper colorings of G if G is bipartite. Thus the statement fails when $k=2$ if G is bipartite (if G is not bipartite, then it still holds when $k=2$).

5.3.9. $\chi(G;x+y)=\sum_{U\subseteq V(G)}\chi(G[U];x)\chi(G[\overline{U}];y)$. Polynomials of degree n that agree at $n+1$ points are equal everywhere. Hence it suffices to prove the claim when x and y are nonnegative integers. We show that then each side counts the proper $(x+y)$-colorings of G.

In each proper $(x+y)$-coloring, the first x colors are used on some subset $U\subseteq V(G)$, and $\overline{U}$ receives colors among the remaining y colors. Since there is no interference between the colors, we can put an arbitrary x-coloring on $G[U]$ and an arbitrary y-coloring on $G[\overline{U}]$ and form such a coloring in $\sum_{U\subseteq V(G)}\chi(G[U];x)\chi(G[\overline{U}];y)$ ways. Furthermore, the set U that receives colors among the first x colors is uniquely determined by the coloring. Hence summing over U counts each coloring exactly once. The left side by definition is the total number of colorings.

5.3.10. *If G is a connected n-vertex graph with $\chi(G;k)=\sum_{i=0}^{n-1}(-1)^i a_{n-i}k^{n-i}$, then $a_i\ge\binom{n-1}{i-1}$ for $1\le i\le n$.* In order to prove this inductively using the chromatic recurrence, we must guarantee that the graphs in the recurrence are connected and appear “earlier”. We use induction on n, and to prove the induction step we use induction on $e(G)-n+1$.

The statement holds for the only 1-vertex graph, so consider $n>1$. If $e(G)=n-1$ and G is connected, then G is a tree and has chromatic polynomial $k(k-1)^{n-1}$. The term involving k^i is $k\binom{n-1}{i-1}k^{i-1}(-1)^{n-i}$, so the magnitude of the coefficient is $\binom{n-1}{i-1}$, as desired.

Now consider $e(G) > n-1$. If G is connected and has more than $n-1$ edges, then G has a cycle, and deleting any edge of the cycle leaves a connected graph. Let e be such an edge, and define $\{b_i\}$, $\{c_i\}$ in the chromatic polynomials by $\chi(G-e;k) = \sum_{i=0}^{n-1}(-1)^i b_{n-i}k^{n-i}$ and $\chi(G\cdot e;k) = \sum_{i=0}^{n-1}(-1)^i c_{n-1-i}k^{n-1-i}$.

The recurrence $\chi(G;k) = \chi(G-e;k) - \chi(G\cdot e;k)$ implies that $(-1)^{n-i}a_i$ is the sum of the coefficients of k^i in the other two polynomials. Since $G-e$ and $G\cdot e$ are connected, the induction hypothesis implies $a_i = b_i - (-1)c_i = b_i + c_i \ge \binom{n-1}{i-1} + \binom{n-2}{i-1} > \binom{n-1}{i-1}$ for $1 \le i \le n-1$. Indeed, equality holds in the bound for any of these coefficients only if G is a tree.

5.3.11. *The coefficients of $\chi(G;k)$ sum to 0 unless G has no edges.* The sum of the coefficients of a polynomial in k is its value at $k=1$. The value of $\chi(G;1)$ is the number of proper 1-colorings of G. This is 0 unless G has no edges. (The inductive proof from the chromatic recurrence is longer.)

5.3.12. *The exponent in the last nonzero term in the chromatic polynomial of G is the number of components of G.* We use induction on $e(G)$. When $e(G)=0$, we have $\chi(G;k)=k^{n(G)}$, and G has $n(G)$ components. Let $c(G)$ count the components in G. Both $G-e$ and $G\cdot e$ have fewer edges than G. Also $G\cdot e$ has the same number of components as G, and $G-e$ has the same number or perhaps one more. Since $n(G\cdot e) = n(G-e)-1$ and coefficients alternate signs, the coefficients of $k^{c(G)}$ have opposite signs in $\chi(G\cdot e;k)$ and $\chi(G-e;k)$. Thus we have positive α and nonnegative α' such that

$$\begin{array}{rlllll}
\chi(G-e;k): & k^n & -[e(G)-1]k^{n-1}+ & \cdots & +(-1)^{n-c(G)}\alpha k^{c(G)} & \\
-\chi(G\cdot e;k): & -(& k^{n-1}- & \cdots & +(-1)^{n-c(G)-1}\alpha' k^{c(G)} &) \\
\hline
=\chi(G;k): & k^n & -e(G)k^{n-1}+ & \cdots & +(-1)^{n-c(G)}(\alpha+\alpha')k^{c(G)} &
\end{array}$$

Since $\alpha+\alpha' > 0$, the last coefficient of $\chi(G;k)$ is as claimed.

Alternatively, one can reduce to the case of connected graphs by observing that the chromatic polynomial of a graph is the product of the chromatic polynomials of its components. Since an n-vertex tree has chromatic polynomial $k(k-1)^{n-1}$, its last nonzero term is the linear term. For a connected graph that is not a tree, the chromatic recurrence can be applied as above to obtain the result inductively.

If $p(k) = k^n - ak^{n-1} + \cdots \pm ck^r$ with $a > \binom{n-r+1}{2}$, then p is not a chromatic polynomial. If p is a chromatic polynomial of a (simple) graph G, then G has n vertices, a edges, and r components. The maximum number of edges in a simple graph with n vertices and r components is achieved by $r-1$ isolated vertices and one clique of order $n-r+1$. This has $\binom{n-r+1}{2}$ edges (Exercise 1.3.40), which is less than a.

5.3.13. *Chromatic polynomials and clique cutsets.* Let $F = G\cup H$, with $S = V(G)\cap V(H)$ being a clique. Every proper k-coloring of F yields proper k-colorings of G and H, and proper k-colorings of G and H together yield a proper k-coloring of F if they agree on S. Since S induces a clique, in every proper k-coloring of G or H the vertices of S have distinct colors. Therefore, given a proper k-coloring of $G\cap H$, the number of ways to extend it to a proper k-coloring of H [or G, or F] is independent of which proper k-coloring of $G\cap H$ is used.

For each $k\ge 0$, the value of the chromatic polynomial simply counts proper colorings. We have partitioned the proper k-colorings of these graphs into equal-sized classes that agree on S. For a fixed coloring f of $G\cap H$, the number of ways to extend it to a coloring of G, H, or F is thus $\chi(G;k)/\chi(G\cap H;k)$, $\chi(H;k)/\chi(G\cap H;k)$, or $\chi(F;k)/\chi(G\cap H;k)$, respectively. Since every extension of f to G is compatible with every extension of f to H to yield an extension of f to F, the product of the first two of these equals the third, and $\chi(G\cup H;k) = \chi(G;k)\chi(H;k)/\chi(G\cap H;k)$. (*Comment*: 1) When G and H intersect in a clique, it need not be true that $\chi(G;k) = \chi(G-G\cap H;k)\chi(G\cap H;k)$; for example, let G and H be 4-cycles sharing a single vertex.)

When $G\cap H$ is not a clique, this argument breaks down. For example, consider $G=H=P_3$, $F=G\cup H=C_4$, $G\cap H = 2K_1$. We have

$$\chi(F;k)\chi(G\cap H;k) = k^3(k-1)(k^2-3k+3) \ne k^2(k-1)^4 = \chi(G;k)\chi(H;k)$$

5.3.14. *Minimum vertex partitions of the Petersen graph into independent sets.* Let P be the Petersen graph. The Petersen graph P has odd cycles, so it requires 3 colors, and it is easy to partition the vertices into 3 independent sets using color classes of size 4,3,3, as described below.

a) If S is an independent 4-set, then $P-S=3K_2$. The three neighbors of a vertex have among them an edge to every other vertex, so S cannot contain all the neighbors of a vertex. Hence $P-S$ has no isolated vertex. Deleting S deletes 12 edges, so $P-S$ has 3 edges and 6 vertices. With no isolated vertices, this yields $P-S=3K_2$.

b) P has 20 partitions into three independent sets. Since P has 10 vertices, every such partition has an independent set of size at least four. There is no independent 5-set, because we have seen that every independent 4-set has two edges to each remaining vertex. For each independent 4-set S, there are 4 ways to partition the vertices of the remaining $3K_2$ into two independent 3-sets. Hence it suffices to count the independent 4-sets and multiply by 4. The number of independent 4-sets containing a specified vertex is 2, since deleting that vertex and its neighbors leaves C_6, which has two independent 3-sets. Summing this over all vertices counts each independent 4-set four times. Hence there are $2 \cdot 10/4 = 5$ independent 4-sets and 20 partitions of the vertices.

c) If $r = \chi(G)$, then $V(G)$ has $\chi(G;r)/r!$ partitions into r independent sets. Each such partition can be converted into a coloring in exactly $r!$ ways.

5.3.15. *A graph with chromatic number k has at most k^{n-k} vertex partitions into k independent sets, with equality achieved only by $K_k + (n-k)K_1$ (complete graph plus isolated vertices.* For $K_k + (n-k)K_1$, the sets of the partition are identified by the vertex of the clique that they contain, and the isolated vertices can be assigned to these sets arbitrarily, so this is the correct number of vertex partitions for this graph.

If G has only k vertices, then G has be a k-clique, and there is only one partition. If $n > k$, choose a vertex $v \in V(G)$. We consider two cases; $\chi(G - v) = k$ and $\chi(G - v) = k - 1$.

If $\chi(G - v) = k$, then partitions of $G - v$ can be extended to partitions of G by putting v in any part to which it has no edges. Thus it extends in at most k ways, with equality only if v is an isolated vertex.

If $\chi(G - v) = k - 1$, then G has a k-partition in which v is by itself and is adjacent to vertices $X = \{x_1, \dots, x_{k-1}\}$ of the other parts. Let R be the independent set containing v in an arbitrary k-partition, and suppose $|R| = 1 + r$. Note that $\chi(G - R) = k - 1$. By the induction hypothesis, $G - R$ has at most $(k-1)^{n-r-k}$ partitions into $k-1$ independent sets. Allowing R to be an arbitary subset of $G - (X \cup \{v\})$, we obtain at most $\sum_{r=0}^{n-k} \binom{n-k}{r}(k-1)^{n-k-r}$ partitions of G into k independent sets, which equals k^{n-k} by the binomial theorem. For equality, we must have $N(v) = X$ and $G - (X \cup \{v\}) = (n-k)K_1$ for each such choice of v, which again yields $G = K_k + (n-k)K_1$.

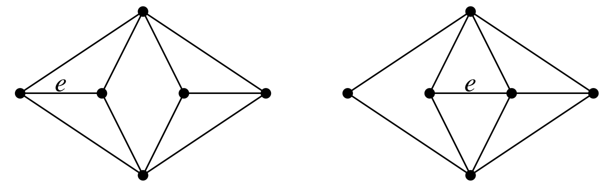
5.3.16. *If G is a simple graph with n vertices and m edges, then G has at most $\frac{1}{3}\binom{m}{2}$ triangles.* Each triangle has three pairs of incident edges, and each edge pair of incident edges appears in at most one triangle. Hence the number of triangles is at most 1/3 of the number of pairs of edges.

The coefficient of k^{n-2} in $\chi(G;k)$ is positive unless G has at most one edge. In the expression for the chromatic polynomial in Theorem 5.3.10, contributions to the coefficient of k^{n-2} arise from spanning subgraphs with $n-2$ components. These include all ways to choose two edges (weighted positively) and all ways to choose three edges forming a triangle (weighted negatively). With m edges and t triangles, the coefficient is $\binom{m}{2} - t$. Since $t \le \frac{1}{3}\binom{m}{2}$, the coefficient is positive unless G has at most one edge.

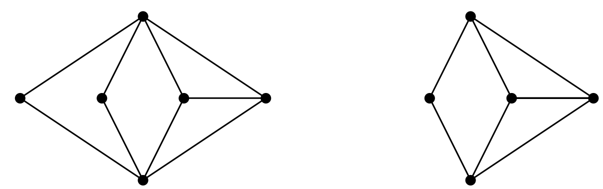
5.3.17. *Chromatic polynomial via the inclusion-exclusion principle.* In the universe of all k-colorings of G, let A_i be the set of colorings that assign the same color to the endpoints of edge e_i. The proper k-colorings of G are the k-colorings outside all the sets A_i. By the inclusion-exclusion formula,

the number of these is $\sum_{S \subseteq E(G)} (-1)^{|S|} g(S)$, where $g(S)$ is the number of k-colorings in $\bigcap_{e_i \in S} A_i$. These are the colorings in which every edge in S has its endpoints given the same color. To count these, we can choose a color independently for each component of the spanning subgraph of G with edge set S. Hence $g(S) = k^{c(G(S))}$, where $c(G(S))$ is the number of these components. We have obtained the formula of Theorem 5.3.10.

5.3.18. *Two chromatic polynomials.*

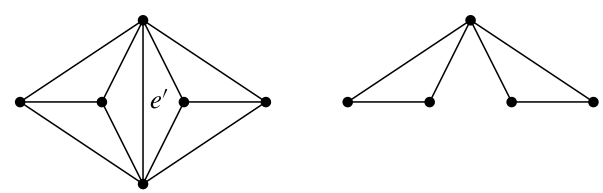


a) The graphs G, H above have the same chromatic polynomial. Applying the chromatic recurrence using the edge labeled e shows that each of these graphs has a chromatic polynomial that is the difference of the chromatic polynomials of the two graphs below.



b) The chromatic polynomial of G. The first graph G is $G' - e'$, where G' is the graph on the left below and e' is the indicated edge. The graph $G' \cdot e'$ appears on the right. Each of these graphs is chordal, as shown by exhibiting a simplicial eliminationordering. For each, the chromatic polynomial is a product of linear factors arising from the reverse of a simplicial elimination ordering. Thus

$$\begin{aligned}\chi(G;k) &= \chi(G' - e';k) = \chi(G';k) + \chi(G' \cdot e;k)\\ &= k(k-1)(k-2)^2(k-3)^2 + k(k-1)^2(k-2)^2\\ &= k(k-1)(k-2)^2(k^2 - 5k + 8)\end{aligned}$$



5.3.19. *The chromatic polynomial of the graph G obtained from K_6 by subdividing one edge is a product of linear factors, although G is not a chordal graph.* Let v be the vertex of degree 2 in G, and let e be an edges incident to e. The cycle consisting of v, its incident edges, and the edges from its neighbors to one other vertex form a chordless 4-cycle, so G is not chordal.

To compute $\chi(G;k)$, observe that $G-e$ consists of a 5-clique Q, an additional vertex w adjacent to four vertices of Q, and v adjacent to w. Hence $G-e$ is a chordal graph, with $\chi(G-e;k) = k(k-1)(k-2)(k-3)(k-4)(k-4)(k-1)$. Let $f(k) = \prod_{i=0}^{4}(k-i)$. The graph $G\cdot e$ is K_6, with $\chi((;K)_6) = f(k)(k-5)$. Thus

$$\begin{aligned}\chi(G;k) &= \chi(G-e;k) - \chi(G\cdot e;k) = f(k)[(k-4)(k-1)-(k-5)]\\ &= f(k)[k^2-6k+9] = k(k-1)(k-2)(k-3)(k-4)(k-3)^2\end{aligned}$$

5.3.20. *Properties of a chordal graph G with n vertices.*

a) G has at most n maximal cliques, with equality if and only if G has no edges. As each vertex v is added in the reverse of a simplicial elimination ordering, it creates one new maximal clique (containing v) if $N(v)$ is not already a maximal clique. If $N(v)$ is already a maximal clique, then the clique grows. No other maximal clique appears or changes. Thus there is at most one new maximal clique for each vertex. The first time an edge is added, a maximal clique is enlarged, not created, so there is a new clique at most $n-1$ times if G has an edge. (Comment: A more formal version of this argument uses the language of induction on n.)

b) Every maximal clique of G that contains no simplicial vertex of G is a separating set of G.

b) Every maximal clique of G that contains no simplicial vertex of G is a separating set of G.

Proof 1 (construction ordering, following part (a).) When a maximal clique Q of G acquires its last vertex v in the construction ordering, v is then simplicial. If all vertices of Q that are simplicial when Q is created are not simplicial in G, then the rest of the construction gives them additional neighbors that are separated by Q from each other and from the vertices of $G-Q$ that are earlier than v. If there are no such earlier vertices, then Q has at least two simplicial vertices at the time it is formed; each of these acquires a later neighbor, so Q separates those later neighbors.

Proof 2 (induction on n.) When $G=K_n$, there is no separating set, but all the vertices are simplicial, so the statement holds. When $G\neq K_n$, let Q be a maximal clique containing no simplicial vertex of G. Every chordal graph that is not a complete graph has two nonadjacent simplicial vertices (this follows, for example, from Lemma 5.3.16). Let u and v be such vertices. Note that Q cannot contain both u and v; we may assume that $v\notin Q$. Hence Q is a maximal clique in $G-v$.

If Q contains no simplicial vertex of $G-v$, then the induction hypothesis implies that Q separates $G-v$. All neighbors of v in G lie in one component of $G-v-Q$, since $N(v)$ is a clique in $G-v$. Hence Q is also a separating set in G.

If Q contains at least one simplicial vertex of $G-v$, then all such vertices lie in $N(v)$, since they are not simplicial in G. Therefore $u\notin Q$, and Q separates v from u.

5.3.21. *A graph G is chordal if and only if $s(H)=\omega(H)$ for every induced subgraph H of G*, where $s(H)$ is the *Szekeres–Wilf number* of H, defined to be $1+\max_{H\subseteq G}\delta(H)$.

Sufficiency. We prove the contrapositive. If G is not chordal, then G has a chordless cycle with length at least 4. Such a cycle is an induced subgraph. Its clique number is 2, and its Szekeres–Wilf number is 3.

Necessity. Since every induced subgraph of a chordal graph is chordal, it suffices to show that $s(G)=\omega(G)$ (the argument for G also applies to each induced subgraph). Since always $s(G)\geq\omega(G)$, it suffices to show that $s(G)\leq\omega(G)$.

Let H be an induced subgraph of G such that $\delta(H)=\max_{G'\subseteq G}\delta(G')$, so $s(G)=1+\delta(H)$. Let x be the first vertex of H that is deleted in some simplicial elimination ordering of G. Since the neighbors of x in H complete a clique with x, we have $\omega(H)\geq 1+d_H(x)\geq 1+\delta(H)=s(G)$.

5.3.22. *If $k_r(G)$ is the number of r-cliques in a connected chordal graph G, then $\sum_{r\geq1}(-1)^{r-1}k_r(G)=1$.* We use induction on $n(G)$. When $n(G)=1$, the only graph is K_1, which has one 1-clique and no larger clique; this satisfies the formula.

For $n(G)>1$, we know that G has a simplicial elimination ordering. Let v be a simplicial vertex in G. By the induction hypothesis, $\sum_{n\geq1}(-1)^{r-1}k_r(G-v)=1$. All cliques in $G-v$ appear also in G, so the contribution to the sum from these cliques is the same in G. Thus it suffices to show that the net contribution from cliques containing v is 0.

Each clique of size r containing v consists of v and $r-1$ vertices from $N(v)$. Since v is simplicial, $N(v)$ is a clique, and thus every selection of $r-1$ vertices from $N(v)$ forms an r-clique with v. Therefore, the contribution from these cliques is $\sum_{r\geq1}(-1)^{r-1}\binom{d(v)}{r-1}$.

The binomial theorem states that $(1+x)^m=\sum_{s=1}^{m}x^s\binom{m}{s}$. Setting $m=d(v)$ and $x=-1$ yields our sum on the right; on the left it yields 0 (since $m>0$). Thus the contribution from cliques containing v is 0, as desired.

5.3.23. *If C is a cycle of length at least 4 in a chordal graph G, then G has a cycle whose vertex set is $V(C)$ minus one vertex.* Given a simplicial elimination ordering of G, let v be the first vertex of C that is deleted. Since the remaining neighbors of v at the time of deletion form a clique, the neighbors of v on C are adjacent. Hence deleting v from the cyclic order of vertices on C yields a shorter cycle.

5.3.24. *If e is an edge of a cycle C in a chordal graph, then e forms a triangle with a third vertex of C.* We use induction on the length of C. If C is a triangle, then we have nothing to do. If C is longer, then because the graph is chordal there is a chord f of C. This splits C into two paths, one of which contains e. Combining this path with f yields a shorter cycle containing e, with all its vertices still in C. Applying the induction hypothesis to this shorter cycle yields the desired vertex of C.

5.3.25. *If Q is a maximal clique in a chordal graph G and $G-Q$ is connected, then Q contains a simplicial vertex.* (Equivalently, a maximal clique containing no simplicial vertex is a separating set.) We use induction on $n(G)$. When $n(G)\le 2$, G is a union of disjoint cliques, and the claim holds. For $n(G)\ge 3$, let Q be a maximal clique of G containing no simplicial vertex. Let v be a simplical vertex of G, and consider $G-v$. Still Q is a maximal clique in $G-v$.

If Q contains no simplicial vertex of $G-v$, then by the induction hypothesis Q is a separating set of $G-v$. If Q is not a separating set of G, then v has a neighbor in each component of $G-v-Q$, which contradicts v being simplicial in G.

Hence we may assume that Q contains a simplicial vertex u of $G-v$ that is not simplicial in G. This requires $v\leftrightarrow u$. If Q is not a separating set, then also v has a neighbor x outside Q. Since $u\leftrightarrow v$ and v is simplicial in G, also $x\leftrightarrow u$. Now since $x,u\in V(G-v)$ and u is simplicial in $G-v$, all of Q must also be adjacent to x. This contradicts the maximality of Q. Hence Q must indeed be a separating set in G.

5.3.26. *Chromatic polynomials of chordal graphs.*

a) If $G\cup H$ is a chordal graph, then $\chi(G\cup H;k)=\frac{\chi(G;k)\chi(H;k)}{\chi(G\cap H;k)}$, regardless of whether $G\cap H$ is a complete graph. We use induction on $n(G\cup H)$; the claim is immediate when there is one vertex. When $G\cup H$ is larger, let v be a simplicial vertex in $G\cup H$. By symmetry, we may assume that $v\in V(G)$. Since $N_{G\cup H}(v)$ is a clique, it cannot intersect both $V(G)-V(H)$ and $V(H)-V(G)$, since $G\cup H)$ has no edges joining these two sets. Hence we may assume that $N_{G\cup H}(v)\subseteq V(G)$.

Since v is simplicial, we have $\chi(G\cup H;k)=(k-d(v))\chi((G\cup H)-v;k)$. Note that $(G\cup H)-v=(G-v)\cup(H-v)$ and $(G\cap H)-v=(G-v)\cap(H-v)$. Since $(G\cup H)-v$ is chordal, the induction hypothesis yields $\chi((G\cup H)-v;k)=\frac{\chi(G-v;k)\chi(H-v;k)}{\chi((G-v)\cap(H-v);k)}$. Since $N_{G\cup H}(v)\subseteq V(G)$, we have $\chi(G-v;k)=\chi(G;k)/(k-d(v))$.

If $v\in V(G)\cap V(H)$, then $d_H(v)=d_{G\cap H}(v)$, and v is simplicial in every induced subgraph of $G\cup H$ containing it, so $\chi(H;k)/\chi(G\cap H;k)=\chi(H-v;k)/\chi((G\cap H)-v;k)$. If $v\in V(G)-V(H)$, then this ratio also holds, because in this case $H-v=H$ and $(G\cap H)-v=G\cap H$.

Hence we have

$$\begin{aligned}\chi(G\cup H;k)&=(k-d(v))\chi((G\cup H)-v;k)\\&=(k-d(v))\frac{\chi(G-v;k)\chi(H-v;k)}{\chi((G\cap H)-v;k)}=\frac{\chi(G;k)\chi(H;k)}{\chi(G\cap H;k)}\end{aligned}$$

b) If x is a vertex in a chordal graph G, then

$$\chi(G;k)=\chi(G-x;k)k\frac{\chi(G[N(x)];k-1)}{\chi(G[N(x)];k)}.$$

We apply part (a) with $G=F\cup H$, where $H=G[N(x)\cup x]$ and $F=G-x$. Observe that $F\cap H=G[N(x)]$. Also, since x is adjacent to all other vertices in H, we form all proper colorings of H by choosing a color for x and then forming a proper coloring of H from the remaining $k-1$ colors. Hence $\chi(H;k)=k\chi(H-x;k-1)=k\chi(G[N(x)];k)$. Now we simply substitute these expressions into the formula from part (a).

5.3.27. *Characterization of chordal graphs by minimal vertex separators,* where a *minimal vertex separator* in a graph G is a set $S\subseteq V(G)$ that for some pair x,y is a minimal set whose deletion separates x and y.

a) If every minimal vertex separator in G is a clique, then the same property holds in every induced subgraph of G. Let H be an induced subgraph of G. If S is a minimal x,y-separator in H, then $S\cup(V(G)-V(H))$ separates x and y in G. Hence $S\cup(V(G)-V(H))$ contains a minimal x,y-separator of G. Such a set T must contain S, since otherwise $G-T$ contains an x,y-path within H. By hypothesis, T is a clique in G, and hence S is a clique in H.

b) A graph G is chordal if and only if every minimal vertex separator is a clique. Necessity. For two vertices u,v in a minimal x,y-separator S, find shortest u,v-paths through the components of $G-S$ containing x and y. The union of these paths is a cycle of length at least 4, and its only possible chord is uv. Hence the vertices in S are pairwise adjacent.

Sufficiency. By part (a) and induction on $n(G)$, it suffices to show that G has a simplicial vertex if every minimal vertex separator of G is a clique. By induction on $n(G)$, we prove the stronger statement that if every minimal vertex separator of G is a clique and G is not a clique, then G has two nonadjacent simplicial vertices. The basis is vacuous (small cliques).

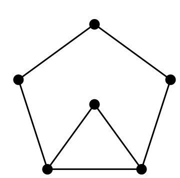
For larger G, let x_1,x_2 be a nonadjacent pair of vertices in G, let S be a minimal x_1,x_2-separator, and let G_i be the S-lobe of G (Definition 5.2.17) containing x_i. Since C holds for induced subgraphs, it holds for G_i. By the induction hypothesis, G_i has a simplicial vertex $u_i \notin S$ (whether or not G_i is a clique). Since no edge connect $V(G_1)$ to $V(G_2)$, the vertices u_1, u_2 are also simplicial in G, and they are nonadjacent in G.

5.3.28. *Every interval graph is a chordal graph and is the complement of a comparability graph.* Consider an interval representation of G, with each v represented by the interval $I(v) = [a(v), b(v)]$. Let v be the vertex with largest left endpoint $a(v)$. The intervals for all neighbors of v contain $a(v)$, so in the intersection graph the neighbors of v form a clique. Hence v is simplicial. If we delete v and procede with the remainder of the representation, which is an interval representation of $G - v$, we inductively complete a perfect elimination ordering.

Alternatively, let C be a cycle in G. Let u be the vertex in C whose right endpoint is smallest, and let v be the vertex whose left endpoint is largest. If u, v are nonadjacent, then the intervals for the two u, v-paths in C must cover $[b(u), a(v)]$. Hence the intersection graph has a chord of C between them. We conclude that an interval graph has no chordless cycle.

If $uv \in E(\overline{G})$, then $I(u)$ and $I(v)$ are disjoint. Orient the edge uv toward the vertex whose interval is to the left. This yields a transitive orientation of $\overline{G}$; if $I(u)$ is to the left of $I(v)$, and $I(v)$ is to the left of $I(w)$, then $I(u)$ is to the left of $I(w)$.

5.3.29. *The smallest imperfect graph G such that $\chi(G) = \omega(G)$.* The only imperfect graph with at most five vertices is C_5. Thus the graph below is the smallest imperfect graph with $\chi(G) = \omega(G)$.



5.3.30. An edge in an acyclic orientation of G is *dependent* if reversing it yields a cycle.

a) Every acyclic orientation of a connected n-vertex graph G has at least $n - 1$ independent edges. We use induction on n. When $n = 1$, we have no edges and need none. Consider $n > 1$. Since the orientation has no cycles, every maximal path starts with a source (indegree 0). Hence G has a source v. Define a digraph H with vertex set $N^+(v)$ by putting $x \leftrightarrow y$ in H if G has an x, y-path. Since a closed walk in the digraph G would contain a cycle, H must be acyclic. Let x be a source in H. The edge vx is independent; reversing it cannot create a cycle, since no path in G from v reaches x except the edge vx itself.

Let $G' = G - v$. Edges of G' are independent if and only if they are also independent in G, because there is no path in G through v from one vertex of G' to another. Also, G' is acyclic. Hence we can apply the induction hypothesis to G' to obtain another $n - 2$ independent edges.

b) If $\chi(G)$ is less than the girth of G, then G has an orientation with no dependent edges. Given an optimal coloring f, orient edge xy from x to y if and only if $f(y) > f(x)$. The maximum path length in this orientation is less than $\chi(G)$, and hence it is smaller by at least two than the length of any cycle.

An edge in an acyclic orientation is dependent if and only if there is another path from its tail to its head. The length of such a path would be one less than the length of the resulting cycle, but we have shown that our orientation has no paths this long.

5.3.31. *Comparison between acyclic orientations and spanning trees.* The number $\tau(G)$ satisfies the recurrence $\tau(G) = \tau(G - e) + \tau(G \cdot e)$. This is the recurrence satisfied by $a(G)$, but the initial conditions are different. A graph with no edges has one acyclic orientation, but it has no spanning tree unless it has only one vertex. A connected graph containing a loop has spanning trees but no acyclic orientation. A tree of order n has one spanning tree and 2^{n-1} acyclic orientations. A clique of order n has n^{n-2} spanning trees and $n!$ acyclic orientations; $n^{n-2} > n!$ if $n \geq 6$.

5.3.32. *Compatible pairs: $\eta(G; k) = (-1)^n \chi(G; -k)$.* Suppose D is an acyclic orientation of G and f is a coloring of $V(G)$ from the set $[k]$. We say that (D, f) is a *compatible pair* if $u \to v$ in D implies $f(u) \leq f(v)$. Let $\eta(G; k)$ be the number of compatible pairs. If $f(u) \neq f(v)$ for every adjacent pair u, v, then only one orientation is compatible with f. Therefore, $\chi(G; k)$ counts the pairs (D, f) under a slightly different condition: D is acyclic and $u \to v$ in D implies $f(u) < f(v)$ (*equality forbidden*). We know that $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$ for any edge of G; we claim that $\eta(G; k) = \eta(G - e; k) + \eta(G \cdot e; k)$.

The two conditions on pairs being counted are the same when there are no edges, so the two recurrences have the same boundary conditions: $k^n = \eta(\overline{K}_n; k) = \chi(\overline{K}_n; k)$. From this and the recurrence, we obtain $\eta(G; k) = (-1)^n \chi(G; k)$ by induction on $e(G)$. We compute $\eta(G; k) = \eta(G - e; k) + \eta(G \cdot e; k) = (-1)^{n(G)} \chi(G - e; k) + (-1)^{n(G)-1} \chi(G \cdot e; k) = (-1)^{n(G)} \chi(G; k)$. Evaluating η at 1 or χ at -1 yields $(-1)^n \chi(G; -1)$. Because there is only one labeling in which all vertices get label 1, and this is compatible with every acyclic orientation, $\eta(G; 1)$ is the number of acyclic orientations.

It remains only to prove the recurrence for η. Let $e = uv$. As in the recurrence for the chromatic polynomial, we begin with the compatible pairs for $G - e$ and consider the effect of adding e. If (D, f) is a compatible pair for $G - e$ such that $f(u) \neq f(v)$, say $f(u) < f(v)$, then e must be oriented from u to v to obtain an orientation of G compatible with f. The result is indeed acyclic, else it has a directed v, u-path along which the value f must step downward at some point. Conversely, we can delete e from a compatible pair for G with $f(u) \neq f(v)$ to obtain a compatible pair for $G - e$. Hence the compatible pairs with differing labels for u and v are in 1-1 correspondence in G and $G - e$.

Now consider pairs with $f(u) = f(v)$. It suffices to show that each such pair for $G - e$ becomes a compatible pair for G by adding e oriented in at least one way, and that for $\eta(G \cdot e, k)$ of these, *both* orientations of e yield compatible pairs for G. For the first statement, consider an arbitrary compatible pair (D', f) with $f(u) = f(v)$ for $G - e$, and suppose neither orientation for e yields a compatible pair for G. This requires D' to have both a u, v-path and a v, u-path, which cannot happen since D' is acyclic. For the second statement, suppose that (D, f) is a compatible pair for G with $f(u) = f(v)$ and that the orientation obtained by reversing e is also compatible with f. Then $D - e$ has neither a u, v-path nor a v, u-path, and contracting e yields a compatible pair for $G \cdot e$. Conversely, given a compatible pair for $G \cdot e$, we can split the contracted vertex to obtain a compatible pair for $G - e$ with $f(u) = f(v)$ so that orienting e in either direction yields a compatible pair for G.

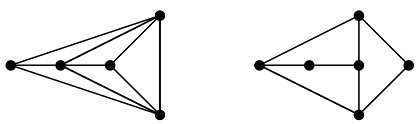
6.PLANAR GRAPHS

6.1. EMBEDDINGS & EULER'S FORMULA

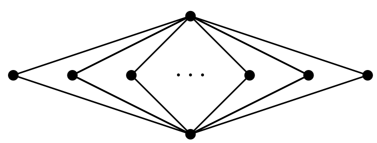
6.1.1. *a) Every subgraph of a planar graph is planar—TRUE.* Given a planar embedding of G, deleting edges or vertices does not introduce crossings, so an embedding of any subgraph of G can be obtained.

b) Every subgraph of a nonplanar graph is nonplanar—FALSE. $K_{3,3}$ is nonplanar, but every proper subgraph of $K_{3,3}$ is planar.

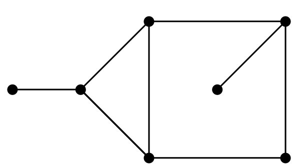
6.1.2. *The graphs formed by deleting one edge from K_5 and $K_{3,3}$ are planar.*



6.1.3. *$K_{r,s}$ is planar if and only if* $\min\{r, s\} \le 2$. If G contains the nonplanar graph $K_{3,3}$, then G is nonplanar; hence $K_{r,s}$ is nonplanar when $\min\{r, s\} \ge 3$. When $\min\{r, s\} = 2$, the drawing below suggests the planar embedding, and $K_{1,s}$ is a subgraph of this.



6.1.4. *The number of isomorphism classes of planar graphs that can be obtained as planar duals of the graph below is 4.*



The 4-cycle C can be embedded in only one way. Let e be the pendant edge incident to it, and let f be the pendant edge incident to the triangle D. We may assume that e immediately follows the edges D when we traverse C clockwise, because the other choice corresponds to reflecting the plane, and the resulting duals will be isomorphic to these.

We may embed e inside or outside C, we may embed D inside or outside C, and we may embed f inside or outside D. This yields eight possible embeddings, all with three faces. These come in pairs yielding the same dual, because flipping the choices involving C (while maintaining the same choice of whether f is inside D) has the effect of exchanging the inside and outside of C without affecting the dual.

In the four pairs, the resulting degree lists for the dual are $(9, 4, 3)$, $(7, 6, 3)$, $(7, 5, 4)$, and $(6, 5, 5)$. These are distinct, so there are four isomorphism classes of duals.

6.1.5. *A plane graph has a cut-vertex if and only if its dual has a cut-vertex—FALSE.* There are many counterexamples. The duals of trees and unicyclic graphs have at most two vertices and hence no cut-vertices. The duals of disconnected graphs without cut-vertices have cut-vertices.

6.1.6. *A plane graph with at least three vertices is 2-connected if and only if for every face, the bounding walk is a cycle.* If multiple edges are being allowed, the restriction to at least three vertices eliminates the cycle of length 2.

A disconnected plane graph has a face whose boundary consists of more than one closed walk, so we restrict our attention to a connected plane graph G. If G has a cut-vertex x, then considering the edges incident to x in clockwise order, there must be two consecutive edges in different $\{x\}$-lobes. For the face incident to these two edges, the boundary intersects more than one $\{x\}$-lobe and hence cannot be a cycle.

Now suppose that G is 2-connected. For a vertex x on the boundary of a face F, there are points inside F near x. By the definition of "face", all the nearby points between two rotationally consecutive incident edges at x are in F. Let e and e' be two such edges. Since G is 2-connected e and e' lie on a common cycle C.

By the Jordan Curve Theorem and the definition of "face", all points interior to F are inside C, or they are all outside C. In either case, as we follow the boundary of F after e and e', C prevents the boundary from visiting x again. Thus every vertex on the boundary of F is incident to exactly two edges of the boundary, and the boundary is a cycle.

6.1.7. *Every maximal outerplanar graph with at least three vertices is 2-connected.* Let G be a planar graph embedded with every vertex on the unbounded face. If G is not connected, then adding an edge joining vertices of distinct components still leaves every vertex on the unbounded face. If G has a cut-vertex v, and u and w are the vertices before and after v in a walk around the unbounded face, then adding the edge uw still leaves every vertex on the unbounded face, since v is visited at another point in the walk. We have shown that an outerplanar graph that is not 2-connected is not a maximal outerplanar graph.

6.1.8. *Every simple planar graph has a vertex of degree at most 5.* Every simple planar graph with n vertices has at most $3n - 6$ edges (Theorem 6.1.23). Hence the degree sum is at most $6n - 12$, and by the pigeonhole principle there is a vertex with degree less than 6.

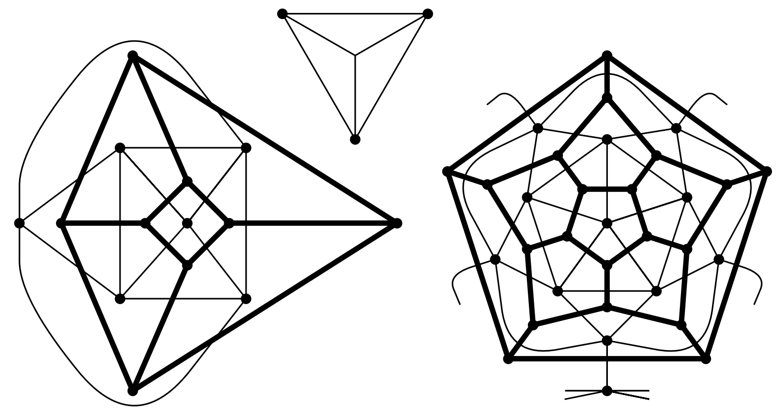
6.1.9. *Every simple planar graph with fewer than 12 vertices has a vertex of degree at most 4.* By Theorem 6.1.23, every simple planar graph with n vertices has at most $3n - 6$ edges and degree-sum at most $6n - 12$. If $12 > n$, then this degree sum is less than $5n$, and the pigeonhole principle implies that some vertex has degree at most 4.

6.1.10. *There is no simple bipartite planar graph with minimum degree at least 4—TRUE.* Since every face of a simple bipartite planar graph has length at least 4, it has at most $2n-4$ edges and degree sum at most $4n-8$. Hence the average degree of a simple bipartite planar graph is less than 4, and its minimum degree is less than 4.

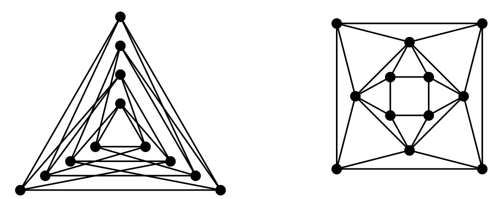
6.1.11. *The dual of a maximal planar graph is 2-edge-connected and 3-regular.* By definition, a maximal planar graph G is a simple planar graph to which no edge can be added without violating planarity. Consider an embedding of G. Every dual G^* is connected. If G^* has a cut-edge, then the edge of G corresponding to this edge is a loop in G, which cannot occur in a simple graph. Thus G^* is 2-connected.

Since G is simple, every face has length at least 3. If some face has length exceeding 3. Let w, x, y, z be four vertices in order on this face. If wy is an edge (outside this face), then xz cannot be an edge. Thus we can add wy or xz, contradicting maximality. This implies that every face has length 3, which is the statement that the dual is 3-regular.

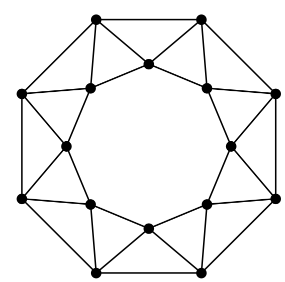
6.1.12. *Drawings of the five regular polyhedra as planar graphs, with the octahedron as the dual of the cube and the icosahedron as the dual of the dodecahedron.* The edges incident to the vertex of the icosahedron corresponding to the unbounded face of the dodecahedron are not fully drawn.



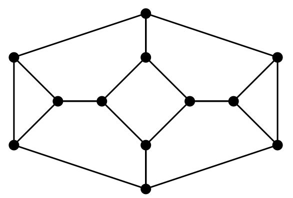
6.1.13. *Planar embedding of a graph.* The drawing on the right is a planar embedding of the graph on the left. (Of course, the isomorphism should be given explicitly.)

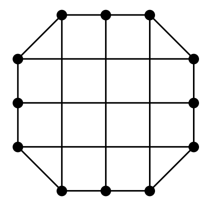


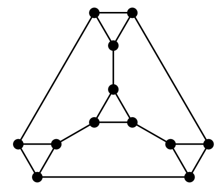
6.1.14. *For each $n \in \mathbb{N}$, there is a simple connected 4-regular planar graph with more than n vertices—TRUE.* When $n \geq 3$, we can form a simple connected 4-regular plane graph with $2n$ vertices by using an inner n-cycle, an outer n-cycle, and a cycle in the region between them that uses all $2n$ vertices. Below we show this for $n = 8$.



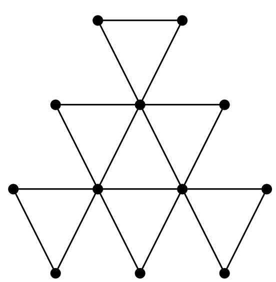
6.1.15. *A 3-regular planar graph of diameter 3 with 12 vertices.* By inspection, the graphs below are 3-regular and planar. To show that that they have diameter 3, we conduct a breadth-first search (Dijkstra's Algorithm) to compute distances from each vertex. By symmetry, this need only by done for one vertex each of "type" (orbit under automorphisms). In this sense, the rightmost graph is the best answer, since it is vertex transitive, and the distances need only be checked from one vertex. The graph on the left has five types of vertices, and the graph in the middle has two.

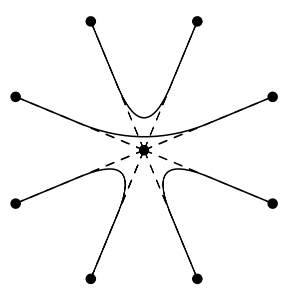






6.1.16. *An Eulerian plane graph has an Eulerian circuit with no crossings.* As the graph on the left below illustrates, it is not always possible to do this by splitting the edges at each vertex into pairs that are consecutive around the vertex. The figure on the right illustrates a non-consecutive planar splitting. We give several inductive proofs. The result does not require the graph to be simple.



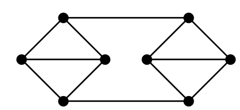


Proof 1 (induction on $e(G)-n(G)$). A connected Eulerian plane graph G has at least as many edges as vertices; if $e(G) = n(G)$, it is a single cycle and has no crossings. If $e(G) > n(G)$, then G has a vertex x on the outer face with degree at least 4. Form G' by splitting x into two vertices: a vertex y incident to two consecutive edges of the outer face that were incident to x, and a vertex z incident to the remaining edges that were incident to x. Locating y in the unbounded face of G yields a planar embedding of G'. Since G' is an even connected plane graph and $e(G') - n(G') = e(G) - n(G) - 1$, the induction hypothesis applies to G'. Since the edges incident to y are consecutive at x, the resulting Eulerian circuit of G' translates back into an Eulerian circuit without crossings in G. This proof converts to an algorithm for obtaining the desired circuit.

Proof 2 (induction on $e(G)$). The basis is a single cycle. If G has more than one cycle, find a cycle C with an empty interior. Delete its edges and apply induction to the resulting components $G_1, \ldots, G_k$. For $i = 1, \ldots, k$, absorb G_i into C as follows. Find a vertex x where G_i intersects C; all edges incident to x but not in C belong to G_i. Consider a visit $e1, x, e_2$ that the tour on G_i makes to x using an edge e_1 next to an edge f_1 of C in the embedding. Let f_2 be the other edge of C at x. Then replace the visits e_1, e_2 and f_1, f_2 by e_1, f_1 and e_2, f_2. This absorbs G_i into C while maintaining planarity of the circuit. Each component of $G - C$ is inserted at a different vertex, so no conflicts arise.

Proof 3 (local change). Given an Eulerian circuit that has a crossing in a plane graph G, we modify it to reduce the number of crossings formed by pairs of visits to vertices. By symmetry, it suffices to consider four edges a, b, c, d incident to v in counterclockwise order in the embedding such that one visit to v enters on a and leaves on c, and a later visit enters on b and leaves on d. We eliminate this crossing by traversing the portion $c, \ldots, b$ of the circuit in reverse order, as $b, \ldots, c$. Crossings at other vertices are unchanged by this operation. At v itself, if a passage e, f through v now crosses a, b, then e, f crossed a, c or b, d before, and if it now crosses a, b and c, d, then it crossed both a, c and b, d before. Thus there is no increase in other crossings, and we obtain a net decrease by un-crossing a, b, c, d.

6.1.17. *The dual of a 2-connected simple plane graph with minimum degree 3 need not be simple.* For the 2-connected 3-regular plane graph G below, G^* has a double edge joining the vertices of degree 6.



6.1.18. *Duals of connected plane graphs.*

a) If G is a plane graph, then G^* is connected. One vertex of G^* is placed in each face of G. If $u, v \in V(G^*)$, then any curve in the plane between u and v (avoiding vertices) crosses face boundaries of G in its passage from the face of G containing u to the face of G containing v. This yields a u, v-walk in G^*, which contains a u, v-path in G^*.

b) If G is connected, and G^* is drawn by placing one vertex inside each face of G and placing each dual edge in G^* so that it intersects only the corres ponding edge in G, then each face of G^* contains exactly one vertex of G. The edges incident to a vertex $v \in V(G)$ appear in some order around v. Their duals form a cycle in G^* in this order. This cycle is a face of G^*. If w is another vertex of G, then there is a v, w-path because G is connected, and this path crosses the boundary of this face exactly once. Hence every face of G^* contains at most one vertex of G. Equality holds because the number of faces of G^* equals the number of vertices of G: since both G and G^* are connected, Euler's formula yields $n-e+f=2$ and $n^*-e^*+f^*=2$. We have $e=e^*$ and $n^*=f$ by construction, which yields $f^*=n$.

c) For a plane graph G, $G^{**}\cong G$ if and only if G is connected. Since G^{**} is the dual of the plane graph G^*, part (a) implies that G^{**} is connected. Hence if G^{**} is isomorphic to G, then G is connected.

Conversely, suppose that G is connected. By part (b), the usual drawing of G^* over the picture of G has exactly one vertex of G inside each face of G^*. Associate each vertex $x\in V(G)$ with the vertex x' of G^{**} contained in the face of G that contains x; by part (b), this is a bijection.

Consider $xy\in E(G)$. Because the only edge of G^* crossing xy is the edge of G^* dual to it, we conclude that the faces of G^* that contain x and y have this edge as a common boundary edge. When we take the dual of G^*, we thus obtain $x'y'$ as an edge. Hence the vertex bijection from G to G^{**} that takes x to x' preserves edges. Since the number of edges doesn't change when we take the dual, G^{**} has no other edges and thus is isomorphic to G.

6.1.19. *For a plane graph G, a set $D\subseteq E(G)$ forms a cycle in G if and only if the corresponding set $D^*\subseteq E(G^*)$ forms a bond in G^*, by induction on $e(G)$.* We prove also that if D forms a cycle, then the two sides of the edge cut that is the bond in G^* corresponding to D are the sets of dual vertices corresponding to the sets of faces inside and outside D.

Basis step: $e(G)=1$. When G and G^* have one edge, in one it is a loop (a cycle), and in the other it is a cut-edge (a bond).

Induction step: $e(G)>1$. If D is a loop or a cut-edge, then the statement holds. Otherwise, D has more than one edge. If D forms a cycle, then let e be an edge of the cycle, and let G' be the graph obtained from G by contracting e. In G', the contracted set D' forms a cycle. Also, the set of faces in G' is the same as the set of faces in G; the only change is that the lengths of the faces bordering e (there are two of them since e is not a cut-edge) have shrunk by 1.

Since $e(G')=e(G)-1$, the induction hypothesis implies that in the dual $(G')^*$, the edges dual to D' form a bond, and the sets of vertices separated by the bond are those corresponding to the faces inside and outside D. By Remark 6.1.15, the effect of contracting e in G was to delete e^* from G^*. Since e^* joins vertices for faces that are inside and outside D, replacing it would reconnect G^*. Hence D^* forms a bond as claimed, and the sets of vertices on the two sides are as claimed.

Now consider the induction step for the converse. We assume that D^* forms a bond, so D^*-e^* forms a bond in G^*-e^* separating the same two vertex sets that D^* separates in G^*. By Remark 6.1.15, G^*-e^* is the dual of G', and the edges of D^*-e^* are the duals to D'. By the induction hypothesis, D' forms a cycle in G', and the two sides of the bond D^*-e^* in G^*-e^* correspond to the faces inside and outside D'. Since e^* joins vertices from these two sets, e (when we re-expand it in G) must bound faces from these two sets. With D being the boundary between two sets of faces, we can argue that D is a cycle.

6.1.20. *A plane graph is bipartite if and only if every face length is even.* A face of G is a closed walk, and an odd closed walk contains an odd cycle, so a bipartite plane graph has no face of odd length.

Conversely, suppose that every face length is even; we prove by induction on the number of faces that G is bipartite. If G has only one face, then by the Jordan Curve Theorem G is a forest and is bipartite.

If G has more than one face, then G has an edge e on a cycle. This edge belongs to two faces F_1, F_2 of even length; these faces are distinct because the cycle embeds as a closed curve, and by the Jordan Curve Theorem the regions on the inside and outside are distinct. Thus deleting e yields a combined face F whose length is the sum of the lengths of F_1 and F_2, less two for the absence of e from each. Hence F has even length. Lengths of other faces remain the same. Thus every face of $G-e$ has even length, and we apply the induction hypothesis to conclude that $G-e$ is bipartite.

To show that G also is bipartite, we replace e. Since F_1 has even length, there is an odd walk in $G-e$ connecting the endpoints of e, so they lie in opposite parts of the bipartition of $G-e$. Hence when we add e to return to G, the graph is still bipartite.

(Comment: Since we deleted one edge to obtain $G-e$, we could phrase this as induction on $e(G)$. Then we must either put all forests into the basis step or consider the case of a cut-edge in the induction step.)

6.1.21. *A set of edges in a connected plane graph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^*.* Since $(G^*)^*=G$ when G is connected, it suffices to prove one direction of the equivalence; the other direction is the same statement applied to G^*. Let T be a spanning tree of G, where G has n vertices and f faces. Let T^* be the spanning subgraph of G^* consisting of the duals of the remaining edges; T^* has f vertices.

Proofs 1, 2, 3. (Properties of trees). It suffices to prove any two of (1) T^* has $f-1$ edges, (2) T^* is connected, (3) T^* is acyclic.

(1) By Euler's Formula, $e(G)=n+f-2$; hence if T has $n-1$ edges there are $f-1$ edges remaining.

(2) Since T has no cycles, the edges dual to T contain no bond of G^* (by Theorem 6.1.14). Hence T^* is connected.

(3) Since T is spanning and connected, the remaining edges contain no bond of G. Thus T^* contains no cycle in G^* (by Theorem 6.1.14 for G^*).

Proof 4 (extremality and duality). A spanning tree of a graph is a minimal connected spanning subgraph. "Connected" is equivalent to "omits no bond" (see Exercise 4.1.29). Hence the remaining edges form a maximal subgraph containing no bond. By Theorem 6.1.14, the duals of the remaining edges form a maximal subgraph of G^* containing no cycle. A maximal subgraph of G^* containing no cycle is a spanning tree of G^*.

Proof 5 (induction on the number of faces). If G has one face, then G is a tree, $G^* = K_1$, and T^* is empty and forms a spanning tree of G^*. If G has more than one face, then G is not a tree, and hence G has an edge e not in the given tree T. Since e lies on a cycle (in $T + e$) and is not a cut-edge, $G - e$ is a connected plane graph with one less face. Let $G' = G - e$.

The induction hypothesis implies that the duals of $E(G') - E(T)$ form a spanning tree in $(G')^*$. Note that $(G - e)^* = G^* \cdot e^*$; we obtain the dual of G' by contracting the edge dual to e in G^*. Returning to G keeps e^* in $E(G) - E(T)$, so what happens to the duals of the edges outside T is that the vertex of $(G - e)^*$ representing the two faces that merged when e was deleted splits into two vertices joined by e^*. This operation turns a tree into a tree with one more vertex, and it has all the vertices of G^*, so it is a spanning tree.

6.1.22. *The weak dual of an outerplane graph is a forest.* A cycle in the dual graph G^* passes through faces that surround a vertex of G. When every vertex of G lies on the unbounded face, every cycle of G^* therefore passes through the vertex v^* of G^* that represents the unbounded face in G. Hence $G^* - v^*$ is a forest when G is an outerplane graph.

6.1.23. *Directed plane graphs.* In following an edge from tail to head, the dual edge is oriented so that it crosses the original edge from right to left.

a) If D is strongly connected, then D^* has no directed cycle. Such a cycle C^* encloses some region R of the plane. Let S be the set of vertices of D corresponding to the faces of D^* contained in R. Since C^* has a consistent orientation, the construction implies that all the edges of D corresponding to C^* are oriented in the same direction across C^* (entering R or leaving R). This contradicts the hypothesis that D is strongly connected.

b) If D is strongly connected, then $\delta^-(D^*) = \delta^+(D^*) = 0$. A finite acyclic directed graph has $\delta^- = \delta^+ = 0$, because the initial vertex of a maximal directed path can have no entering edge, and the terminal vertex of such a path can have no exiting edge.

c) If D is strongly connected, then D has a face on which the edges form a clockwise directed cycle and a face on which the edges form a counter-clockwise directed cycle. A vertex of D^* with indegree 0 corresponds to a face of D on which the bounding edges must form a clockwise directed cycle, and a vertex of D^* with outdegree 0 corresponds to a face of D on which the edges must form a counter-clockwise directed cycle.

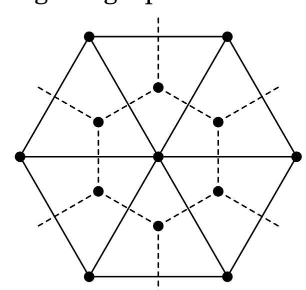
6.1.24. *Alternative proof of Euler's Formula.*

a) Faces of trees. Given a planar embedding of a tree, let x, y be two points of the plane not in the embedding. If the segment between them does not intersect the tree, then x and y are in the same face. If the segment does intersect the tree, then we create a detour for it closely following the embedding. Induction on the number of vertices yields a precise proof that this is possible. Using the detour yields a polygonal x, y-path that does not cross the embedding, so again x and y are in the same face.

b) Euler by edge-deletion. Euler's formula states that for a connected n-vertex plane graph with m edges and f faces, $n - m + f = 2$. If every edge of such a graph is a cut-edge, then the graph is a tree. This implies $m = n - 1$ and $f = 1$, in which case the formula holds. For an induction on e, we need only consider graphs that are not trees in the induction step. Such a graph G has an edge that is not a cut-edge. If e lies on a cycle, then both the interior and the exterior of the cycle have e on their boundary, and hence e is on the boundary of two faces. Therefore, deleting e reduces the number of faces by one but does not disconnect G. By the induction hypothesis, $n - (m - 1) + (f - 1) = 2$, and hence also $n - m + f = 2$.

6.1.25. *Every self-dual plane graph with n vertices has $2n - 2$ edges.* If G is isomorphic to G^*, then G must have the same number of vertices as faces. Euler's formula then gives $n - e + n = 2$ (and hence $e = 2n - 2$) if G is connected. Every self-dual graph is connected, because the dual of any graph contains a path to the vertex for the outside face of the original.

For every $n \geq 4$, the n-vertex "wheel" is self-dual. This is a cycle on $n - 1$ vertices, plus an nth vertex joined to all others. The triangular faces becomes a cycle, and each is adjacent to the remaining face; this is the same description as the original graph.



6.1.26. *The maximum number of edges in a simple outerplanar graph of order n is $2n - 3$.* For the lower bound, we provide a construction. A simple cycle on n vertices together with the chords from one vertex to the $n-3$ vertices not adjacent to it on the cycle forms an outerplanar graph with $2n-3$ edges. For the upper bound, we give three proofs.

a) (induction on n). When $n=2$, such a graph has at most 1 edge, so the bound of $2n-3$ holds. When $n>2$, recall from the text that every simple outerplanar graph G with n vertices has a vertex v of degree at most two. The graph $G' = G - v$ is an outerplanar graph with $n-1$ vertices; by the induction hypothesis, $e(G') \le 2(n-1)-3$. Replacing v restores at most two edges, so $e(G) \le 2n-3$.

b) (using Euler's formula). The outer face in an outerplanar graph has length at least n, since each vertex must be visited in the walk traversing it. The bounded faces have length at least 3, since the graph is simple. With $\{f_i\}$ denoting the face-lengths, we have $2e(G) = \sum f_i = n + 3(f-1)$, where f is the number of faces. Substituting $f = e - n + 2$ from Euler's formula yields $2e = n + 3(e-n+1)$, or $e(G) = 2n-3$. (Comment: If one restricts attention to a maximal outerplanar graph, then equality holds in both bounds: the outer face is a spanning cycle, and the bounded faces are triangles.)

c) (graph transformation). Add a new vertex in the outer face and an edge from it to each vertex of G. This produces an $n+1$-vertex planar graph G' with n more edges than G. Since $e(G') \le 3(n+1)-6$ edges, we have $e(G) \le 3(n+1)-6-n = 2n-3$.

Comment: If G has no 3-cycles, then the bound becomes $(3n-4)/2$.

6.1.27. *A 3-regular plane graph G with each vertex incident to faces of lengths 4,6, and 8 has 26 faces.* Let n be the number of vertices in the graph. Since each vertex is incident to one face of length 4, one face of length 6, and one face of length 8, there are n incidences of vertices with faces of each length. Since every face of length l is incident with l vertices, there are thus $n/4$, $n/6$, and $n/8$ faces of lengths $4, 6, 8$, respectively. Hence there are $n(\frac{1}{4}+\frac{1}{6}+\frac{1}{8})$ faces.

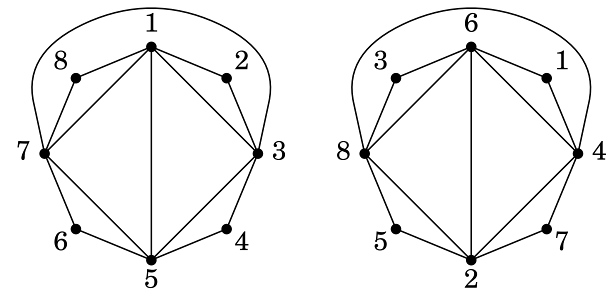
Also, the graph is 3-regular, it has $3n/2$ edges. By Euler's formula, $n - \frac{3}{2}n + n\frac{13}{24} = 2$. Multiplying by 24 yields $(-12+13)n = 48$, so $n = 48$. Hence the number of faces is $48\frac{13}{24}$.

6.1.28. *When m chords with distinct endpoints and no triple intersections form p points of intersection inside a convex region, the region is cut into $m+p+1$ smaller regions.* Form a planar graph G by establishing a vertex at each of the p points of intersection and at each endpoint of each chord. The $2m$ endpoints of chords have degree 3 in G, and the p points of intersection have degree 4. By the degree-sum formula, G has $3m+2p$ edges. Since it has $2m+p$ vertices, Euler's Formula yields $m+p+1$ as the number of bounded regions.

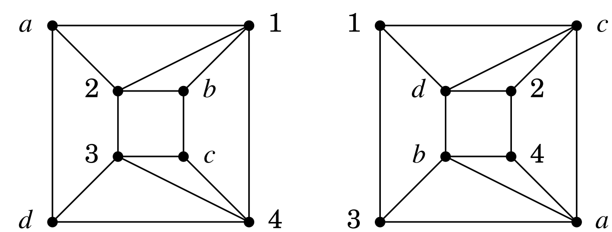
6.1.29. *Complements of planar graphs.*

a) The complement of each simple planar graph with at least 11 vertices is nonplanar. A planar graph with n vertices has at most $3n-6$ edges. Hence each planar graph with 11 vertices has at most 27 edges. Since K_{11} has 55 edges, the complement of each planar subgraph has at least 28 edges and is non-planar. For $n(G) > 11$, any induced subgraph with 11 vertices shows that $\overline{G}$ is nonplanar. There is also no planar graph on 9 or 10 vertices having a planar complement, but the easy counting argument here is not strong enough to prove that.

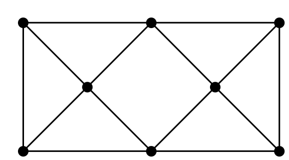
b) A self-complementary planar graph with 8 vertices.



The example below has a different degree sequence.



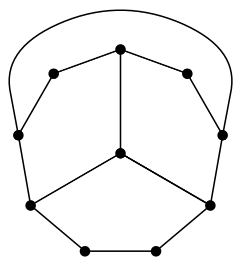
The graph below is planar and has the same degree sequence as that above, but it is not self-complementary.



6.1.30. *A 2-edge-connected n-vertex planar graph G with no cycle of length less than k has at most $(n-2)k/(k-2)$ edges.* Since adding edges will make G connected without reducing face lengths, we may assume that G is connected. Consider an embedding of G in the plane. Each face length is at least k, and each edge contributes twice to boundaries of faces. Therefore, counting the appearances of edges in faces grouped according to the e edges or according to the f faces yields $2e \ge kf$.

Since G is connected, we can apply Euler's formula, $n - e + f = 2$. Substituting for f in the inequality yields $2e \ge k(2 - n + e)$ and thus $e \le k(n-2)/(k-2)$. Note that when $k = 2$, multiple edges are available, and there is no limit on the number of edges.

The Petersen graph has 10 vertices, 15 edges, and girth 5. It has girth 5, so the size of a planar subgraph is at most $\lfloor 5 \cdot 8/3 \rfloor$, which equals 13. Since $15 > 13$, the Petersen graph is not planar, and at least two edges must be deleted to obtain a planar subgraph. The figure below shows that deleting two edges suffices.

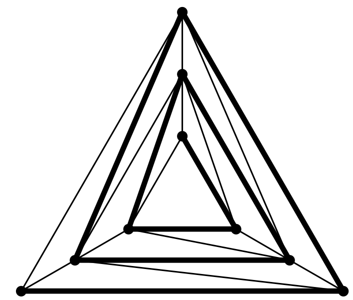


6.1.31. *The simple graph G with vertex set $v_1, \ldots, v_n$ and edge set $\{v_i v_j \colon |i - j| \le 3\}$ is a maximal planar graph.* The maximal planar graphs with n vertices are the simple n-vertex planar graphs with $3n - 6$ edges, so it suffices to prove by induction that G is a planar graph with $3n - 6$ edges. To facilitate the induction, we prove the stronger statement that G has a planar embedding *with all of $\{v_{n-2}, v_{n-1}, v_n\}$ on one face.*

Basis step: $n = 3$. The triangle has $3 \cdot 3 - 6$ edges and has such an embedding.

Induction step: $n > 3$. The graph G' obtained by deleting vertex n from G is the previous graph. By the induction hypothesis, it has $3(n-1) - 6$ edges and has an embedding with $\{v_{n-3}, v_{n-2}, v_{n-1}\}$ on one face. We add edges from v_n to these vertices to obtain G. Thus $e(G) = 3n - 6$. To embed G we place v_n inside the face of the embedding of G' having $\{v_{n-3}, v_{n-2}, v_{n-1}\}$ on its boundary. When we add the edges from n to those vertices to complete the embedding, we form a face with $\{v_{n-2}, v_{n-1}, v_n\}$ on the boundary.

The resulting embedding is illustrated below, with the bold path being $v_1, \ldots, v_n$ in order. The special face remains the outside face as the induction proceeds.



6.1.32. *If G is a maximal plane graph, and S is a separating 3-set of G^*, then $G^* - S$ has two components.* A maximal plane graph is a triangulation and has no loops or multiple edges. Hence its dual is 3-regular and 3-edge-connected. The connectivity of a 3-regular graph equals its edge-connectivity (Theorem 4.1.11). If G^* has a separating 3-set S, then it is a minimal separating set, and each vertex of S has a neighbor in each component of $G^* - S$. Extract a portion of a spanning tree in each component of $G^* - S$ that links the chosen neighbors of S. Combine these with the edges from S to the chosen neighbors. If $G^* - S$ has at least three components, then we obtain a subdivision of $K_{3,3}$. Since G^* is planar, we conclude that $G^* - S$ has at most two components.

6.1.33. *If G is a triangulation, and n_i is the number of vertices of degree i in G, then $\sum(6-i)n_i = 12$.* A triangulation with n vertices has $3n-6$ edges and hence degree-sum $6n - 12$. The sum $\sum in_i$ also equals the degree-sum. Hence $6(\sum n_i) - 12 = \sum in_i$, as desired.

6.1.34. *An infinite family of planar graphs with exactly twelve vertices of degree 5.* Begin with (at least two) concentric 5-cycles; call these "rungs". For each consecutive pair of rungs, add the edges of a 10-cycle in the region between the two 5-cycles. Inside the innermost rung, place a single vertex adjacent to the 5 vertices of the rung. Outside the outermost rung, place a single vertex adjacent to the 5 vertices of the rung. The vertices of degree 5 are the innermost vertex, the outermost vertex, and the vertices of the innermost and outermost rungs. The other vertices have degree 6. The case with exactly two 5-cycles is the icosahedron.

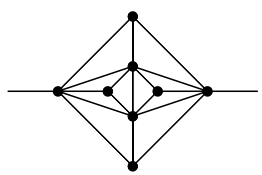
6.1.35. *Every simple planar graph with at least four vertices has at least four vertices with degree less than 6.* It suffices to prove the result for maximal planar graphs, since deleting an edge from a graph cannot make the statement become false. Let G be a maximal planar graph with n vertices.

In a maximal planar graph with at least four vertices, every vertex has degree at least 3.

The sum of the vertex degrees is $6n-12$. Therefore, the sum of $6-d(v)$ over the vertices with degree less than 6 is at least 12. Since $\delta(G) \ge 3$, each term contributes at most 3, so we must have at least four such vertices.

For each even value of n with $n \ge 8$, there is an n-vertex simple planar graph G that has exactly four vertices with degree less than 6. By the analysis above, such a graph must be a triangulation with four vertices of degree 3 and the rest of degree 6.

The graph sketched below has eight vertices. If we extend the two half-edges at the left and right to become a single edge, then we have the desired 8-vertex graph. To enlarge the graph, we could instead place vertices at the ends of the two half-edges, make them adjacent also to the top and bottom vertices, and extend half-edges from the top and bottom. If those half-edges become a single edge, then we have the desired 10-vertex graph. Otherwise, we can continue adding pairs of vertices to obtain the sequence of examples.

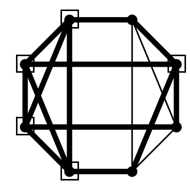


6.1.36. *If S is a set of n points in the plane such that the distance in the plane between any pair of points in S is at least 1, then there are at most $3n-6$ pairs for which the distance is exactly 1.* If two unit-distances cross, then one of the other distances among these four points is less than 1. Hence the condition implies that the graph of unit distances is a planar graph with n vertices. A planar graph with n vertices has at most $3n-6$ edges.

6.1.37. *Given integers $k \ge 2$, $l \ge 2$, and kl even, there is a planar graph with exactly k faces in which every face has length l.* (For $l = 1$ and $k > 2$, this does not work.) When $l > 1$ and k is even, use two vertices with degree k joined by k paths of lengths $\lceil l/2 \rceil$ and $\lfloor l/2 \rfloor$ (alternating) through vertices of degree 2. Each face is formed by a path of length $\lceil l/2 \rceil$ and a path of length $\lfloor l/2 \rfloor$. When k is odd, l is even and $\lceil l/2 \rceil = \lfloor l/2 \rfloor$, so k paths of this length suffice.

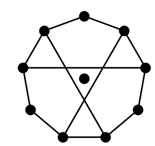
6.2. CHAR'ZN OF PLANAR GRAPHS

6.2.1. *The complement of the 3-dimensional cube Q_3 is nonplanar.* The vertices of Q_3 are the binary triples. Those with an odd number of 1s form an independent set, as do those with an even number of 1s. Each vertex is adjacent to three in the other independent set. Hence $\overline{Q}_3$ consists of two 4-cliques with a matching between them. This graph contains a subdivision of K_5 in which four branch vertices lie in one of the 4-cliques.



6.2.2. *Nonplanarity of the Petersen graph.*

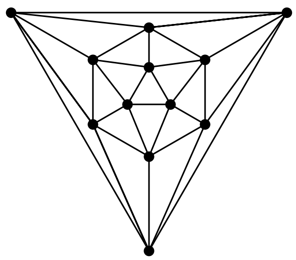
a) via Kuratowski's Theorem. Since the Petersen graph has no vertices of degree at least 4, it contains no K_5-subdivision. Below we show a $K_{3,3}$-subdivision.



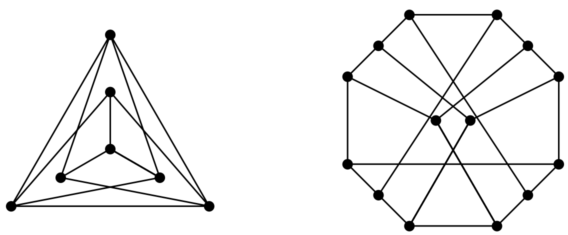
b) via Euler's Formula. To apply Euler's formula, assume a planar embedding. Since the Petersen graph has no cycle of length less than 5, each face has at least 5 edges on its boundary. Each edge contributes twice to boundaries of faces. Counting the appearances of edges in faces grouped by edges or by faces yields $2e \ge 5f$. Since the graph is connected, Euler's formula yields $n - e + f = 2$. Substituting for f in the inequality yields $2e \ge 5(2 - n + e)$, or $e \le (5/3)(n-2)$. For the Petersen graph, $15 \le (5/3)8$ is a contradiction.

c) via the planarity-testing algorithm. We may start with any cycle. When we start with a 9-cycle C as illustrated, every C-fragment can go inside or outside, so we can pick one of the chords and put it inside. Now the other two chords can only go outside, but after embedding one of them, the remaining chord cannot go on any face. This occurs because this cycle has three pairwise crossing chords.

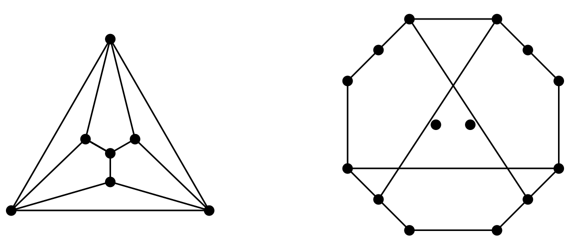
6.2.3. *A convex embedding.* This is the graph of the icosahedron. It is 3-connected and has a convex embedding in the plane.



6.2.4. *Planarity of graphs.*

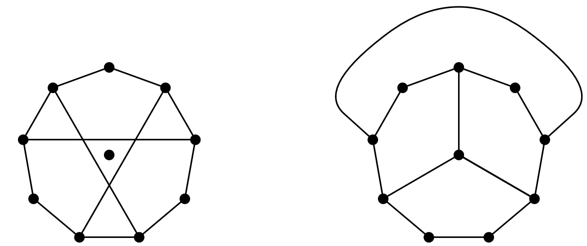


The first graph is planar; a straight-line embedding with every face convex appears below. The second graph is nonplanar. It has many subgraphs that are subdivisions of $K_{3,3}$; one is shown below.



6.2.5. *The minimum number of edges that must be deleted from the Petersen graph to obtain a planar subgraph is 2.* The drawing on the left below illustrates a subdivision of $K_{3,3}$ in the Petersen graph. Since this does not use every edge of the Petersen graph, the graph obtained by deleting one edge from the Petersen graph is still nonplanar (all edges are equivalent, by the disjointness description of the Petersen graph).

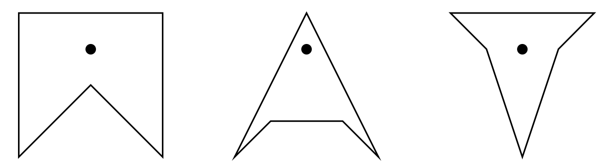
Deleting two edges from the Petersen graph yields a planar subgraph as shown on the right below.



6.2.6. *Fary's Theorem.*

a) Every simple polygon with at most 5 vertices contains a point that sees every point in the polygon. In a convex polygon, by definition, the segment joining any pair of points lies entirely in the polygon. Hence every point in a convex polygon sees the entire polygon.

Proof 1. If a 4-gon is not convex, then the vertex opposite the interior reflex angle (exceeding 180 degrees) sees the entire polygon. A non-convex pentagon has one or two reflex angles, and if two they may be consecutive or not. The cases are illustrated below.



Proof 2. Triangulate the polygon by adding chords between corners that can see each other. This can be done by adding one chord to a 4-gon and by adding two to a 5-gon, with cases as illustrated above. The resulting triangles have one common vertex. Since a corner of a triangle sees the entire triangle, the common corner sees the entire region.

b) Every planar graph has a straight-line embedding. By induction on $n(G)$, we prove the stronger statement that the edges of any plane graph G can be "straightened" to yield a straight-line embedding of G without changing the order of incident edges at any vertex. The statement is true by inspection for $n(G) \le 3$.

For $n(G) \ge 4$, we may assume that G is a triangulation, since any plane graph can be augmented to a maximal plane graph, and deleting extra edges in a straight-line embedding of the maximal planar supergraph yields a straight-line embedding of the original graph. Every planar graph has a vertex of degree at most 5; let x be such a vertex in G.

Since G is a triangulation, the neighborhood of G is a cycle C, and $G - x$ has C as a face boundary. By the induction hypothesis, $G - x$ has a straight-line embedding with C as a polygonal face boundary. By part (a), we can place x at a point inside C and draw straight lines from x to all vertices of C without crossings.

6.2.7. *G is outerplanar if and only if G contains no subdivision of K_4 or $K_{2,3}$.* Let $G' = G \vee K_1$ denote the graph obtained by from G by adding a single vertex x joined to all vertices of G. Then G has an embedding with all vertices on a single face $\Leftrightarrow$ G' is planar $\Leftrightarrow$ G' has no subdivision of K_5 or $K_{3,3}$ $\Leftrightarrow$ G has no subdivision of K_4 or $K_{2,3}$.

Additional details for these statements of equivalence:

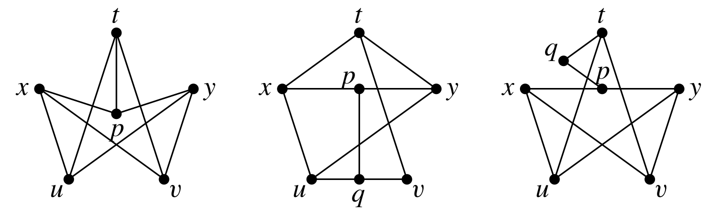
1) If G is outerplanar, then we place x in the unbounded face of an outerplanar embedding of G and join it to all vertices on the face to obtain a planar embedding of G'. Conversely, if G' is planar, then it has a planar embedding in which x lies on the unbounded face. Deleting x from this embedding yields an outerplanar embedding of G, because it has an unobstructed curve from each vertex to the point that had been occupied by x and is now in the unbounded face.

2) Kuratowski's Theorem.

3) If G has a subdivision of K_4 or $K_{2,3}$, then adding x as a additional branch vertex yields a subdivision of K_5 or $K_{3,3}$ in G'. Conversely, if G' has a subdivision F of K_5 or $K_{3,3}$, then deleting x destroys at most one branch vertex or one path of F, which leaves a subdivision of K_4 or $K_{2,3}$ in G.

6.2.8. *Every 3-connected graph with at least 6 vertices that contains a subdivision of K_5 also contains a subdivision of $K_{3,3}$.* Let H be a K_5-subdivision in G, with branch vertices x, y, t, u, v. If H itself has only five vertices, then G has another vertex p, and G has a $p, V(H)$-fan of size 3. By symmetry, we may assume that the paths of the fan arrive at x, y, t. Then G has a subdivision of $K_{3,3}$ with branch vertices x, y, t in one partite set and p, u, v in the other partite set.

If H has more than five vertices, then by symmetry we may assume that the x, y-path P in H has length at least two. Since G is 3-connected, $G - \{x, y\}$ has a shortest path Q from $V(P) - \{x, y\}$ to $V(H) - V(P)$. Let the endpoints of Q be p on P and q in $H' = H - V(P)$. If q is on the cycle in H' through t, u, v, then by symmetry we may assume q is on the branch path between u and v and not equal to u. In $H \cup Q$ we now have a subdivision of $K_{3,3}$ with branch vertices x, y, q in one partite set and p, t, u in the other partite set. On the other hand, if q is not on the cycle through t, u, v, then by symmetry we may assume q is on the x, t-path in H' (and not equal to x). Now $H \cup Q$ has a subdivision of $K_{3,3}$ with branch vertices x, y, t in one partite set and p, u, v in the other partite set.



6.2.9. *For $n \geq 5$, the maximum number of edges in a simple planar n-vertex graph not having two disjoint cycles is $2n - 1$.* For the construction, begin with a copy of P_3 and $n - 5$ isolated vertices. Add two vertices x, y adjacent to all of these and each other. In a set of pairwise-disjoint cycles, at most one cycle can avoiding using both x and y, so no two cycles are disjoint. The number of edges is $2 + 2(n - 2) + 1 = 2n - 1$.

For the upper bound, we use induction on n. Basis step ($n = 5$): There is no 5-vertex planar graph with 10 edges, so the bound holds.

Induction step ($n \geq 6$): We need only consider a planar graph G with exactly $2n$ edges and no disjoint cycles. If any vertex has degree at most 2, then we delete it and apply the induction hypothesis to the smaller graph. Hence $\delta(G) \geq 3$. Since G is planar, $e(G) \geq 2n - 4$ forces a triangle on some set $S \subseteq V(G)$. Since G does not have disjoint cycles, $G - S$ is a forest H.

If H has three isolated vertices, then $\delta(G) \geq 3$ yields a copy of $K_{3,3}$ with S as a partite set. Hence H has a nontrivial component.

Main case. If x, y are vertices in a nontrivial component of H, and z is a vertex of H not on the unique x, y-path, and z has two neighbors in S other than a vertex of $N(x) \cap N(y)$, then we form one cycle using the x, y-path in H and a vertex of $N(x) \cap N(y)$, and we form a second cycle using z and the rest of S.

Any two vertices of degree 1 in H have a common neighbor in S. If H has an isolated vertex z, then using two leaves x, y from a nontrivial component of H yields the main case. Hence H has no isolated vertex.

Suppose that H has a component with at least three leaves x, y, z. If x and y both have a neighbor in S outside $N(z)$, then the main case occurs. Otherwise, symmetry yields $N(y) \cap S = N(z) \cap S$, and the main case occurs unless x, y, z all have the same two neighbors in S. Now G contains a subdivision of $K_{3,3}$ with one partite set being $\{x, y, z\}$ and the other consist of their two common neighbors in S and the vertex that is the common vertex of the x, y-, y, z-, and z, x-paths in H. Hence every component of H is a nontrivial path.

If any component of H has endpoints with a common neighbor in S distinct from a common neighbor of the endpoints of another component, then we obtain two disjoint cycles. Hence there is a single vertex $t \in S$ that is adjacent to all endpoints of components in H. In each component the two ends have distinct second neighbors in S; otherwise $n(G) \geq 6$ yields the main case.

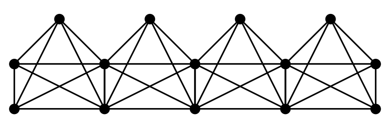
If H has at least two components, we now form one cycle using one component of H plus t and another cycle using another component of H plus the rest of S. Hence H is a single path.

If any internal vertex u of H has a neighbor w in S other than t, then let v be the leaf of H that also neighbors w. We now obtain one cycle using w and the u,v-path in H, and we obtain another cycle using the other endpoint of that component of H plus $S - \{w\}$. Hence every internal vertex in H is adjacent only to t in S.

We now have determined H exactly. Every cycle in H contains t or avoids only t. In fact, G is the wheel $C_{n-1} \vee K_1$, where K_1 is the vertex t. However, this graph has only $2n - 2$ edges. This final contradiction completes the proof.

6.2.10. *Simple n-vertex graphs containing no $K_{3,3}$-subdivision.* Let $f(n)$ be the maximum number of edges in such a graph.

a) If $n-2$ is divisible by 3, then $f(n) \geq 3n-5$. We form G by pasting together $(n-2)/3$ copies of K_5 as shown. Since $K_{3,3}$ is 3-connected, a subdivision of $K_{3,3}$ cannot have branch vertices in different S-lobes when $|S| = 2$. This confines the branch vertices to a single S-lobe and yields an inductive proof that this graph has no $K_{3,3}$-subdivision.



b) $f(n) = 3n-5$ when $n-2$ is divisible by 3, and otherwise $f(n) = 3n-6$ (for $n \geq 2$). Note that $f(n) \geq 3n-6$ for all n by using maximal planar graphs. For the upper bound, we use induction on n, checking the small cases ($2 \leq n \leq 5$) by inspection.

If $e(G) \geq 3n-5$, then G is nonplanar. By Kuratowski's Theorem, G contains a subdivision of K_5 or $K_{3,3}$. If G is 3-connected, then a subdivision of K_5 also yields a subdivision of $K_{3,3}$, by Exercise 6.2.8. Hence we may assume that G has a separating 2-set S. We avoid a $K_{3,3}$-subdivision in G if and only if each S-lobe *with the addition of the edge joining the vertices of* S has no such subdivision. Since we are maximizing $e(G)$, we may assume that this edge is present in G.

Now the number of edges and existence of $K_{3,3}$-subdivisions is unaffected by how we add the S-lobes. If there are more than two, then we can paste one onto an edge in one of the other S-lobes and maintain the same properties. Hence we may assume that there are only two S-lobes.

Let the two S-lobes have n_1 and n_2 vertices, repectively. The induction hypothesis yields $e(G) \leq f(n_1) + f(n_2) - 1$. Since $n_1 + n_2 = n + 2$ and we count the shared edge only once, this total is $3n - c$, where c depends on the congruence classes of n and n_1 modulo 3. If n_1 and n_2 are congruent to 2 modulo 3, then the sum is $3n_1 - 5 + 3n_2 - 5 - 1 = 3n - 5$. In other cases, at least one of the contributions is smaller by one. Hence $3n-5$ is achievable only when $n \equiv 2 \pmod 3$, and otherwise $3n-6$ is an upper bound.

Comment. When $n-2$ is divisible by 3, the *only* way to achieve the bound is by pasting together copies of K_5 at edges.

6.2.11. *If $\Delta(H) \leq 3$, then a graph G contains a subdivision of H if and only if G contains a subgraph contractible to H.* A H-subdivision in G is a subgraph of G contractible to H, so the condition is necessary.

For the converse, it suffices to show that if H' is contractible to H then H' contains a subdivision of H.

Proof 1. In contracting H' to H, each vertex of H arises by contracting the edges in some connected subgraph of H'. Let T_v be a spanning tree of the subgraph that is contracted to v. Since $\Delta(H) \geq 3$, at most three edges of H depart from T_v. Let T_v' be the union of the paths in T_v that connect the vertices of T_v from which edges of H depart. In particular, if x, y, z are the vertices of departure for the paths leaving T_v, we can let T_v' be the x,y-path P in T_v plus the path in T_v from z to P. Discard from H' all edges except those of each T_v' and those that in the paths that contract to edges of H. The remaining graph is a subdivision of H in H'.

Proof 2. An alternative proof follows the process from H (that is, $K_{3,3}$) itself back to G, undoing the sequence of deletions and contractions (in the reverse order), keeping only a graph that is a subdivision of H and at the end is H', a subdivision of H contained in G. Deletions are undone by doing nothing (don't add the edge back). Undoing a contraction is splitting a vertex v. At most three edges incident to v have been kept in the current subdivision of H. If u and w are the adjacent vertices resulting from the split, then at least one of them, say w, inherits at most one of these important edges. Keeping that edge and the edge uw allows u to become the vertex playing the role of v in the subdivision, with the same number of paths entering as entered v, going to the same places. If a path went off along an edge now incident to w, then that path is one edge longer.

Comment. The claim fails for graphs with maximum degree 4. Consider the operation of *vertex split*, which replaces a vertex x with two new adjacent vertices x_1 and x_2 such that each former neighbor of x is adjacent to x_1 or x_2. Contracting the edge x_1x_2 in the new graph produces the original graph. In applying a split to a vertex x of degree 4, the two new vertices may each inherit edges to two of the neighbors of x and thus wind up with degree 3. If H has maximum degree 4, then applying vertex splits to vertices of maximum degree can produce a graph G in which each new vertex has degree at most 3. This graph G has H as a minor, but G contain no H-subdivision since G has no vertex of degree 4.

6.2.12. *Wagner's characterization of planar graphs.* The condition is that neither K_5 nor $K_{3,3}$ can be obtained from G by performing deletions and contractions of edges.

a) Deletion and contraction of edges preserve planarity. Given an embedding of G, deleting an edge cannot introduce a crossing. Also, there is a dual graph G^*. Contracting an edge e in G has the effect of deleting the dual edge e^* in G^*. In other words, $G^* - e^*$ is planar, and $G \cdot e$ is its planar dual, so $G \cdot e$ is also planar. Alternatively, one can follow the transformation that shrinks one endpoint of e continuously into the other and argue that at no point is a crossing introduced.

Since deletion and contraction preserve planarity and K_5 and $K_{3,3}$ are not planar, we cannot obtain these graphs from a planar graph by deletions and contractions. Hence the condition is necessary.

b) Kuratowski's Theorem implies Wagner's Theorem. We prove sufficiency by proving the contrapositive: if G is nonplanar, then K_5 or $K_{3,3}$ can be obtained by deletions and contractions.

If G is nonplanar, then by Kuratowski's Theorem, G contains a subdivision of K_5 or $K_{3,3}$. Every graph containing a subdivision of a graph F can be turned into F by deleting and contracting edges (delete the edges not in the subdivision, then contract edges incident to vertices of degree 2). Hence K_5 or $K_{3,3}$ can be obtained by deleting and contracting edges.

6.2.13. *G is planar if and only if every cycle in G has a bipartite conflict graph.* The condition is necessary because in any planar embedding a cycle C separates the plane into two regions, and the C-bridges embedded in each of the regions must form an independent set in the conflict graph. Conversely, if G is non-planar, then by the preceding theorem it is K_5 (with conflict graph C_5), or it has a cycle C with three crossing chords that produce a triangle in the conflict graph of C.

6.2.14. *If x and y are vertices of a planar graph G, then there is a planar embedding with x and y on the same face if and only if G has no cycle C avoiding $\{x, y\}$ such that x and y belong to conflicting C-fragments.*

If there is a cycle C such that x and y belong to conflicting C-fragments, then in every planar embedding of G, one of these fragments goes inside G and the other goes outside it. Hence in every embedding, x and y are separated by C. (This argument applies when C does not contain x or y, but it suffices to consider such cycles.)

Conversely, suppose there is no such cycle; we show that $G + xy$ is planar. If not, then $G + xy$ contains a Kuratowski subgraph using xy. If this is a $K_{3,3}$-subdivision H with xy on the path between branch vertices u and v, then x and y belong to fragments with alternating vertices of attachment on the cycle in H through the other four branch vertices. If this is a K_5-subdivision H with xy on the path between branch vertices u and v if and only if x and y belong to fragments with three common vertices of attachment on the cycle in H through the other three branch vertices. In either case, x and y belong to conflicting C-fragments for some cycle C.

6.2.15. *A cycle C in a 3-connected plane graph G is the boundary of a face in G if and only if G has exactly one C-fragment.* If G has exactly one C-fragment, then it must be embedded inside C or outside C, and the other of these regions is a face with boundary C.

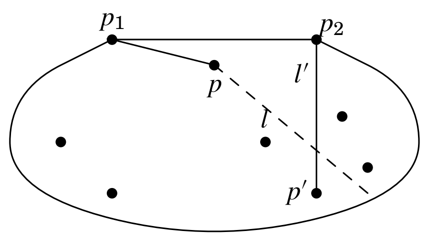
If C is a face boundary, then all C-fragments are embedded on one side of C, say the inside. This prevents two C-fragments H_1, H_2 from having alternating vertices of attachment along C. This means that there is a path P along C that contains all vertices of attachment of H_1 and none of H_2. Now the endpoints of P separate G, which contradicts the assumption of 3-connectedness.

6.2.16. *If G is an n-vertex outerplanar graph and P is a set of n points in the plane with no three on a line, then G has a straight-line embedding with its vertices mapped onto P.* It suffices to consider maximal outerplanar graphs. We prove the stronger statement that if v_1, v_2 are two consecutive vertices of the unbounded face of a maximal outerplanar graph G, and p_1, p_2 are consecutive vertices of the convex hull of P, then G has a straight-line embedding f on P such that $f(v_1) = p_1$ and $f(v_2) = p_2$.)

The statement is trivial for $n = 3$; assume $n > 3$. Let $v_1, v_2, \ldots, v_n$ denote the counterclockwise ordering of the vertices of G on the outside face in a particular embedding. Let v_i be the third vertex on the triangle containing v_1, v_2.

Claim: *there is a point $p \in P - \{p_1, p_2\}$ with the two properties (a) no point of P is inside $p_1 p_2 p$, and (b) there is a line l through p that separates p_1 from p_2, meets P only at p, and has exactly $i-2$ points of P on the side of l containing p_2.* To obtain p, we rotate the line $p_1 p_2$ about p_2 until we reach a line $l' = p_2 p'$ with $p' \in P$ such that exactly $i-3$ points of P are separated from p_1 by l'. Among the points of $P - \{p_1, p_2\}$ in the closed halfplane determined by l' that contains p_1, let p be the point minimizing the angle $p_2 p_1 p$. By this choice, p satisfies (a), and there are at most $i-2$ points of P on the side of $p_1 p$ containing p_2. If we rotate this line about p, then before it becomes parallel to l' it reaches a position l satisfying (b).

Let H_1 and H_2 denote the closed halfplanes determined by l containing p_1 and p_2, respectively. By the induction hypothesis, the subgraphs of G induced by $\{v_1, v_2, \dots, v_i\}$ and $v_i, v_{i+1}, \dots, v_n, v_1\}$ can be straight-line embedded on $H_1 \cap P$ and $H_2 \cap P$ so that v_1, v_2, v_i are mapped to p_1, p_2, p. Combining these embeddings yields a straight-line embedding of G with the desired properties.



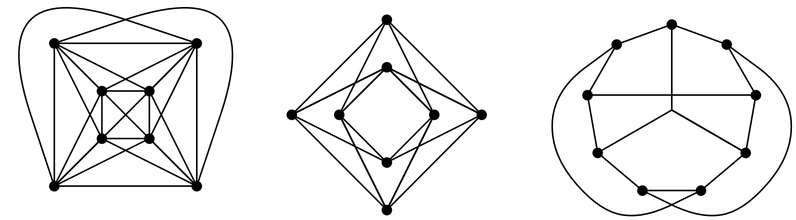
6.3. PARAMETERS OF PLANARITY

6.3.1. *A polynomial-time algorithm to properly color a planar graph G.* First find an embedding in the plane and augment to a maximal plane graph G'. Now delete a vertex v of degree at most 5. Recursively find a proper 5-coloring of $G' - v$. To extend the coloring to v, use Kempe chains if necessary to remove a color from the neighborhood of v.

6.3.2. *If every subgraph of G has a vertex of degree at most k, then G is k+1-colorable.* We use induction on $n(G)$. For the basis, K_1 is $k+1$-colorable whenever $k \ge 0$. For the induction step, let v be a vertex of degree at most k in a graph G with at least two vertices. By the definition of k-degenerate, every subgraph of a k-degenerate graph is k-degenerate. He the induction hypothesis yields a proper $k+1$-coloring of $G - v$. Extend the coloring to v by giving v a color that does not appear on its neighbors.

6.3.3. *Every outerplanar graph G is 3-colorable, by the Four Color Theorem.* Adding a vertex v adjacent to all of G yields a planar graph G', which is 4-colorable. A proper 4-coloring of G' restricts to a proper 3-coloring of G, because the colors used on the vertices of G must all be different from the color used on v.

6.3.4. *Crossing number of $K_{2,2,2,2}$, $K_{4,4}$, and the Petersen graph.* Let $k = \lfloor (n-2)g/(g-2) \rfloor$. The maximum number of edges in a planar n-vertex graph with girth g is k, so $\nu(G) \ge e(G) - k$ if G has girth g. This yields $\nu(K_{2,2,2,2}) \ge 6$ and $\nu(K_{4,4}) \ge 4$, and $\nu(G) \ge 2$ when G is the Petersen graph. The drawings below achieve these lower bounds.



6.3.5. *Every planar graph G decomposes into two bipartite graphs.* By the Four Color Theorem, G is 4-colorable. Let the four colors be $0, 1, 2, 3$. Let H consist of all edges of G joining a vertex of odd color with a vertex of even color. Let H' consist of all edges joining two vertices whose color has the same parity. Now H and H' are bipartite and have union G.

6.3.6. *Small planar graphs.* We use induction on $n(G)$; every graph with at most four vertices is planar. A planar graph G with at most 12 vertices has degree-sum at most $6 \cdot 12 - 12$, with equality only for triangulations. The bound is 60. Hence $\delta(G) \le 4$ unless G is a 5-regular triangulation. The only such graph is the icosahedron, which is 4-colorable explicitly.

Hence $\delta(G) \le 4$. The same conclusion holds for graphs with at most 32 edges that have more than 12 vertices: the average vertex degree is at most $64/13$, which is less than 5.

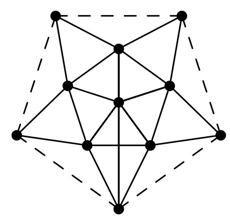
Thus we have $\delta(G) \le 4$. Let v be a vertex of minimum degree. A proper 4-coloring of G can be obtained from a proper 4-coloring of $G - v$. The case of $d(v) = 4$ uses Kempe chain arguments as in the proof in the text that a minimal 5-chromatic planar graph has no vertex of degree at most 4.

6.3.7. *A configuration H in a planar triangulation can be retrieved from the partially labeled subgraph H′ obtained by labeling the neighbors of the ring vertices with their degrees and then deleting the ring vertices.* For each vertex v on the external face of H', the data $d_H(v)$ is given. Append $d_H(v) - d_{H'}(v)$ new edges at v, extending to new vertices in the extremal face. When H' is 2-connected, the new neighbors for vertices of H' appear in the same order cyclically in the outside face of H' as the vertices of H' themselves. The requirement that H' is a block is necessary for this reason; when $H' = P_3$, it is easy to construct a counterexample to the desired statement.

For each consecutive pair v, w on the external face of H', the edge vw is on the boundary of a face with a new edge from v and a new edge from w. Since each face of H is a triangle, the endpoints of these edges other than $\{v, w\}$ must merge into a single vertex to complete the face. (When $H' = K_2$, this occurs for both sides of the edge.)

Finally, pass a cycle through the resulting vertices outside H' to form the ring. This is forced, since the bounded faces must all be triangles.

6.3.8. *Configurations with ring size 5 in planar triangulations such that every internal vertex has degree at least five.* The intent was to seek a configuration with more than one internal vertex. For example:



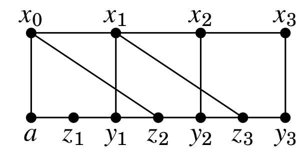
6.3.9. *A minimal non-4-colorable planar graph has no separating cycle of length at most 4.* Let G be a minimal 5-chromatic graph with a vertex cut S that induces a cycle. Let A and B be the S-lobes of G; A is the subgraph induced by S and the vertices inside the cycle, and B is the subgraph induced by S and the vertices outside the cycle. By the minimality of G, A and B are 4-colorable. If $|S| = 3$, then S receives three distinct colors in each of these colorings, and we can permute the names to agree on S.

Hence we may assume that $S = \{v_1, v_2, v_3, v_4\}$, indexed in order around the cycle. Let f and g be proper 4-colorings of $A + v_1v_3$ and $B + v_1v_3$, respectively. We may rename colors in f and g so that $f(v_i) = g(v_i) = i$ for $i \in \{1, 2, 3\}$. If $f(v_4) = g(v_4)$, then we are finished. Otherwise we may assume that $f(v_4) = 4$ and $g(v_4) = 2$. If the subgraph of A induced by vertices with colors 2 and 4 under f does not have a v_2, v_4-path, then we can interchange colors 2 and 4 on the component containing v_4 to obtain a coloring that agrees with g on S.

Otherwise, v_1 and v_3 are in different components of the subgraph of A induced by vertices with colors 1 and 3, and we may interchange colors 1 and 3 on the component containing v_3 to obtain a new proper 4-coloring f' of A that assigns colors 1,2,1,4 to v_1, v_2, v_3, v_4. Now consider $B + v_2v_4$; this planar graph also has a proper 4-coloring g'. Since g' assigns distinct colors to v_1, v_2, v_4, we may assume by renaming colors that these are 1,2,4, respectively. Now g' agrees with f on S if $g'(v_3) = 3$, and g' agrees with f' on S if $g'(v_3) = 1$. In either case, we have 4-colorings of A and B that combine to provide a proper 4-coloring of G.

6.3.10. *Triangle-free planar graphs may have independence number arbitrarily close to $n(G)/3$, so no greater lower bound can be guaranteed.* Consider the sequence of graphs G_k defined as follows: G_1 is

the 5-cycle, with vertices a, x_0, x_1, y_1, z_1 in order. For $k > 1$, G_k is obtained from G_{k-1} by adding the three vertices x_k, y_k, z_k and the five edges $x_{k-1}x_k, x_ky_k, y_kz_k, z_ky_{k-1}, z_kx_{k-2}$. The graph G_3 is shown below. Moving the edges $x_{i-2}z_i$ outside yields a planar embedding.



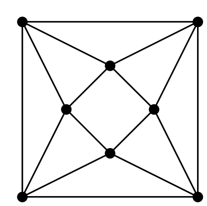
We prove by induction on k that $\alpha(G_k) = k+1 = (n(G)+1)/3$ and that, furthermore, every maximum stable set in G_k contains x_k or y_k or $\{z_k, x_{k-1}\}$. In G_1 the maximum stable sets are the nonadjacent pairs of vertices; the only one not containing x_1 or y_1 is $\{z_1, x_0\}$, so the stronger statement holds.

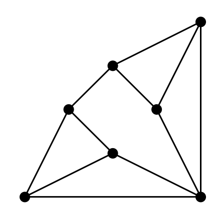
Suppose that the claim holds for G_{k-1}. A maximum stable set S in G_k uses at least one vertex not in G_{k-1}. If it uses two, then they are x_k and z_k. Since $y_{k-1}, x_{k-1}, x_{k-2} \in N(\{x_k, z_k\})$, S cannot contain a maximum stable set from G_{k-1}, and hence $|S| \le k+1$. Furthermore, when S contains x_k it satisfies the stronger statement.

To complete the proof of the stronger statement, we must show that a stable set S of size $k+1$ in G_k that contains z_k but not x_k also contains x_{k-1}. Since $|S| = k+1$, S must contain a stable set of size k in G_{k-1}, which contains x_{k-1} or y_{k-1} or $\{z_{k-1}, x_{k-2}\}$, by the induction hypothesis. Since z_k is adjacent to x_{k-2} and y_{k-1}, the only possibility here is $x_{k-1} \in S$, which completes the proof of the statement.

6.3.11. *For the graph G_n defined below, when n is even, every proper 4-coloring of G_n uses each color on exactly n vertices.* Let G_1 be C_4. For $n > 1$, obtain G_n from G_{n-1} by adding a new 4-cycle surrounding G_{n-1}, making each vertex of the new cycle also adjacent to two consecutive vertices of the previous outside face.

Each two consecutive "rungs" of G_n form a subgraph isomorphic to G_2, shown below on the left. This graph is 4-chromatic but not 4-critical, since it contains the 4-chromatic graph shown on the right. Since the remaining graph after deleting any one vertex still needs four colors, every color must appear at least twice (and hence exactly twice) in each copy of G_2.





6.3.12. *Every outerplanar graph is 3-colorable.* The fact that every induced subgraph of an outerplanar graph is outerplanar yields inductive proofs.

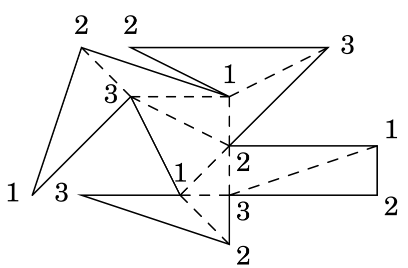
Proof 1 (induction on $n(G)$). If every edge of G is on the outside face, then every block of G is an edge or a cycle, and G is 3-colorable. Otherwise, suppose xy is an internal edge. Then $S = \{x, y\}$ is a separating set. The S-lobes of G are outerplanar; by the induction hypothesis, they are 3-colorable. Since S induces a clique, we can make these colorings agree on S, which yields a 3-coloring of G.

Proof 2 (induction on $n(G)$). Every simple outerplanar graph has a vertex of degree at most 2 (proved in the text); we can delete such a vertex x, 3-color $G - x$ by the induction hypothesis, and extend the coloring to x.

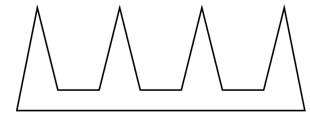
Proof 3 (prior results). Every graph with chromatic number at least 4 contains a subdivision of K_4 (Dirac's Theorem), but a graph containing a subdivision of K_4 cannot be outerplanar.

Every art gallery laid out as a polygon with n segments can be guarded by $\lfloor n/3 \rfloor$ guards so that every point of the interior is visible to some guard. The art gallery is a drawing of an n-cycle in the plane. We add straight-line segments to obtain a maximal outerplanar graph with n vertices. To do this, observe that 3-gons are already triangulated without adding segments. For $n > 3$, some corner can see some other corner across the interior of the polygon. We add this segment and proceed inductively on the two resulting polygons with fewer corners.

Consider a proper 3-coloring of the resulting maximal outerplanar graph (outerplanar graphs are 3-colorable). Since each bounded region is a triangle, its vertices are pairwise adjacent and receive distinct colors. Thus each color class contains a vertex of each triangule. Any point in a triangle, such as a corner, sees all points in the triangle. Thus guards at the vertices of a color class can see the entire gallery. Since the three classes partition the set of vertices, the smallest class has at most $\lfloor n/3 \rfloor$ elements.

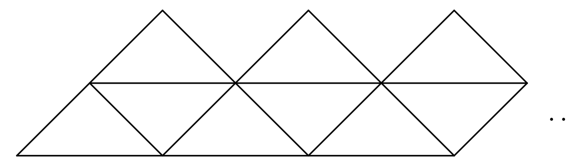


The bound of $\lfloor n/3 \rfloor$ guards is best possible. The alcoves in the polygon below require their own guards; no guard can see into more than one of them. There are $\lfloor n/3 \rfloor$ alcoves. When n is not divisible by 3, we can add the extra vertex (or two) anywhere.



6.3.13. *Every art gallery with walls whose outer boundary is an n-gon can be guarded by $\lfloor (2n - 3)/3 \rfloor$ guards, where $n \geq 3$, and this is sharp.* Adding walls cannot make it easier to guard the gallery, so we may assume that the polygon is triangulated by nonintersecting chords. A guard in a doorway can see the two neighboring triangles; we use such guards and guards on the outside walls.

The bound is achieved by an art gallery of the type below.



Proof 1. The embedded n-gon plus the interior walls form a planar embedding of an outerplanar graph whose vertices are the corners; it has $n + (n - 3) = 2n - 3$ edges. Every outerplanar graph is 3-colorable (this can be proved inductively by cutting along chords formed by walls, as in Thomassen's proof of 5-choosability, or by using the existence of a vertex of degree at most 2, which can be proved inductively or by Euler's Formula.)

From a proper 3-coloring of the vertices of the outerplanar graph, 3-color the edges of the graph by assigning to each edge the color *not used* on its endpoints. Now each triangle has each color appearing on its incident edges. If we put guards at the edges occupied by the least frequent color, then each room is guarded, and we have used at most $\lfloor (2n - 3)/3 \rfloor$ guards.

Proof 2. Again triangulate the region to obtain an outerplanar graph G. In the dual graph G^*, let v denote the vertex corresponding to the unbounded face of G. The graph $G^* - v$ is a tree with $n - 2$ vertices and maximum degree at most 3. Each edge corresponds to a guard in a doorway, so an edge cover (a set of edges covering the vertices) corresponds to a set of guards in doorways that together can see all the rooms.

It suffices to show that a tree T with $n - 2$ vertices and maximum degree at most 3 has an edge cover with at most $(2n - 3)/3$ edges, for $n \geq 4$. We study $n \in \{4, 5, 6\}$ explicitly. For larger n, consider the endpoint x of a longest path in T. By the choice of x, its neighbor y has one non-leaf neighbor and at most two leaf neighbors. We use the pendant edges at y in the edge cover and delete y and its leaf neighbors to obtain a smaller tree

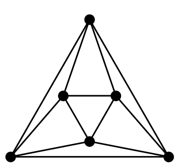
T'. We have placed k edges in the cover and deleted $k+1$ vertices, where $k \in \{1,2\}$. Using the induction hypothesis and $k \le 2$, we obtain an edge cover of size at most

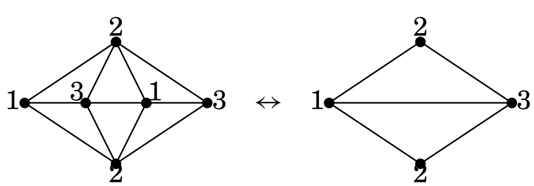
$$\tfrac{2(n-k-1)-3}{3} + k = \tfrac{2n-3}{3} - \tfrac{2(k+1)}{3} + k = \tfrac{2n-3}{3} + \tfrac{k-2}{3} \le \tfrac{2n-3}{3}.$$

6.3.14. *A maximal planar graph is 3-colorable if and only if it is Eulerian.* Let G be a maximal plane graph; G is connected, and every face is a triangle. Suppose that G is 3-colorable. The three colors $\{1,2,3\}$ appear on each face, in order clockwise or counterclockwise. When two faces share an edge, the colors appear clockwise around one face and counterclockwise around the other. This defines a proper 2-coloring of the faces of G, using the colors "clockwise" and "counterclockwise". Hence G^* is bipartite. The degree of a vertex in a connected plane graph G equals the length of the corresponding face in G^*. Since G^* has no odd cycle, G has even degrees and is Eulerian.

Conversely, suppose that G is Eulerian, meaning that each vertex has even degree. **Proof 1**. We use induction on $n(G)$ to obtain a proper 3-coloring. The smallest Eulerian triangulation is K_3, which is 3-colorable. A 2-valent vertex in a larger simple triangulation would belong to two triangles, forcing a double edge. Since G is Eulerian, we may thus assume that $\delta(G) \ge 4$. Since K_5 is nonplanar, the next smallest Eulerian triangulation is the octahedron, with six vertices of degree 4; this is 3-colorable, as illustrated below.

For the induction step, suppose that $n(G) \ge 6$. Since every planar graph has a vertex of degree less than 6, G has a 4-valent vertex. If G has a triangle T of 4-valent vertices, then G the neighbors of T induce a 3-cycle containing T, as in the octahedron. Deleting T reduces the degrees of the neighboring vertices by 2 each, so we can apply the induction hypothesis to the resulting subgraph G'. The coloring assigns distinct colors to the neighbors of T, and this proper coloring extends also to T.

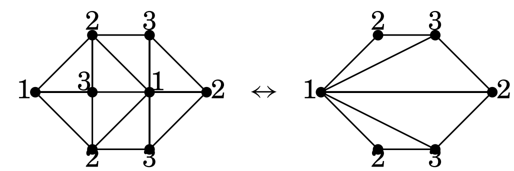




Hence we may assume that when G has two adjacent 4-valent vertices x, y, their two common neighbors a, b have degree greater than 4. Suppose G has adjacent 4-valent vertices $\{x, y\}$, with u and v being the fourth neighbors of x and y, respectively. Form G' by deleting $\{x, y\}$ and adding the edge uv. Because $d(a), d(b) > 4$, u and v are not already adjacent. All vertices

still have even degree; hence G' is an Eulerian triangulation. We apply the induction hypothesis and extend the resulting coloring to a coloring of G, as indicated above.

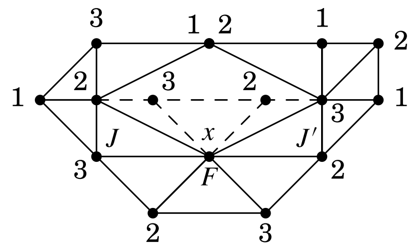
Finally, suppose that G has no adjacent 4-valent vertices. Choose a 4-valent vertex x with neighbor y, and define a, b, u as before. Form G' by deleting $\{x, y\}$ and adding edges from u to all of $N(y) - N(x)$. Because $d(a), d(b) > 4$, z is not already adjacent to any vertex of $N(y) - N(x)$. All vertices still have even degree; hence G' is an Eulerian triangulation. We apply the induction hypothesis and extend the resulting coloring to a coloring of G, as indicated below.



Proof 2. All faces are triangles; we start with an arbitrary 3-coloring on some face F. The color of the remaining vertex on any neighboring face is forced. We claim that iterating this yields a proper 3-coloring f. Otherwise, a contradiction is reached at some vertex v. This means there are two paths of faces from F, distinct after some face F' (we start at F' to obtain disjoint dual paths), that reach v but assigning different colors to v. Let C be the cycle enclosing the faces on these two paths and the regions inside them. Choose an example in which C encloses the smallest possible number of vertices.

The contradiction cannot arise when C encloses only one vertex x. In this case, the faces causing the conflict are only those incident to x, and C is the cycle through the neighbors of x. Since $d(x)$ is even, the colors alternate on C when following the path of faces, and there is no conflict.

We obtain a contradiction by finding such a cycle enclosing fewer vertices. Since the initial face starts two distinct paths of faces, one of its vertices (x below) must be enclosed by C and not on C. Together, the two paths contain a portion of the faces containing x, say from J to J' around x. We replace these by the other faces involving x, but keeping J, J'. Because $d(x)$ is even, the coloring forced on J by $f(V(F))$ forces onto J' the same coloring that $f(V(F))$ forced onto J' directly. From J, J' outward, the paths of faces lead to the same conflict as before. Hence we can start with one of the inner faces involving x and obtain a conflict using paths that enclose fewer vertices than before (x is no longer inside).

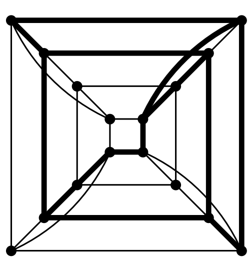


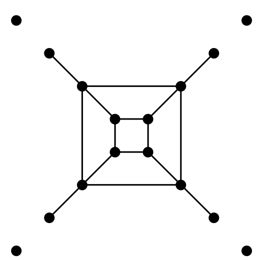
6.3.15. *The vertices of a simple outerplanar graph can be partitioned into two sets so that the subgraph induced by each set is a disjoint union of paths.* Let one set be the set of vertices with even distance from a fixed vertex u, and let the other set be the remainder; call these "color classes". Since no adjacent vertices can have distance from x differing by more than 1, each component of the graph induced by one color class consists of vertices with the same distance from u. Let H be such a component.

To show that H is a path, it suffices to show that H has no cycle and has no vertex of degree at least 3. Given three vertices x_1, x_2, x_3 in H, let P_i be a shortest x_i, u-path in G. Since x_1, x_2, x_3 have the same distance from u, each P_i has only x_i in H. Also, since the paths eventual merge, $P_1 \cup P_2 \cup P_3$ contains a subdivision of a claw; call this F (note that F need not contain u, as the paths may meet before reaching u).

If H contains a cycle C, let x_1, x_2, x_3 be three vertices on C. Now $F \cup C$ is a subdivision of K_4. If H contains a vertex w of degree 3, let x_1, x_2, x_3 be neighbors of w. Now F together with the claw having center w and leaves x_1, x_2, x_3 is a subdivision of $K_{2,3}$. Since an outerplanar graph has no subdivision of K_5 or $K_{2,3}$, H is a path.

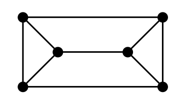
6.3.16. *The 4-dimensional cube Q_4 is nonplanar and has thickness 2.* The graph is isomorphic to $C_4 \square C_4$. On the left below, we show a subdivision of $K_{3,3}$ in bold. The graph is also isomorphic to $Q_3 \square K_2$, consisting of two 3-cubes with corresponding vertices adjacent. Taking one of the 3-cubes and the edges to the other 3-cube from one of its 4-cycles yields a planar graph that is isomorphic to the subgraph consisting of the remaining edges.





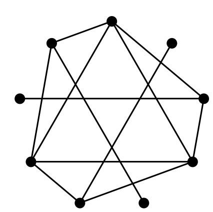
6.3.17. *Thickness. a) The thickness of K_n is at least $\lfloor (n+1)/6 \rfloor$.* Each planar graph used to form G has at most $3n(G)-6$ edges, so the thickness of G is at least $e(G)/[3n(G)-6]$. For $G = K_n$, this yields $\lceil n(n-1)/[6(n-2)] \rceil$, since thickness must be an integer. We compute $n(n-1)/(n-2) = n(1 + 1/(n-2)) = n + n/(n-2) = n + 1 + 2/(n-2)$. Since $\lceil x/r \rceil = \lfloor (x+r-1)/r \rfloor$, the thickness is at least $\lceil [n+1+2/(n-2)]/6 \rceil = \lfloor [n+6+2/(n-2)]/6 \rfloor = \lfloor (n+7)/6 \rfloor$. The last equality holds because there is no integer between these two arguments to the floor function. (Comment: this lower is the exact answer except for $n = 9, 10$.)

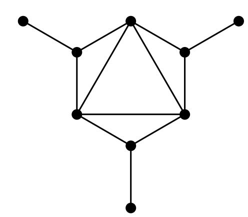
b) A self-complementary planar graph with 8 vertices. See the solution to Exercise 6.1.29 for examples of self-complementary planar graphs with 8 vertices. To show that the thickness of K_8 is 2, it suffices to present any 8-vertex planar graph with a planar complement. Many examples are possible. A natural approach is to use a triangulation to eliminate the most possible edges from the complement. An example is $C_6 \vee 2K_1$, putting one vertex inside and one vertex outside a 6-cycle. The complement is $(C_3 \square K_2) + K_2$, which is planar as shown below.



Since K_8 is nonplanar, these examples show that K_8 has thickness 2, and that in fact K_5, K_6, K_7 also have thickness 2. The bound in (a) implies that the thickness of K_n is at least 3 when $n \ge 11$, which is the same as saying there is no planar graph with more than 10 vertices having a planar complement. In fact, there is also no planar graph on 9 or 10 vertices having a planar complement, but the counting argument in (a) is not strong enough to show that.

6.3.18. *Decomposition of K_9 into three pairwise-isomorphic planar graphs.* View the vertices as the congruence classes of integers modulo 9. Group them into triples by their congruence class modulo 3. The graph below consists of a triangle on one triple, a 6-cycle between that and a second triple, and a matching from the second triple to the third. Rotating the picture on the left yields three pairwise isomorphic graphs decomposing K_9. The drawing on the right shows that the graph is planar.





6.3.19. *The chromatic number of the union of two planar graphs is at most 12.* Let G be a graph with n vertices that is the union of two planar graphs H_1 and H_2. For coloring problems, we may assume that G, H_1, H_2 are simple. We claim that G has a vertex of degree at most 11. Since each H_i has at most $3n-6$ edges, G has at most $6n-12$ edges. The degree-sum in G is at most $12n-24$, and by the pigeonhole principle G has a vertex of degree at most 11.

It now follows by induction on $n(G)$ that $\chi(G) \le 12$. This holds trivially for $n(G) \le 12$. For $n(G) > 12$, we delete a vertex x of degree at most 11 to obtain G'. Since $G' = (H_1 - x) \cup (H_2 - x)$, we can apply the induction hypothesis to G' to obtain a proper 12-coloring. Since $d(x) \le 11$, we can replace x and give it one of these 12 colors to obtain $\chi(G) \le 12$.

The chromatic number of the union of two planar graphs may be as large as 9. The graph $C_5 \vee K_6$ has chromatic number 9, since the three colors on the 5-cycle must be distinct from the six colors on the 6-clique, and such a coloring is proper. It thus suffices to show that $C_5 \vee K_6$ is the union of two planar graphs. Since $C_5 \vee K_6$ contains K_8 and $K_9 - e$ (for some edge e) as induced subgraphs, it is reasonable to start with one of these and then try to add the missing vertices with their desired neighbors to the two graphs.

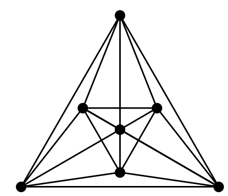
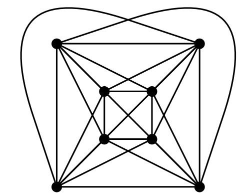
Let the vertices of the C_5 be a, b, c, d, e in order, and let the vertices of the K_6 be $1, 2, 3, 4, 5, 6$. Exercise 6.1.29 requests a self-complementary graph with 8 vertices; in other words, an expression of K_8 as the union of two planar graphs.

6.3.20. *Thickness of $K_{r,s}$.* Let X, Y be the partite sets of $K_{r,s}$, with $|X| = r$. The graph $K_{2,s}$ is planar. Taking all of Y and two vertices from X yields a copy of $K_{2,s}$. Taking two vertices at a time from X thus yields $r/2$ planar subgraphs decomposing $K_{r,s}$.

Since $K_{r,s}$ is triangle-free, a planar subgraph of $K_{r,s}$ has at most $2(r+s)-4$ edges. Thus the number of planar subgraphs needed in a decomposition is at least $\frac{rs}{2r+2s-4} = \frac{r}{2+(2r-4)/s}$. As s increases, the denominator decreases and the quotient increases. Thus when $s > (r-2)^2/2$, the value of the lower bound is larger than the result of setting $s = (r-2)^2/2$ in the formula. Since $(2r-4)/s = 4/(r-2)$ when s has this value,our lower bound is bigger than $\frac{r}{2+4/(r-2)} = \frac{r(r-2)}{2r-4+4} = r/2 - 1$. Since the crossing number is an integer bigger than $r/2 - 1$ when r is even and $s > (r-2)^2/2$, it is at least $r/2$. Hence our construction is optimal.

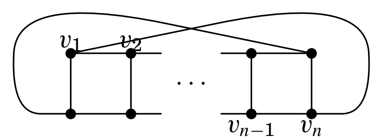
6.3.21. *Crossing number of $K_{1,2,2,2}$.* This simple graph has 7 vertices and 18 edges. The maximum number of edges in a simple planar graph with 7 vertices is $3 \cdot 7 - 6 = 15$. Hence in any drawing of this graph, a maximal plane subgraph has at most 15 edges, and the remaining edges each yield at least one crossing with the maximal plane subgraph. Hence $\nu(K_{1,2,2,2}) \ge 3$, and the drawing of this graph on the left below shows that equality holds.

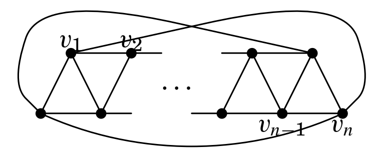
Crossing number of $K_{2,2,2,2}$. Deleting any vertex in a drawing of $K_{2,2,2,2}$ yields a drawing of $K_{1,2,2,2}$, which must have at least 3 crossings. Doing this for each vertex yields a total of at least 24 crossings. Since each crossing is formed by two edges involving 4 vertices, we have counted each crossing $8-4 = 4$ times. Thus the drawing of $K_{2,2,2,2}$ has at least 6 crossings. We have proved that $\nu(K_{1,2,2,2}) \ge 6$, and the drawing of this graph on the right below shows that equality holds.

6.3.22. *$K_{3,2,2}$ has no planar subgraph with 15 edges, and thus $\nu(K_{3,2,2}) \ge 2$.* The graph has 16 edges, so it suffices to show that deleting one edge leaves a nonplanar graph. Let X be the partite set of size 3. Every 6-vertex induced subgraph containing X contains a copy of $K_{3,3}$, which is nonplanar. Since every edge e is incident to a vertex not in X, the 6-vertex induced subgraph avoiding such an endpoint remains when e is deleted.

6.3.23. *The crossing number of the graph M_n obtained from the cycle C_n by adding chords between vertices that are opposite (if n is even) or nearly opposite (if n is odd) is 0 if $n \le 4$ and 1 otherwise.* For $n \le 4$, all n-vertex graphs are planar. For $n = 5$, $M_n = K_5$. For $n \ge 6$, the cycle with vertices $v_1, \dots, v_n$ plus the chords $v_1 v_{1+\lfloor n/2 \rfloor}$, $v_2 v_{2+\lfloor n/2 \rfloor}$, $v_3 v_{3+\lfloor n/2 \rfloor}$ is a subgraph of M_n that is a subdivision of $K_{3,3}$, so the crossing number is at least 1. The drawings below, by avoiding crossings among the chords and allowing a crossing within the drawing of the cycle, show that one crossing is enough.

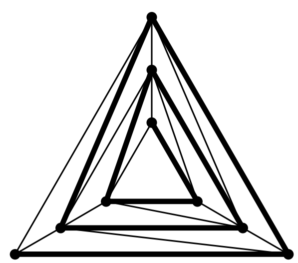




6.3.24. *a) P_n^3 is a maximal planar graph.* The graph P_4^3 is K_4, with six edges. Note that $6 = 3 \cdot 4 - 6$. Each successive vertex in P_n is adjacent to the last three of the earlier vertices, so $e(P_n^3) = 3n - 6$. Together with having $3n - 6$ edges, showing that P_n^3 is planar implies that it is a maximal planar graph. An embedding is obtained by drawing the path in a spiral as suggested below.

Alternatively, we can prove planarity by proving inductively that there is a planar embedding *with all of* $\{n-2, n-1, n\}$ *on the same face*. This holds for an embedding of P_4^3 (we could also start with $n=3$ as the basis). For the induction step ($n>4$), take such an embedding of P_{n-1}^3. Since all of $\{n-3, n-2, n-1\}$ lie on a single face, we can place n in that face and draw edges to all three. This yields a planar embedding of P_n with all of $\{n-2, n-1, n\}$ on a single face.

b) $\nu(P_n^4) = n-4$. The graph P_5^4 is K_5, with 10 edges. Each additional vertex provides four more edges, so $e(P_n^4) = 4n-10$. In any drawing, a maximal plane subgraph H has at most $3n-6$ edges and thus leaves at least $n-4$ edges that each cross an edge of H. That bound is achieved with equality by adding the second diagonals of the trapezoids in the picture below, making each vertex adjacent to the vertex four earlier on the path.



(Alternatively, the earlier induction proof can be strengthened to guarantee an embedding with all of $\{n-2, n-1, n\}$ on a single face and $n-4$ on an adjacent face across the edge joining $n-1$ and $n-3$. That enables the $n-4$ additional edges to be added so that each crosses only the specified edge of P_n^3 and no added edges cross each other.)

6.3.25. *There are toroidal graphs with arbitrarily large crossing number in the plane.* The cartesian product of two cycles, $C_m \square C_n$, embeds naturally on the torus; each face is a 4-cycle. The graph has mn vertices and $2mn$ edges. View the copies of C_m as vertical slices (columns) and the copies of C_n as horizontal slices (rows).

A subgraph of $C_m \square C_n$ consisting of three full columns and three full rows is a subdivision of $C_3 \square C_3$. Since $C_3 \square C_3$ contains a subdivision of $K_{3,3}$, it is nonplanar. Therefore, a planar subgraph of $C_m \square C_n$ cannot contain three full columns and three full rows. This means it must omit at least $\min\{m-2, n-2\}$ edges. By Proposition 6.3.13, $\nu(C_m \square C_n) \ge \min\{m-2, n-2\}$. By making m and n at least $k+2$, we make the crossing number at least k while having a toroidal graph.

6.3.26. *Lower bounds on crossing numbers.* As stated correctly in Example 6.3.15 (not stated correctly in this exercise), the crossing number of $K_{6,n}$ is $6\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor$. *a)* $\nu(K_{m,n}) \ge m\frac{m-1}{5}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. Consider copies of $K_{6,n}$ in a drawing of $K_{m,n}$, with the partite set of size 6 in the subgraph selected from the partite set of size m in the full graph. There are $\binom{m}{6}$ such copies, and each has at least $6\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor$ crossings. Each crossing appears in $\binom{m-2}{4}$ of the subgraphs.

Hence $\nu(K_{m,n}) \ge \binom{m}{6}6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor/\binom{m-2}{4}$. Cancellation of common factors in the numerator and denominator yields the bound claimed.

b) $\nu(K_p) \ge \frac{1}{80}p^4 + O(p^3)$. Consider copies of $K_{6,p-6}$ in a drawing of K_p. There are $\binom{p}{6}$ of these copies, and each has at least $6\lfloor (p-6)/2\rfloor\lfloor (p-7)/2\rfloor$ crossings. Each crossing appears in $4\binom{p-4}{4}$ of these subgraphs, since the four vertices involved in the crossing can contribute to the smaller partite set in four ways (assuming that $n>12$), and then four vertices not involved in the crossing must be chosen to fill that partite set.

Hence $\nu(K_p) \ge \binom{p}{6}6\left\lfloor\frac{p-6}{2}\right\rfloor\left\lfloor\frac{p-7}{2}\right\rfloor/[4\binom{p-4}{4}]$. The numerator has four more linear factors than the denominator, so the growth is quartic. The leading coefficient is $\frac{6}{6!}\frac{1}{2}\frac{1}{2}\frac{4!}{4}$, which simplifies to $1/80$.

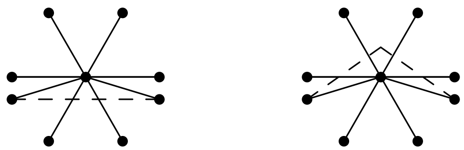
6.3.27. *If the conjecture that* $\nu(K_{m,n}) = \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. *holds for* $K_{m,n}$ *and* m *is odd, then the conjecture holds also for* $K_{m+1,n}$. In a drawing of $K_{m+1,n}$, there are $m+1$ copies of $K_{m,n}$ obtained by deleting a vertex of the partite set of size m. Each crossing in the drawing of $K_{m+1,n}$ appears in $m-1$ of these copies. Hence $(m-1)\nu(K_{m+1,n}) \ge (m+1)\nu(K_{m,n})$.

Since m is odd, $\lfloor m/2\rfloor\lfloor (m-1)/2\rfloor = (m-1)^2/4$, and $\lfloor (m+1)/2\rfloor\lfloor m/2\rfloor = (m+1)(m-1)/4$. Therefore,

$$\nu(K_{m+1,n}) \ge \frac{m+1}{m-1}\nu(K_{m,n}) = \frac{m+1}{2}\frac{m-1}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor = \left\lfloor\frac{m+1}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$$

6.3.28. *If m and n are odd, then in all drawings of $K_{m,n}$, the parity of the number of pairs of edges that cross is the same.* (We consider only drawings where edges cross at most once and edges sharing an endpoint do not cross.) Any drawing of $K_{m,n}$ can be obtained from any other by moving vertices and edges. The pairs of crossing edges change only when an edge e is moved through a vertex v not incident to it (or vice versa). Let S be the set of edges incident to v other than those also incident to endpoints of e. When e is moved through v (or vice versa), the set of edges incident to v that e crosses is exchanged for its complement in S.

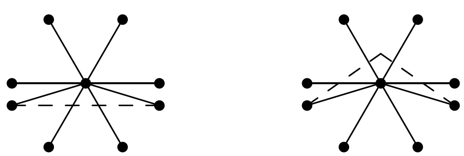
Since the degree of each vertex is odd and v is adjacent to exactly one endpoint of e, the parity of these two sets is the same. Hence the parity of the number of crossings never changes.



If m and n are odd, then $\nu(K_{m,n})$ is odd when $m-3$ and $n-3$ are divisible by 4 and even otherwise. In the naive drawing of $K_{m,n}$ with the vertices on opposite sides of a channel and the edges drawn as segments across the channel, the number of crossings is $\binom{m}{2}\binom{n}{2}$. It suffices to determine the parity of this, since the parity is the same for all other drawings including those with fewest crossings.

The binomial coefficient $r(r-1)/2$ is odd if and only if r is congruent to 2 or 3 modulo 4. Since we require m and n odd, the additional requirement for $\binom{m}{2}\binom{n}{2}$ being odd is thus m and n being congruent to 3 modulo 4.

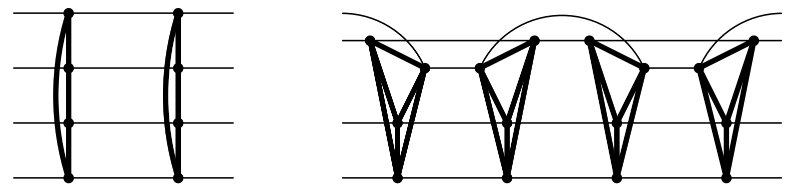
6.3.29. *If n is odd, then in all drawings of K_n, the parity of the number of pairs of edges that cross is the same.* (We consider only drawings where edges cross at most once and edges sharing an endpoint do not cross.) Any drawing of $K_{m,n}$ can be obtained from any other by moving vertices and edges. The crossing pairs change only when an edge e moves through a vertex v not incident to it (or vice versa). Let S be the set of edges incident to v that are not incident to endpoints of e. When e moves through v (or vice versa), the set of edges incident to v that e crosses is exchanged for its complement in S. Since $d(v)$ is even and v is adjacent to both endpoints of e, we have $|S|$ even, so the sizes of complementary subsets of S have the same parity. Hence the parity of the number of crossings does not change.



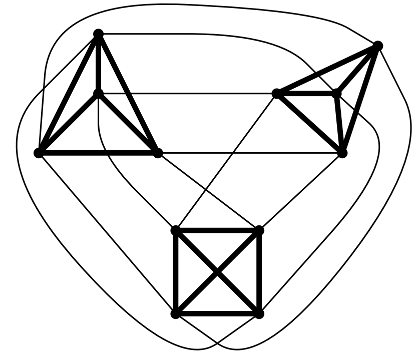
$\nu(K_n)$ is even when n is congruent to 1 or 3 modulo 8 and is odd when n is congruent to 5 or 7 modulo 8. Since the parity is the same in all drawings of K_n, we need only look at one, such as the straight-line drawing with the vertices on a circle. Its number of crossings is $\binom{n}{4}$, which equals $n(n-1)(n-2)(n-3)/24$. When the congruence class is 1 or 3, the numerator has a multiple of 8 and an odd multiple of 2, so it has four factors of 2, and only three are canceled by the denominator. Hence $\binom{n}{4}$ is even. When the class is 5 or 7, the numerator has an odd multiple of 4, an odd multiple of 2, and two odd factors, so the factors of 2 are canceled out, and $\binom{n}{4}$ is odd.

6.3.30. *$\nu(C_m \square C_n) \le (m-2)n$ and $\nu(K_4 \square C_n) \le 3n$.* For $C_m \square C_n$, we draw the copies of C_m along concentric circles. The vertices arising from a single copy of C_n are laid out along a spoke. The "long" edge in each copy of C_n generates $m-2$ crossings, as shown below on the left.

For $K_4 \square C_n$, we make the cycles concentric again, almost: the two outside cycles weave in and out of each other, as shown on the right below. We draw each copy of K_4 around a spoke, but each copy is the reflection of its neighbors. For each copy of K_n, the two outer cycles cross, and the central cycle crosses two edges in that copy of K_n.



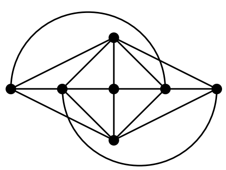
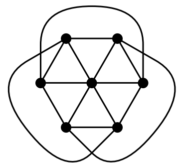
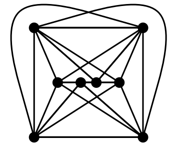
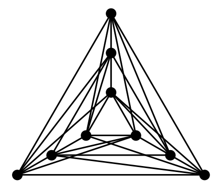
The weaving in and out requires n to be even. When n is odd, a special construction is needed for the case $n = 3$ (shown below); there is one crossing in a copy of K_4, and the inner two triangles and outer two triangle each provide four crossings. Copies of K_4 can then be added in pairs by breaking the four edges joining two "neighboring" copies of K_4 and inserting two copies of K_4 with six crossings as in the middle of the figure above.



6.3.31. *Crossing number of complete tripartite graphs.* Let $f(n) = \nu(K_{n,n,n})$. We prove that $n^3(n-1)/6 \le f(n) \le (9/16)n^4 + O(n^3)$.

a) $3\nu(K_{n,n}) \le f(n) \le 3\binom{n}{2}^2$. The lower bound follows from the existence of three pairwise edge-disjoint copies of $K_{n,n}$ in $K_{n,n,n}$. The upper bound follows from a straight-line drawing with the vertices of each part placed on a ray leaving the origin.

b) $\nu(K_{3,2,2}) = 2$. Lower bounds: Since a triangle-free 7-vertex planar graph has at most $2n-4 = 10$ edges, $\nu(K_{3,4}) \ge 2$, and $K_{3,4}$ is a subgraph of $K_{3,2,2}$. Alternatively, a counting argument for the crossings shows both $\nu(K_{3,2,2}) \ge 2$ and $\nu(K_{3,3,1}) \ge 3$. Consider each vertex-deleted subgraph for some embedding; if it contains $K_{3,3}$, its includes a crossing, and each crossing is counted $n-4 = 3$ times. Hence $\nu(K_{3,2,2}) \ge \lceil 4/3 \rceil = 2$ and $\nu(K_{3,3,1}) \ge \lceil 7/3 \rceil = 3$. Extending this approach yields $\nu(K_{3,3,2}) \ge \lceil 18/4 \rceil = 5$ and $\nu(K_{3,3,3}) \ge \lceil 45/5 \rceil = 9$. Constructions for $\nu(K_{3,2,2}) \le 2$, $\nu(K_{3,3,1}) \le 3$, $\nu(K_{3,3,2}) \le 7$, and $\nu(K_{3,3,3}) \le 15$ appear below.

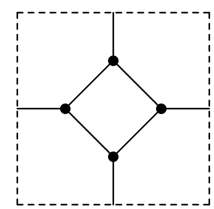
c) Recurrence to improve the lower bound in (a): $K_{n,n,n}$ has n^3 copies of $K_{n-1,n-1,n-1}$. In totalling this, each crossing formed between vertices of two parts counts $n(n-2)^2$ times, but each crossing using vertices of all three parts counts $(n-1)^2(n-2)$ times. We don't know how many crossings of each type there may be, but each crossing is counted at most $(n-1)^2(n-2)$ times, so there are at least $f(n-1)\frac{n^3}{(n-1)^2(n-2)}$ crossings. This recursive bound for $f(n)$ expands to a telescoping product: $f(n) \ge \frac{n^3}{(n-1)^2(n-2)} \frac{(n-1)^3}{(n-2)^2(n-3)} \cdots \frac{5^3}{(5-1)^2(5-2)} \frac{4^3}{(4-1)^2(4-2)} f(3)$. After cancellation, we have $f(n) \ge n^3(n-1)f(3)/54$. Since in (b) we found $f(3) \ge 9$, we at least have $f(n) \ge (1/6)n^4 + O(n^3)$. If in fact $f(3) = 15$, then we have $f(n) \ge (5/18)n^4 + O(n^3)$, which is approximately a factor of 2 from the upper bound below.

d) Improving the upper bound $(3/4)n^4 + O(n^3)$ *of (a) to* $f(n) \le (9/16)n^4 + O(n^3)$. The layout on the tetrahedron splits the n vertices of each part into two sets of size $n/2$ laid out along opposite edges. For the points on a given edge, the four neighboring edges of the tetrahedron contain all points of the other two parts, to which these points have edges laid out directly on the surface of the tetrahedron. Crossings on a face of the tetrahedron are formed by pairs of vertices from two incident edges or by a pair from one edge with one vertex each from the other two edges. If the parts have sizes l, m, n and $l' = \binom{l/2}{2}$, $m' = \binom{m/2}{2}$, $n' = \binom{n/2}{2}$, then the total number of crossings on a single face of the tetrahedron is $[l'm' + l'n' + m'n'] + [l'(m/2)(n/2) + m'(l/2)(n/2) + n'(l/2)(m/2)]$. Over the four identical faces,

this sums to $\frac{1}{16}(l^2m^2 + l^2n^2 + m^2n^2 + 2l^2mn + 2m^2ln + 2n^2lm)$, plus lower order terms. When $l = m = n$, this becomes $(9/16)n^4$.

For the other construction, begin with an optimal drawing of K_n. Turn each vertex into an independent set consisting of one vertex from each part. When there are three parts, each edge of the original drawing has now become a bundle consisting of 6 edges. For each crossing in the drawing of K_n, we get 36 crossings between the two bundles. For each edge in the drawing of K_n, we get at most 15 crossings within the bundle. Near a vertex of the original drawing, we get at most 36 crossings (actually slightly less) between the bundles corresponding to incident edges. There are $\binom{n}{2}$ edges in K_n, and $n\binom{n-1}{2}$ pairs of incident edges, but always $\Omega(n^4)$ crossings, so the other contributions are of smaller order. Therefore, we have only $36\nu(K_n) + O(n^3)$ crossings. With the best known bound of $\nu(K_n) \le n^4/64 + O(n^3)$, we get the same constant $9/16$. This generalizes easily to $\binom{r}{2}^2/16n^4$ for the complete multipartite graph with r parts of size n.

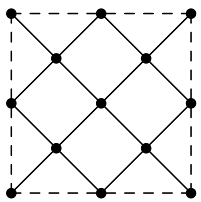
6.3.32. *An embedding of a 3-regular nonbipartite simple graph on the torus such that every face has even length.* It suffices to use K_4 as shown below. Larger examples can be obtained from this.

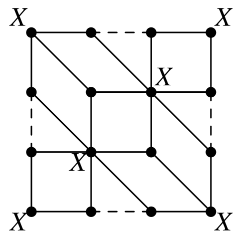


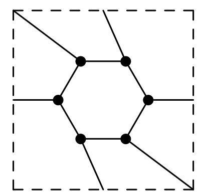
6.3.33. *If n is at least 9 and n is not a prime or twice a prime, then there is a 6-regular toroidal graph with n vertices.* Given these conditions, express n as rs with r and s both at least 3. Now form $C_r \square C_s$; this 4-regular graph embeds naturally on the torus with each face having length 4. On the combinatorial description of the torus as a rectangle, the embedding looks like the interior of $P_{r+1} \square P_{s+1}$, but the top and bottom rows of vertices are the same, and the left and right columns of vertices are the same.

Now add a chord in each face from its lower left corner in this picture to its upper right corner. The resulting graph is 6-regular, toroidal, and has n vertices.

6.3.34. *Regular embeddings of $K_{4,4}$, $K_{3,6}$, and $K_{3,3}$ on the torus.* The number of faces times the face-length is twice the number of edges, and the number of faces is the number of edges minus the number of vertices. For $K_{4,4}$, we need eight 4-faces. For $K_{3,6}$, we need nine 4-faces. For $K_{3,3}$, we need three 6-faces. Such embeddings appear below.

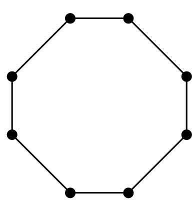




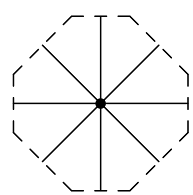


6.3.35. *Euler's Formula for genus γ: For every 2-cell embedding of a graph on a surface with genus γ, the numbers of vertices, edges, and faces satisfy $n - e + f = 2 - 2\gamma$.* We use induction on $e(G)$ via contraction of edges. For the basis step, we need the number of edges required to cut S_γ into 2-cells. Each face in an embedding is a 2-cell; lay it flat. Combining neighboring faces along shared edges yields a large 2-cell R. Identifying shared edges reassembles the surface. The number of edge-pairs needed on the boundary of R is at least the number of cuts required to lay the surface flat, because that is what these boundary edges do.

It takes two cuts to lay a handle flat. If every cut is one edge, then every cut is a loop and there is only one vertex. So, the only 2-cell embeddable graphs on S_γ that have at most 2γ edges are those with 1 vertex and 2γ edges, and the resulting embeddings have 1 face. The polygonal representation of the surface is itself such an embedding, if we view the vertices of the polygonal as copies of the single vertex in the graph, and the edges of the polygon as paired loops. Since $1 - 2\gamma + 1 = 2 - 2\gamma$, all is well.



or

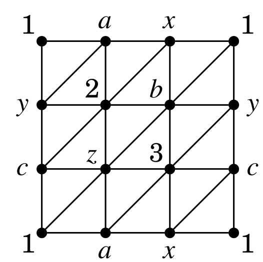


Given a 2-cell embedding with more than 2γ edges, contract an edge e that is not a loop surrounding another loop of the embedding. If e is not a loop, then following the boundaries of the face(s) bounded by e shows they are still 2-cells, and we now have a 2-cell embedding of $G \cdot e$ on the same surface. The induction hypothesis provides Euler's Formula for $G \cdot e$. Since G has one more vertex and edge than $G \cdot e$ but the same number of faces, Euler's Formula holds also for G. On the other hand, if e is a loop, then G has one more edge and face than $G \cdot e$ but the same number of vertices; again the formula holds.

An n-vertex simple graph embeddable on S_γ has at most $3(n - 2 + 2\gamma)$ edges. If G embeds on S_γ, then G has a 2-cell embedding on $S_{\gamma'}$ for some γ' with $\gamma' \le \gamma$, so we may assume that G has a 2-cell embedding on S_γ. Since the sum of the face-lengths is $2e(G)$ and each face in an embedding of a simple graph has length at least three, we have $2e \ge 3f$. Substituting in Euler's Formula $n - e + f = 2 - 2\gamma$ yields $e \le 3(n - 2 + 2\gamma)$.

6.3.36. *Genus of $K_{3,3,n}$.* Since $K_{3,3,n}$ has $n + 6$ vertices and $6n + 9$ edges, Euler's formula yields $\gamma(K_{3,3,n}) \ge 1 + (6n + 9 - 3n - 18)/6 = (n-1)/2$. This can be improved by apply Euler's Formula to the bipartite subgraph $K_{6,n}$. Here the genus is at least $1 + (6n - 2n - 12)/4$, which simplifies to $n - 2$.

For $0 \le n \le 3$, the genus is 1, since $K_{3,3}$ is nonplanar and $K_{3,3,3}$ embeds in the torus. Such an embedding is obtained by adding vertices for the third part in the faces of a regular embedding of $K_{3,3}$ as found in Exercise 6.3.34 (each of the three faces in the regular embedding is incident to all six vertices of $K_{3,3}$). Alternatively, a pleasing triangular embedding of $K_{3,3,3}$ can be found directly as shown below.

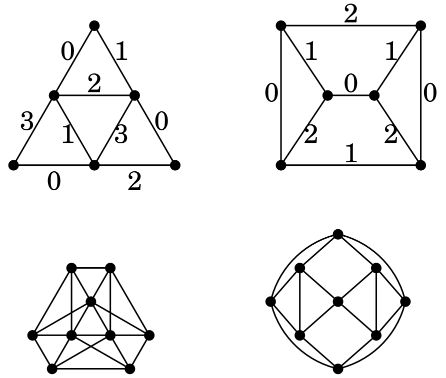


6.3.37. *For every positive integer k, there exists a planar graph G such that $\gamma(G \,\square\, K_2) \ge k$.* Let $H = G \,\square\, K_2$. The definition of cartesian product yields $n(H) = 2n(G)$ and $e(H) = 2e(G) + n(G)$. Since an n-vertex graph embeddable on S_γ has at most $3(n - 2 + 2\gamma)$ edges, we have $\gamma(H) \ge 1 + (e(H) - 3n(H))/6 = 1 + (2e(G) - 5n(G))/6$. If G is a triangulation with n vertices, then $e(G) = 3n - 6$, and we obtain $\gamma(H) \ge -1 + n/12$. It suffices to choose $n \ge 12k + 12$.

7.EDGES AND CYCLES

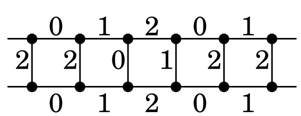
7.1. LINE GRAPHS & EDGE COLORING

7.1.1. *Edge-chromatic number and line graph for the two graphs below.* The labelings are proper edge colorings, the number of colors used is the maximum degree, so the colorings are optimal.



7.1.2. $\chi'(Q_k) = \Delta(Q_k)$, *by explicit coloring.* In the cube Q_k, the edges between vertices differing in coordinate j form a complete matching. Over the k choices of j, these partition the edges.

7.1.3. $\chi'(C_n \,\square\, K_2) = 3$. The lower bound is given by the maximum degree. For the upper bound, when n is even colors 0 and 1 can alternate along the two cycles, with color 2 appearing on the edges between the two copies of the factor C_n. When n is odd, colors 0 and 1 can alternate in this way except for the use of one 2. Color 2 appears on all cross edges except those incident to edges on the cycles with color 2, as shown below.

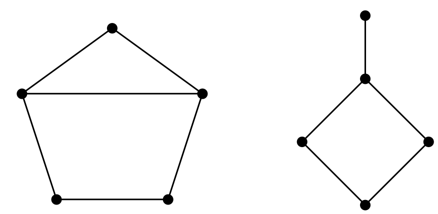


7.1.4. *For every graph G, $\chi'(G) \ge e(G)/\alpha'(G)$.* In a proper edge-coloring, each color class has at most $\alpha'(G)$ edges. The lower bound follows because all $e(G)$ edges must be colored.

7.1.5. *The Petersen graph is the complement of $L(K_5)$.* The vertices of $L(K_5)$ are the edges in K_5, which can be named as the 2-element subsets of [5]. Two such pairs are adjacent in the Petersen graph if and only if they are disjoint, which is the condition for them being nonadjacent in $L(K_5)$.

7.1.6. *The line graph of the Petersen graph has 10 triangles.* For a simple graph G, there is a triangle in $L(G)$ for every set of three edges in G that share one common endpoint and for every set of three edges that form a triangle in G. The Petersen graph has no triangles, so the latter type does not arise. However, the Petersen graph has 10 triples of edges with a common endpoint, one for each of its vertices.

7.1.7. *$\overline{P}_5$ is a line graph.* The complement of P_5 is a 5-cycle with a chord. It is the line graph of a 4-cycle with a pendant edge.



$\overline{P}_5 = L(H)$ $\qquad$ H

7.1.8. *The line graph of $K_{m,n}$ is the cartesian product of K_m and K_n.* For each edge $x_i y_j$ in $K_{m,n}$, we have a vertex (i,j) in $L(K_{m,n})$; these are also the vertices of $K_m \,\square\, K_n$. Pairs (i,j) and (k,l) in $V(K_m \,\square\, K_n)$ are adjacent in $K_m \,\square\, K_n$ if and only if $i = k$ or $j = l$. This is the same as the condition for adjacency in $L(K_{m,n})$, because $x_i y_j$ and $x_k y_l$ share an endpoint in $K_{m,n}$ if and only if $i = k$ or $j = l$.

7.1.9. *A set of vertices in the line graph of a simple graph G form a clique if and only if the corresponding edges in G have one common endpoint or form a triangle.* Let S be the corresponding set of edges in G, and choose $e \in S$. If all other elements of S intersect e at the same endpoint of e, we have one common endpoint. Otherwise, we have edges f and g such that f shares endpoint x with e and g shares endpoint y with e. Since f and g must share an endpoint, they share their other endpoint z and complete a triangle. Since no single vertex lies in all of e, f, g, no additional edge of the simple graph G can share a vertex with all of these.

7.1.10. *If $L(G)$ is connected and regular, then either G is regular or G is a bipartite graph in which vertices of the same partite set have the same degree.* If $L(G)$ is connected, then G is connected (except for isolated vertices, which we ignore). For $e = uv \in E(G)$, the degree of e in $L(G)$ is $d(u) + d(v) - 2$. If the edges incident to v in G have the same degree in $L(G)$, then they must join v to vertices of the same degree in G.

If G is not regular, then G has adjacent vertices u, v with different degrees, since G is connected. By the observation above about maintaining constant degree, every walk from v in G must alternate between vertices of degrees $d(v)$ and $d(u)$. Thus G has no odd walk and is bipartite. Furthermore, the vertices of one partite set have degree $d(v)$, and those of the other partite set have degree $d(u)$.

7.1.11. *Line graphs of simple graphs.*

a) $e(L(G)) = \sum_{v\in V(G)} \binom{d(v)}{2}$.

Proof 1 (bijective argument). The edges of $L(G)$ correspond to the incident pairs of edges in G. Such pairs share exactly one vertex, and each vertex $v \in V(G)$ contributes exactly $\binom{d(v)}{2}$ such incident pairs.

Proof 2 (degree-sum formula). The degree in $L(G)$ of the vertex corresponding to $uv \in E(G)$ is $d_G(u) + d_G(v) - 2$, the number of edges of G sharing an endpoint with it. When this is summed over all edges of G, the term $d_G(u)$ appears $d_G(u)$ times. Hence the degree sum in $L(G)$ is $\sum d(v)^2 - 2e(G)$, and $L(G)$ has $\frac{1}{2}\sum d(u)^2 - e(G)$ edges. Replacing $e(G)$ by $\frac{1}{2}\sum d_i$ yields $\sum \binom{d(v)}{2}$.

(Comment: The formula holds also for graphs with multiple edges under the convention that when edges share both endpoints we have two edges between the corresponding vertices of $L(G)$.)

b) G is isomorphic to $L(G)$ if and only if G is 2-regular.

Sufficiency. A 2-regular graph is a disjoint union of cycles. The line graph of any cycle is a cycle of the same length (successive edges on a cycle in G turn into successive vertices on a cycle in $L(G)$).

Necessity.

Proof 1 (numerical argument). If G is isomorphic to $L(G)$, then $L(G)$ has the same number of vertices and edges as G. Thus $n(G) = n(L(G)) = e(G) = e(L(G))$. By (a), this becomes $n(G) = \sum_{v\in V(G)} \binom{d(v)}{2}$. Using the degree-sum formula, $\sum d(v) = 2e(G) = 2n(G)$.

We have shown that the average degree is 2. When the degrees all equal 2, the sum $\sum \binom{d(v)}{2}$ equals $n(G)$, as desired. It suffices to show that when the average degree is 2 but the individual degrees do not all equal 2, $\sum \binom{d(v)}{2}$ is larger than $n(G)$.

In this case, there is at least one number bigger than 2 (the average) and one smaller than 2. Since $\binom{r}{2} + \binom{s}{2} > \binom{r-1}{2} + \binom{s+1}{2}$ when $r > s+1$, we can iteratively bring the values toward the average while always decreasing $\sum \binom{d(v)}{2}$. Hence the equality $n(G) = \sum \binom{d(v)}{2}$ is achieved only when every vertex degree is 2.

(Comment: This is the discrete version of a calculus argument. Because $\binom{x}{2}$ is quadratic in x with positive leading coefficient, it is convex. For a convex function, the sum of values at a set of n arguments with fixed sum s is minimized by setting each argument to s/n.)

Proof 2 (graph structure). As above, $n(G) = \sum \binom{d(v)}{2}$. If all degrees are at least 2, then equality holds only when all equal 2. Hence it suffices to forbid vertices of degree less than 2.

For a graph H, observe that $L(H)$ is a path if and only if H is a nontrivial path. If G has any component that is a path, then let k be the maximum number of vertices in such a component. In $L(G)$ there is no component isomorphic to P_k. Hence G does not have a component that is a path. In particular, G has no isolated vertex.

Suppose that G has a path $(v_0, \ldots, v_l)$ such that $d(v_0) \ge 3$, $d(v_l) = 1$, and internal vertices have degree 2. Let $e_1, \ldots, e_l$ be the edges of P. In $L(G)$, the vertices $e_1, \ldots, e_l$ form a path such that $d(e_1) \ge 3$, $d(e_l) = 1$, and internal vertices have degree 2. This path is shorter than P. Also, a pendant path in $L(G)$ can only arise in this way.

Let m be the maximum length of a path from a vertex of degree at least 3 through vertices of degree 2 to a vertex of degree 1. By the reasoning above, $L(G)$ has no such path of length m. Hence $L(G)$ cannot be isomorphic to G if G has a vertex of degree 1.

7.1.12. *If G is a connected simple n-vertex graph, then $e(L(G)) < e(G)$ if and only if G is a path.* In the preceding problem, it is shown that $e(L(G)) = \sum_{v\in v(G)} \binom{d(v)}{2}$ and that this is numerically minimized when each $d(v)$ is $2e(G)/n$. Hence we require $n\binom{2e(G)/n}{2} < e(G)$, which simplifies to $e(G) < n$. This holds if and only if G is a tree.

Hence it is necessary that G be a tree, but this is not sufficient, because the degrees may be far from equal (consider $L(K_{1,n-1}$, for example). If G has k leaves, then these contribute 0 to $\sum \binom{d(v)}{2}$. The sum of the other vertex degrees is $2n-2-k$. Again the sum is smallest when these degrees are equal. The requirement for the edge inequality becomes $(n-k)\binom{\frac{2n-2-k}{n-k}}{2} < n-1$. Since $\frac{2n-2-k}{n-k} - 1 = \frac{n-2}{n-k}$, the inequality simplifies to $(2n-2-k)\left(\frac{n-2}{n-k}\right) < 2n-2$, which further simplifies to $(2n-2)\left(\frac{n-2}{n-k} - 1\right) < k\left(\frac{n-2}{n-k}\right)$ and eventually to $k < 4(n-1)/n$.

We conclude that $k \le 3$. If a tree has three leaves, then its actual degree list must consist of one 3, three 1s, and the rest 2s, and $L(G)$ has exactly $n-1$ edges. Hence the only graphs with $e(L(G)) < e(G)$ are paths.

7.1.13. *If G is a simple graph such that $\overline{G} \cong L(G)$, then G is C_5 or the graph consisting of a triangle plus a matching from the triangle to an independent 3-set (shown on the right below).* Since $\overline{G}$ and $L(G)$ have the same number of vertices, $e(G) = n(G)$. Also G has only one nontrivial component, since otherwise $\overline{G}$ would be connected while $L(G)$ would not.

If G has an isolated vertex, then $\overline{G}$ and hence $L(G)$ has a dominating vertex. Hence G has an edge xy incident to all other edges. Since there are $n(G)$ edges, the number of common neighbors of x and y is one more than the number of common nonneighbors. Hence G has a triangle, which means that $\overline{G}$ and hence $L(G)$ has an independent set of size 3. Hence G has three disjoint edges, which contradicts that every edge contains x or y.

Thus we may assume that G is connected. Since $e(G) = n(G)$, there is exactly one cycle in G (since deleting an edge of a cycle, which must exist, leaves a tree). Also, $e(G) = n(G)$ implies that the average vertex degree is 2. If G is 2-regular, then $G \cong L(G) \cong \overline{G}$. The only 2-regular graph isomorphic to its complement is C_5.

Otherwise, G has a vertex of degree 1. Thus $\overline{G}$ and hence $L(G)$ has a vertex adjacent to all but one other vertex. Such a vertex in $L(G)$ corresponds to an edge in G that is incident to all but one other edge; we call such an edge *semidominant*. Indeed, we have argued that the number of semidominant edges in G equals the number of vertices of degree 1. Thus G has a semidominant edge xy. Since G is unicyclic, x and y have at most one common neighbor.

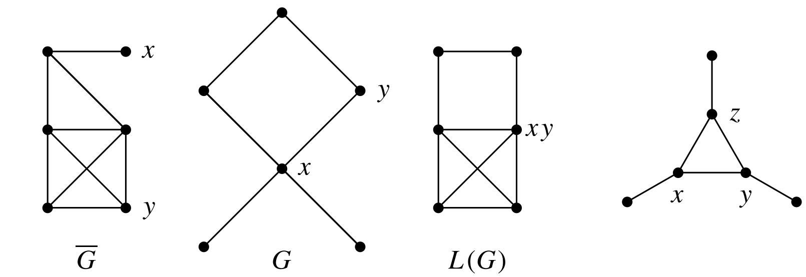
First suppose that x and y have no common neighbor. The one edge not incident to xy now joins neighbors of x and y (creating a 4-cycle) or joins (by symmetry) two neighbors of x.

In the 4-cycle case, semidominant edges other than xy must be on the 4-cycle and incident to xy, and these work only if there are no pendant edges incident to the opposite side of the 4-cycle. Thus there cannot be pendant edges at both x and y, and there must be exactly two pendant edges at one of them, say x. We have now specified G completely, as shown below, but its complement and line graph are not isomorphic.

On the other hand, if an edge zw joins two neighbors of x, all semidominant edges other than xy are incident to x. Since xz and xw are incident to all edges except the pendant edges at y, there must be at least one pendant edge at y. If there is exactly one such edge, then xz and xw are semidominant. If there is more than one, then xy is the only semidominant edge. Both possibilities contradict the equality between the number of semidominant edges and the number of vertices of degree 1.

In the remaining case, x and y have a common neighbor, z. The one edge not incident to xy is incident to z, since G is unicyclic. Since G has only one triangle, $\overline{G}$ and $L(G)$ have only one independent set of size 3.

Hence there is only one way to choose three disjoint edges in G. Hence x and y, like z, have only one neighbor each of degree 1, and the graph is as shown on the right below.



7.1.14. *Connectivity and edge-connectivity of line graphs of k-edge-connected graphs.* Suppose $L(G)$ has a separating t-set S. Then S corresponds to a set of t edges whose deletion disconnects G, because the line graph of a connected graph is connected. Therefore $t \ge k$. This can also be proved using edge-disjoint paths in G, but not as cleanly.

Now consider edge-connectivity. Since $\delta(G) \ge k$, we have $\delta(L(G)) \ge 2k-2$, since each edge is incident to at least $k-1$ others at each endpoint. Let $[T, T']$ be a minimum edge cut of $L(G)$, with κ' edges. Because a minimum edge cut yields only two components, T, T' corresponds to a partition of $E(G)$ into two connected subgraphs, which we call F, F', respectively. There is an edge of $L(G)$ in $[T, T']$ each time an edge of F is incident to an edge of F'.

These incidences take place at vertices of G. At a vertex $x \in V(G)$, there are $d_F(x)$ edges of F (corresponding to vertices of T) and $d_{F'}(x)$ edges of F (corresponding to vertices of T'). Since each such edge of F is incident to each such vertex of F', this vertex x in G yields $d_F(x)d_{F'}(x)$ edges in $[T, T']$. Since $d_F(x) + d_{F'}(x) = d_G(x) \ge k$, this product is at least $k-1$ whenever x is incident to edges of both F and F'.

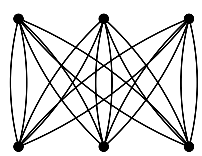
Hence it suffices to show that there are at least two vertices of G that are incident to edges from both F and F'. If F and F' are incident at only one vertex x, then this must be a cut-vertex of G, because any path from F to F' that avoids x would yield another vertex where F and F' are incident. Deleting the edges of F incident to x or the edges of F' incident to x disconnects G. Since G is k-edge-connected, we conclude that $d_F(x), d_{F'}(x)\ H \ge k$ and $\left|[T, T']\right| \ge k^2 > 2k-2$.

7.1.15. *Every connected line graph of even order has a perfect matching.* Note that a graph without isolated vertices has the same number of components as its line graph. Let S' be the set of edges in G corresponding to a set $S \subseteq V(L(G))$. Deleting S from $L(G)$ corresponds to deleting S' from G, but each edge deletion increases the number of components by at most one. Thus $G - S'$ (and $L(G) - S$) have at most $1 + |S|$ components of any sort, odd or otherwise. For a graph of even order, $o(L(G) - S) \le 1 + |S|$ implies Tutte's condition $o(L(G) - S) \le |S|$, since the order is even.

The edges of a connected simple graph of even size can be partitioned into paths of length two. The paired vertices of a perfect matching in $L(G)$ correspond in G to paired edges forming paths of length 2. Since the matching saturates $V(L(G))$, the corresponding paths partition $E(G)$.

7.1.16. *If G is a simple graph, then $\gamma(L(G)) \ge \gamma(G)$, where $\gamma(G)$ denotes the genus of G (Definition 6.3.20).* Consider an embedding of $L(G)$ on a surface S; it suffices to obtain an embedding of G on the same surface. For each $x \in V(G)$, the edges of G with endpoint x form a clique Q_x in $L(G)$. For the embedding of G, locate x at one vertex xx' of Q_x in the embedding of $L(G)$. For each edge xy, embed it along the path in the embedding of $L(G)$ from xx' to xy to yy'. Since xy is used in only one such path, the edges of the new embedding of G on this surface have no crossings.

7.1.17. *The number of proper 6-edge-colorings of the graph below (from a specified set of six colors) is* $900 \cdot 512$.



It suffices to count the ways to assign pairs of colors to the double edges so that the pairs at two double edges with a common endpoint are disjoint, because we can then multiply by 2^9 to assign the colors within the pairs.

We can view such an assignment as a 3-by-3 matrix in which the entry in position (i, j) is the pair assigned to the two edges joining the ith top vertex and the jth bottom vertex. Each color must appear exactly once in some pair in each row and each column. We can choose entry $(1, 1)$ in $\binom{6}{2}$ ways, and for each such way there are $\binom{4}{2}$ choices for entry $(1, 2)$. Thus we can choose the first row in 90 ways, and for each way the number of completions will be the same. Let the pairs in the first row be $\{a, b\}$, $\{c, d\}$, and $\{e, f\}$, in order.

If entry $(2, 1)$ is one of the pairs in the first row, then we have two such pairs to choose from. By symmetry, let it be $\{c, d\}$. Now entry $(2, 2)$ must be $\{e, f\}$, and entry $(2, 3)$ is $\{a, b\}$, and the bottom row is determined.

If entry $(2, 1)$ is not one of the pairs in the first row, then we fill it using one element from entry $(1, 2)$ and one element from entry $(1, 3)$; these can be chosen in 4 ways. For example, suppose that entry $(2, 1)$ is $\{c, e\}$. Now f must appear in entry $(2, 2)$ and d in entry $(2, 3)$, and the second row is completed by deciding which of $\{a, b\}$ goes into entry $(2, 2)$ and which goes into entry $(2, 3)$. There are two ways to make this choice, and again the bottom row is determined.

Thus after choosing the first row, there are two ways to complete the matrix with entry $(2, 1)$ not being a pair from the first row. Since there are two ways when entry $(2, 1)$ is a pair from the first row, the total number of colorings is $10 \cdot 90 \cdot 2^9$, as claimed.

7.1.18. $\chi'(K_{m,n}) = \Delta(K_{m,n})$, *by explicit coloring.* We may assume that $m \le n$, so the maximum degree is n. If the vertices are $X \cup Y$ with $X = x_1, \dots, x_m$ and $Y = y_1, \dots, y_n$, we give the edge $x_i y_j$ the color $i + j \pmod n$. Since incident edges differ in the index of the vertex in X or the vertex in Y, they receive different colors.

7.1.19. *Every simple bipartite graph G has a $\Delta(G)$-regular simple bipartite supergraph.* Let $k = \Delta(G)$, and let X and Y be the partite sets of G.

Construction 1. A huge simple k-regular supergraph of G can be constructed iteratively as follows: If G is not regular, add a vertex to X for each vertex of Y and a vertex to Y for each vertex of X. On the new vertices, construct another copy of G. For each vertex in G with degree less than k, join its two copies in the new graph to get G'. Now k is the same as before, the minimum degree has increased by one, and G' is a supergraph of G. Iterating this $k - \delta(G)$ times yields the desired simple supergraph H. It is connected if G was connected.

Construction 2. We may assume that $|X| = |Y|$ by adding vertices to the smaller side, if necessary. Let $M = nk - \sum_i d(x_i)$; this is the total "missing degree". Add M vertices to both X and Y, and place a $(k-1)$-regular graph H on these, which may be constructed using successively tilted matchings as in the natural 1-factorization of $K_{M,M}$. Now add edges joining deficient vertices of X and Y to vertices of H on the opposite side. Each vertex of H receives one such edge, which remedies the M deficiencies in each of X and Y.

7.1.20. *Edge-coloring of digraphs.* Given a digraph D with indegrees and outdegrees at most d, form a bipartite graph H as follows. The partite sets are $A = \{x^- \colon x \in V(D)\}$ and $B = \{x^+ \colon x \in V(D)\}$. For each edge xy in D, place an edge $x^- y^+$ in H; the vertex x^- inherits the edges exiting x and the vertex x^+ inherits the edges entering x. The resulting bipartite graph H is the "split" of D (Section 1.4).

Since the maximum number of edges entering or exiting a vertex of D is d, $\Delta(H) = d$. Since H is bipartite, $\chi'(H) = d$. The d-edge-coloring on the edges of H is the desired coloring of the corresponding edges in D.

7.1.21. *Algorithmic proof of $\chi'(G) = \Delta(G)$ for bipartite graphs.* Let G be a bipartite graph with maximum degree k. Let f be a proper k-edge-coloring of a subgraph H of G. Let uv be an edge of G not in H. We produce a proper $\Delta(G)$-edge-coloring of the subgraph consisting of H plus the edge uv.

Since uv is uncolored, among the $\Delta(G)$ available colors there is a color α not used at u. Similarly, some color β is not used at v. If α is missing at v or β at u, then we can extend the coloring to uv using α or β. Otherwise, follow the path P from u that alternates in colors α and β. The path is well-defined, since each color appears at most once at each vertex.

Since α does not appear at u, the path P ends somewhere and does not complete a cycle. The path reaches the partite set of v along edges of color β, and it reaches the partite set of u along edges of color α. Hence P cannot reach v, where β is missing. We can now interchange colors α and β on the edges of P to make β available for the edge uv.

7.1.22. *If G is a simple graph with maximum degree 3, then $\chi'(G) \le 4$.* Let $H = L(G)$; since $\chi'(G) = \chi(L(G))$, we seek a bound on $\chi(H)$. By making the same argument for each component, we may assume that G and H are connected. Since $\Delta(G) = 3$, an edge of G intersects at most two other edges at each end, and hence $\Delta(H) \le 4$. If H is 4-regular, then G must be 3-regular. The smallest 3-regular simple graph has 6 edges, so $H \ne K_5$.

By Brooks' Theorem, $\chi(H) \le \Delta(H)$ if H is not a clique or odd cycle. If H is an odd cycle, then $\chi(H) \le 3$. If H is a clique, then it has at most 4 vertices. Otherwise, $x'(G) = \chi(H) \le \Delta(H) \le 4$. (Note: when $\Delta = 3$, $\Delta + 1 = 2(\Delta - 1)$. For larger Δ, Brooks' Theorem is not strong enough to prove $\chi'(G) \le \Delta(G) + 1$.)

7.1.23. *1-factorization of $K(p,q)$, the complete p-partite graph with q vertices in each partite set.* With $G[H]$ denoting composition, we have $K(p,q) = K(p,d)[\overline{K}_{q/d}]$ when d divides q.

a) If G decomposes into copies of F, then $G[\overline{K}_m]$ decomposes into copies of $F[\overline{K}_m]$. Expanding a copy of F in the decomposition of G into a copy of $F[\overline{K}_m]$ uses precisely the copies in $G[\overline{K}_m]$ of the edges in that copy of F. Thus these expansions exhaust the copies in $G[\overline{K}_m]$ of edges in G. Since $\overline{K}_m$ is an independent set, there are no other edges to consider.

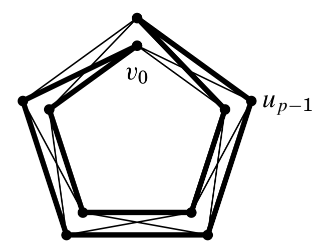
The relation "G decomposes into spanning copies of F" is transitive. If G decomposes into spanning copies of F and H decomposes into spanning copies of G, then the F-decomposition of G can be used on each graph in a G-decomposition of H to decompose H into spanning copies of F.

b) $K(p,q)$ decomposes into 1-factors when pq is even. When p is even, K_p has a 1-factorization - a decomposition into copies of $(p/2)K_2$. By part (a), $K_p[\overline{K}_q]$ decomposes into spanning copies of $(p/2)K_2[\overline{K}_q]$, which equals $(p/2)K_{q,q}$. Since $(p/2)K_{q,q}$ is a regular bipartite graph, it has a 1-factorization. By transitivity, $K(p,q)$ also has a 1-factorization.

When p is odd, we have q even, and thus $K(p,q) = K(p,2)[\overline{K}_{q/2}]$. If $K(p,2)$ has a 1-factorization (into spanning copies of pK_2), then we decompose $K(p,q)$ into spanning copies of $pK_{q/2,q/2}$ and obtain a 1-factorization of $K(p,q)$ by transitivity.

It remains only to decompose $K(p,2)$ into 1-factors when p is odd. Cliques of odd order decompose into spanning cycles; thus it suffices to decompose $C_p[\overline{K}_2]$ into 1-factors. Since this 4-regular graph has an even number of vertices ($2p$), it suffices to decompose it into two spanning cycles. Let the vertices be $v_0, \ldots, v_{p-1}$ and $u_0, \ldots, u_{p-1}$, with $\{u_i, v_i\} \leftrightarrow \{u_{i+1}, v_{i+1}\}$ (indices modulo p). The two desired cycles are

$$(v_0, \ldots, v_{p-1}, u_0, u_{p-1}, \ldots, u_1)$$
$$(u_0, v_1, u_2, \ldots, u_{p-1}, v_0, v_{p-1}, u_{p-2}, \ldots, v_2, u_1).$$



7.1.24. *If $\chi'(H) = \Delta(H)$, then $\chi'(G \square H) = \Delta(G \square H)$.* The graph $G \square H$ consists of a copy of G for each vertex of H and a copy of H for each vertex of G. A vertex $(u,v) \in G \square H$ has neighbors (u,v') for every $v' \in N_H(v)$ and (u',v) for every $u' \in N_G(u)$. Hence $d_{G \square H}(u,v) = d_G(u) + d_H(v)$. With $u \in G$ and $v \in H$ having maximum degree, we obtain $\Delta(G \square H) = \Delta(G) + \Delta(H)$. We construct a proper $(\Delta(G) + \Delta(H))$-edge-coloring.

Use the same proper $(\Delta(G)+1)$-edge-coloring (guaranteed by Vizing's Theorem) on each copy of G. With $\Delta(G) + 1$ colors allowed, some single color i is missing from all edges incident to all copies of the vertex $u \in V(G)$. To color the copy of H on the vertices with first coordinate u, we need only $\Delta(H)$ colors. We use color i and $\Delta(H) - 1$ additional colors. Doing this for each $u \in V(G)$ constructs a proper edge-coloring of $G \square H$ with $\Delta(G) + 1 + \Delta(H) - 1 = \Delta(G \square H)$ colors.

7.1.25. *Kotzig's Theorem on Cartesian products.*

a) $\chi'(G \square K_2) = \Delta(G \square K_2)$. By Vizing's Theorem, we can properly color $E(G)$ with $\Delta(G) + 1$ colors. Use a single such coloring on both copies of G. The two copies of a vertex of G are joined by an edge, but both are missing the same color i in the coloring of G, so color i can be assigned to the edge between them. Hence $G \times K_2$ is $(\Delta(G)+1)$-edge-colorable. We cannot properly color $E(G \times K_2)$ with $\Delta(G)$ colors, because $\chi'(G \square K_2) \ge \Delta(G \square K_2) = \Delta(G)+1$.

b) If G_1, G_2 are edge-disjoint graphs with the same vertex set and H_1, H_2 are edge-disjoint graphs with the same vertex set, then $(G_1 \cup G_2) \square (H_1 \cup H_2) = (G_1 \square H_2) \cup (G_2 \square H_1)$. We view G_1 and G_2 as a red/blue edge-coloring of $G_1 \cup G_2$, and we view H_1 and H_2 as a yellow/green edge-coloring of H. Since every edge of $G \square H$ is a copy of an edge of G or H, this induces a red/blue/yellow/green edge-coloring of the product. The spanning subgraph containing the red and green edges is $G_1 \square H_2$, and the spanning subgraph containing the blue and yellow edges is $G_2 \square H_1$.

c) $G \square H$ is 1-factorable if G and H each have a 1-factor. Let G_1 be a 1-factor of G, $G_2 = G - E(G_1)$, H_1 a 1-factor of H, and $H_2 = H - E(H_1)$. Since $H_1 = mK_2$, we have $G_2 \square H_1 = G_2 \square mK_2 = m(G_2 \square K_2)$. By part (a), there is a proper edge-coloring of $G_2 \square H_1$ with $\Delta(G_2)+1 = \Delta(G)$ colors. Similarly, there is a proper edge-coloring of $G_1 \square H_2$ with $\Delta(H)$ colors. By part (b), these together yield a proper edge-coloring of $G \square H$ with $\Delta(G)+\Delta(H) = \Delta(G \square H)$ colors. (This result is Kotzig's Theorem, usually stated for regular graphs; the proof is from the thesis of J. George.)

7.1.26. *If G is a regular graph with a cut-vertex x, then $\chi'(G) > \Delta(G)$.*

Proof 1. Because G is regular, $\chi'(G) = \Delta(G)$ requires that each color class be a 1-factor. Hence $n(G)$ is even. Since $n(G)-1$ is odd, $G - x$ has a component H of odd order. Let y be a neighbor of x not in H. A 1-factor of G that contains xy must contain a 1-factor of H, which is impossible since H has odd order.

Proof 2. Again each color class must be a 1-factor. Let M_1 and M_2 be color classes containing edges incident to x whose other endpoints are in different components of $G - x$. Since these are perfect matchings, their symmetric difference consists of isolated vertices and even cycles. In particular, it contains a cycle through x that visits different components of $G - x$, but there is no such cycle.

7.1.27. *Density conditions for $\chi'(G) > \Delta(G)$.*

a) If G is regular and has $2m+1$ vertices, then $\chi'(G) > \Delta(G)$. For a regular graph, being $\Delta(G)$-edge-colorable means being 1-factorable, which is impossible with odd order since such graphs have no 1-factor.

b) If G has $2m+1$ vertices and more than $m \cdot \Delta(G)$ edges, then $\chi'(G) > \Delta(G)$. Each color class is a matching, and each matching has size at most m, so Δ matchings cover at most $m\Delta$ edges. Since G has more edges than that, every proper edge-coloring of G requires more than Δ colors.

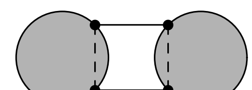
c) If G arises from a k-regular graph with $2m+1$ vertices by deleting fewer than $k/2$ edges, then $\chi'(G) > \Delta(G)$. Since fewer than k vertices have

lost an edge and $k \le 2m$, some vertex of degree k remains; hence $\Delta(G) = k$. Also $e(G) > (2m+1)k/2 - k/2 = m\Delta(G)$, so (b) implies $\chi'(G) > \Delta(G)$.

7.1.28. *The Petersen graph has no overfull subgraph.* A subgraph H is overfull if and only if it has an odd number of vertices and has more than $(n(H)-1)\Delta(G)/2$ edges. Subgraphs of order $3, 5, 7, 9$ would need more than $3, 6, 9, 12$ edges, respectively. Since the Petersen graph has no cycle of length less than 5, the smaller cases are excluded. For the last case, deleting a single vertex leaves a subgraph with 9 vertices and 12 edges, but 12 is not more than 12.

7.1.29. *A non-1-factorable regular graph with high degree.* Let G be the $(m-1)$-regular connected graph formed from $2K_m$ by deleting an edge from each component and adding two edges between the components to restore regularity. If m is odd and greater than 3, then G is not 1-factorable.

To see this, observe that the central edge cut of size 2 leaves an odd number of vertices on both sides. Hence every 1-factor in G includes an edge of this cut. If G is 1-factorable, this forces the degree to be at most 2, and hence $m \le 3$.



$K_m - e$ $K_m - e$

7.1.30. *Overfull Conjecture $\Rightarrow$ 1-factorization Conjecture.* Let G be a k-regular simple graph of order $2m$.

An induced subgraph of G is overfull if and only if the subgraph induced by the remaining vertices is overfull. Let H be the subgraph induced by vertex set S. We have $2e(H) = kn(H) - \left|[S, \overline{S}]\right|$ (Proposition 4.1.12). Overfullness for H is thus the inequality $kn(H) - \left|[S, \overline{S}]\right| > k(n(H)-1)$ (and $n(H)$ odd), since $\Delta(G) = k$. This inequality simplifies to $\left|[S, \overline{S}]\right| < k$, and it is satisfied for S if and only if it is satisfied for $\overline{S}$.

If G has an overfull subgraph, then $k \le 2\lfloor (m-1)/2 \rfloor$. Again, we have $2e(H) = kn(H) - \left|[S, \overline{S}]\right|$. If H is overfull, then $\left|[S, \overline{S}]\right| < k$, by the computation in part (a). Also, $2e(H) \le n(H)[n(H)-1]$, since G is simple. Together, these inequalities yield $n(H)[n(H)-1] < k[n(H)-1]$, or $k < n(H)$. By part (a) we may assume that $n(H) \le m$, since both $V(H)$ and $V(G) - V(H)$ induce overfull subgraphs. Hence we may conclude that $k < m$. Furthermore, since $n(H)$ is odd when H is overfull, we strengthen this to $k < m-1$ when m is even.

If the constraint on k fails, then there is no overfull subgraph, so if the Overfull Conjecture holds, then the 1-factorization Conjecture also holds.

7.1.31. *Optimal edge-colorings.* A k-edge-coloring of a multigraph G is *optimal* if it has the maximum possible value of $\sum_{v\in V(G)} c(v)$, where $c(v)$ is the number of distinct colors appearing on edges incident to v.

a) If G does not have a component that is an odd cycle, then G has a 2-edge-coloring that uses both colors at each vertex of degree at least 2. If G is Eulerian, we follow an Eulerian circuit, alternating between the colors; each visit to a vertex enters and leaves on different colors. If $e(G)$ is even, then the first and last edge also contribute both colors to their common vertex. If $e(G)$ is odd and the starting vertex has degree at least 4, then it receives both colors from another visit. If G has no vertex of degree at least 4 at which the odd circuit can be started, then G is an odd cycle, which is the exceptional case and has no such 2-edge-coloring.

If G is not Eulerian, then we add a new vertex x having an edge to each vertex of G with odd degree. Let C be an Eulerian circuit starting at x in the new graph G'; alternate the two colors along C. The problem of first and last edge having the same color is irrelevant, because we discard the edges incident to x. For vertices other than x, the degree in G' is at least 2, and there is at most one edge to x. Hence each vertex v of degree at least 2 in G has a visit to it in C using only edges of G, and this visit contributes edges of both colors at v.

b) If f is an optimal k-edge-coloring of G, having color a at least twice at $u \in V(G)$ and color b not at u, then in the subgraph of G consisting of edges colored a or b, the component containing u is an odd cycle. Let H be the specified component, consisting of edges reachable from u using paths of colors a and b. If H is not an odd cycle, then part (a) allows us to recolor $E(H)$ with colors a and b so that both colors appear at every vertex of degree at least 2. Now the number of colors appearing at each vertex of H is at least as large as before, and at u the number has increased. This new coloring of G has a larger value of $\sum c(v)$, which contradicts the optimality of f. Hence H must be an odd cycle.

c) If G is bipartite, then G is $\Delta(G)$-edge-colorable, and G has a $\delta(G)$-edge-coloring in which each color appears at every vertex. Consider an optimal $\Delta(G)$-edge-coloring f of G. If f is not a proper edge-coloring, then some color appears at least twice at some vertex. Since the degree of that vertex is at most $\Delta(G)$, some other color must be missing at that vertex. Since f is optimal, part (b) implies that G has an odd cycle, which cannot occur in a bipartite graph. Hence f is a proper edge-coloring.

For the second claim, consider an optimal δ-edge-coloring of G. If some color i is missing at u, then some color j must appear twice, because the number of edges at u is at least δ. By part (b), this requires an odd cycle in G. Thus an optimal δ-coloring must have δ different colors appearing at each vertex of G.

7.1.32. *Every bipartite graph G with minimum degree k has a k-edge-coloring in which at each vertex v, each color appears $\lceil d(v)/k\rceil$ or $\lfloor d(v)/k\rfloor$ times.* Modify G to obtain another bipartite graph H by iteratively splitting each vertex v of G into $\lceil d_G(v)/k\rceil$ vertices, each inheriting k of the edges incident to v, except for one vertex that may receive fewer. Let the resulting graph be H; note that $\Delta(H) = k$. Since a bipartite graph H has a proper $\Delta(H)$-edge-coloring, we can properly color H with the desired number of colors. Each color is used at each vertex that was split from v except possibly the one that received fewer incident edges. Hence we recombine the split vertices to return to G, we have each of the k colors appearing $\lceil d(v)/k\rceil$ or $\lfloor d(v)/k\rfloor$ times at each vertex v. (Comment: The same argument holds for every k, not only the minimum degree.)

7.1.33. *Every simple graph with maximum degree Δ has a proper $(\Delta+1)$-edge-coloring in which each color is used $\lceil e(G)/(\Delta+1)\rceil$ or $\lfloor e(G)/(\Delta+1)\rfloor$ times.* By Vizing's Theorem, there is a proper coloring with $\Delta+1$ colors. If the total usage of some two colors differs by more than one edge, consider the subgraph formed by the edges with these two colors. Since the coloring is proper, this consists of components that are paths and/or cycles alternating between the two colors. The color appearing more often must occur on the end edges of a path of odd length. Switching colors on such a path yields a new proper coloring that is less out of balance. Such improvements can be made until the frequencies differ by at most one, at which point they must all be $\lceil e(G)/(\Delta+1)\rceil$ or $\lfloor e(G)/(\Delta+1)\rfloor$.

7.1.34. *Shannon's bound on $\chi'(G)$, almost.*

a) Every loopless graph G has a $\Delta(G)$-regular loopless supergraph. Given G with vertex set $x_1,\ldots,x_n$, add another copy of G, disjoint from it, with vertex set $y_1,\ldots,y_n$. Add $\Delta(G)-d_G(v_i)$ copies of the edge x_iy_i to complete the construction. (Comment: If G is simple and a simple supergraph is desired, modify the construction by taking $2(\Delta(G)-\delta(G))$ copies of G. Since a complete graph with an even number of vertices is 1-factorable, on the copies of $x \in V(G)$ we can add $\Delta(G)-d_G(x)$ edge-disjoint matchings to raise the degree of these vertices to $\Delta(G)$).

b) If G is a loopless graph with even maximum degree, then $\chi'(G) \le 3\Delta(G)/2$. By part (a), we can find a $\Delta(G)$-regular supergraph H of G; by Petersen's Theorem, we can partition H into $\Delta(G)/2$ 2-factors. Since each 2-factor is a disjoint union of cycles, each 2-factor is 3-edge-colorable. Hence we can color $E(H)$ with $3\Delta(G)/2$ colors, and we can delete the edges of $H-G$ to obtain a proper edge-coloring of G with $3\Delta(G)/2$ colors.

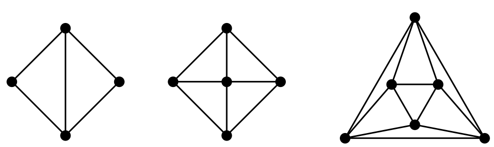
7.1.35. *Bounds on $\chi'(G)$.* Let P denote the set of 3-vertex paths in G, expressed as edges xy and yz, and let $\mu(e)$ denote the multiplicity of edge

e. the last bound below (Anderson–Goldberg) implies the earlier bounds.
Shannon: $\chi'(G) \le \lfloor 3\Delta(G)/2 \rfloor$.
Vizing, Gupta: $\chi'(G) \le \Delta(G) + \mu(G)$.
Ore: $\chi'(G) \le \max\{\Delta(G), \max_P \left\lfloor \frac{1}{2} d(x) + d(y) + d(z) \right\rfloor\}$.
$\chi'(G) \le \max\{\Delta(G), \max_P \left\lfloor \frac{1}{2}(d(x) + \mu(xy) + \mu(yz) + d(z)) \right\rfloor\}$.
The last implies Ore because $\mu(xy)+\mu(yz) \le d(y)$. It implies Vizing–Gupta because $[\mu(xy) + mu(yz)]/2 \le \mu(G)$ and $[d(x) + d(z)]/2 \le \Delta(G)$. It implies Shannon because it implies Ore and $[d(x) + d(y) + d(z)]/2 \le 3\Delta(G)/2$.

7.1.36. $(+)$ *Line graphs of complete graphs.* If $n \ne 8$, prove that $G = L(K_n)$ if and only if G is a $(2n-4)$-regular simple graph with $\binom{n}{2}$ vertices in which nonadjacent vertices have four common neighbors and adjacent vertices have $n-2$ common neighbors. (When $n = 8$, there are three exceptional graphs satisfying the conditions.) (Chang [1959], Hoffman [1960])

7.1.37. $(+)$ *Line graphs of complete bipartite graphs.* Unless $n = m = 4$, prove that $G = L(K_{m,n})$ if and only if G is an $(n+m-2)$-regular simple graph of order mn in which nonadjacent vertices have two common neighbors, $n\binom{m}{2}$ pairs of adjacent vertices have $m-2$ common neighbors, and $m\binom{n}{2}$ pairs of adjacent vertices have $n-2$ common neighbors. (Moon [1963], Hoffman [1964]) (Comment: for $n = m = 4$, there is one exceptional graph - Shrikande [1959].)

7.1.38. *Sufficiency of van Rooij-Wilf condition for connected graphs containing a double triangle with two even triangles.* We claim that the only possibilities for G in this case are the three graphs appearing below. By inspection, these graphs are line graphs, being $L(K_{1,3}+e)$, $L(K_4-e)$, and $L(K_4)$, respectively. Let F be a double triangle of G with two even triangles axy and xyb. If G has another vertex, then G has another edge to one of $\{x, y\}$, say an edge xz, else any edge joining F to $G-F$ creates an odd triangle in F. Now $N(z) \cap \{a, b, y\}$ is $\{y\}$ or $\{a, b\}$, but in the former case $\{x, a, b, z\}$ induces $K_{1,3}$. Hence $z \leftrightarrow \{a, b\}$, and the graph induced by $S = \{x, y, z, a, b\}$ is the wheel $L(K_r - e)$ in the middle below.



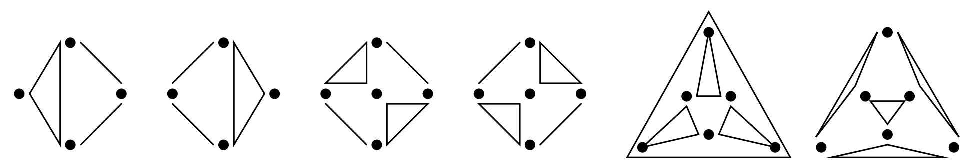
If x or y has another neighbor w, then by the same argument the other neighbors of w in F are $\{a, b\}$. If the edge is yw, then we must have $z \leftrightarrow w$ to avoid making both zxa and zxb odd; this graph is now $L(K_4)$ on the right below. If the edge is xw, then we must have $z \leftrightarrow w$ to avoid inducing $K_{1,3}$ on $\{x, y, z, w\}$. Now $\{y, z, w, a, b\}$ induce the expanded 4-cycle, which we

saw earlier is not a line graph; it violates the hypothesis because y makes both azw and bzw odd. This argument shows that y also has at most one neighbor not in F. The only remaining way to attach additional vertices is $z \leftrightarrow v$ (or, equivalently, $w \leftrightarrow v$ if y does have a neighbor w outside F), but then $\{z, v, a, b\}$ induces $K_{1,3}$.

7.1.39. *Characterization of graphs with the same line graph.* A **Krausz decomposition** of a simple graph H is a partition of $E(H)$ into complete graphs such that each vertex of H is used at most twice.

a) For a connected simple graph H, two Krausz decompositions of H that have a common complete graph are identical. Beginning with the common complete graph Q_1, we iteratively find common complete graphs in the decomposition until no more edges remain. While an edge remains, it has a path to the subgraph that has been decomposed, since G is connected. Thus there is an unabsorbed edge incident to a clique that has been absorbed; call the common vertex v. Since each vertex is used at most twice, all the unabsorbed edges incident to v must lie in the same complete graph in each decomposition. Its vertex set must be v together with the neighbors of v along the remaining incident edges, so no other neighbors of v are available. Hence this complete graph must also be in each decomposition.

b) Distinct Krausz decompositions for the graphs in Exercise 7.1.38.



c) No connected simple graph except K_3 and those in part (b) has two distinct Krausz decompositions. By part (a), it suffices to show that in any other graph G, there is some complete graph that appears in every Krausz decomposition. Call the complete graphs in some Krausz decomposition *K-graphs*.

Suppose first that G has a clique Q of size at least 4. We may assume that three vertices of Q appear together in a some K-graph, since otherwise each vertex of Q is in at least three K-graphs. If three vertices of Q appear together but not with all of Q, then an omitted vertex of Q, since it is used only twice, appears in another K-graph with at least two vertices of Q, and now some edge is covered twice. Therefore, if G has a maximal clique of size at least 4, it appears in every Krausz decomposition.

If G has an edge in no triangle, then it appears in every Krausz decomposition. Hence we may assume that every edge of G is in a triangle and G has no 4-clique. Since there is no 4-clique and every vertex is used at most twice, no edge appears in three triangles.

Suppose that every edge of G appears in exactly one triangle. If there are two triangles sharing a vertex, then there are four edges at that vertex and both triangles appear in every decomposition, since the vertex can only be used twice. Hence $G = K_3$, which has two decompositions.

Therefore, we may assume that some edge e appears in two triangles and there are two decompositions. Now e is the common edge of a double triangle (no K_4), and each triangle is used in one decomposition (since the endpoints are used only once). By symmetry, we may let $e = xy$, let x, y, z be the triangle used, and let w be the other vertex of the double triangle.

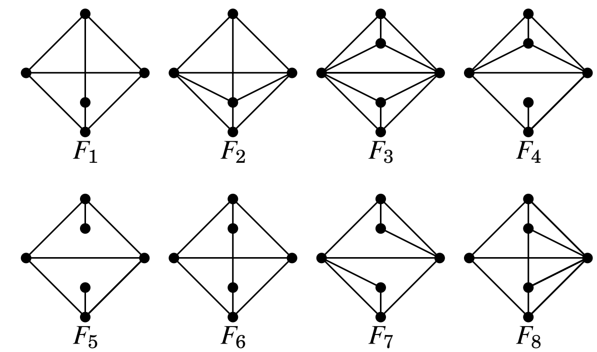
If w has two other neighbors u and v, then each is adjacent to exactly one of $\{x, y\}$, since w has been used twice and there is no K_4. By symmetry, let these edges be ux and vy. Since we have assumed another decomposition using x, y, w in a triangle, the edges uw and vw must appear together in a triangle in that decomposition. But now also yv and yz must lie in a triangle, and similarly xu and xz. Hence u, v, z form another triangle in that decomposition. Now every vertex is in two triangles in both decompositions, there is no room for additional incident edges, and the graph is the last graph in part (a).

If w has exactly one other neighbor, by symmetry we may assume that it is u, adjacent to x. Now since x, y, w form a triangle in the other decomposition, xu and xz must lie in a triangle, so uz is an edge and x, u, z form a triangle in the other decomposition. There are no other neighbors of w or y. Another neighbor of u or z would have to form a triangle with uz, but in the other decomposition these edges could not be absorbed. Hence G is the 5-vertex wheel (the middle graph in part (a)).

Hence w has no other neighbor. This implies that x and y have no other neighbor, since they are already used twice. If z has another neighbor, then we are in one of the cases described above with respect to the other decomposition where x, y, w is the triangle used. Hence z also has no other neighbor, and our graph is the kite (the left graph in part (a)).

d) $K_{1,3}$ and K_3 are the only two nonisomorphic simple graphs with isomorphic line graphs. When G is the line graph of a graph H, the vertices of H correspond to complete subgraphs in a Krausz decomposition of G. Furthermore, given a Krausz decomposition, there is one way to retrieve H satisfying this correspondence, as in Theorem 7.1.16. Thus if G is the line graph of two graphs H_1 and H_2, then G must have distinct Krausz decompositions. For the graphs in part (b), the Krausz decompositions are "isomorphic", retrieving the same graph as H_1 and H_2. For K_3, the decomposition using one triangle yields $L(K_{1,3}) = K_3$, and the decomposition into three edges yields $L(K_3) = K_3$. For every other line graph G, there is only one Krausz decomposition and hence only one solution to $L(H) = G$.

7.1.40. *A simple claw-free graph with has a double triangle with both triangles odd if and only if it some graph below is an induced subgraph.*



In each graph shown, for each triangle of the double triangle T there is a vertex with an odd number of neighbors on that triangle. If such a graph is an induced subgraph of G, then T also has both triangles odd in G.

Conversely, suppose that G has a double triangle T with triangles X and Y both odd. Let $\{u, w, z\}$ and $\{v, w, z\}$ be the vertex sets of X and Y, respectively. A vertex outside T is adjacent to w or z and to nothing else in T would yield an induced claw, so G has no such vertex. However, a vertex can have one neighbor in X or Y and two in the other by being adjacent to w or z and to u or v. A vertex with exactly one neighbor in X or Y and none in the other is adjacent only to u or v in T.

A single vertex outside G cannot be adjacent to three vertices in X or Y and one in the other, but it can be adjacent to one in each or to three in each, which yield F_1 and F_2 above.

Otherwise, we use two vertices x and y, respectively, to make X and Y odd. Suppose first that neither x nor y has one neighbor on one triangle and two on the other. Let $a = |N(x) \cap X|$ and $b = |N(y) \cap Y|$. If $x \not\leftrightarrow y$, then we obtain F_3, F_4, or F_5 when (a, b) is $(1, 1)$, ($(1, 3)$ or $(3, 1)$), or $(3, 3)$, respectively. If $x \leftrightarrow y$ and $(a, b) = (1, 1)$, then we obtain F_6. If $x \leftrightarrow y$ and $b = 3$, then deleting v yields F_1 or F_2, depending on whether a is 1 or 3.

In the remaining case, we may assume by symmetry that y has one neighbor in Y and two neighbors in X, with $N(y) \cap (X \cup Y) = \{z, u\}$. If x has one neighbor in X and none in Y, then $N(x) \cap (X \cup Y) = \{u\}$, and $\{u, x, y, w\}$ induces a claw. If x has three neighbors in X, then $\{z, x, y, v\}$ induces a claw if $x \not\leftrightarrow y$, and deleting v leaves F_2 if $x \leftrightarrow y$.

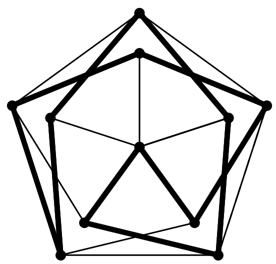
Hence we may assume that x has one neighbor in X and two neighbors in Y. Depending on $N(x) \cap \{y, z\}$, we have these outcomes:

$N(x) \cap \{y, z\}$	$\varnothing$	$\{y\}$	$\{z\}$	$\{y, z\}$
outcome	F_7	$G[\{v, w, x, y, z\}] \cong F_1$	$G[\{z, w, x, y\}] \cong K_{1,3}$	F_8

7.2. HAMILTONIAN CYCLES

7.2.1. *The complete bipartite graph $K_{r,r}$ is Hamiltonian if and only if $r \geq 2$.* Since $K_{1,1}$ has no cycle, we exclude it. For $r \geq 2$, we list the vertices alternately from the two partite sets. Consecutive vertices are adjacent, and the last is adjacent to the first, so we obtain a spanning cycle.

7.2.2. *The Grötzsch graph is Hamiltonian.*



7.2.3. *$K_{n,n}$ has $n!(n-1)!/2$ Hamiltonian cycles.* Specifying the order in which the vertices of each partite set will be visited determines a cycle starting at a given vertex x. Since there are n vertices in the other partite set and $n-1$ remaining to be visited in the same partite set as x, there are $n!(n-1)!$ ways to specify these orderings. This counts each cycle twice, since each cycle can be followed in either direction from x.

7.2.4. *If a graph G has a Hamiltonian path, then for every vertex set S, the number of components in $G - S$ is at most $|S| + 1$.* Let $c(H)$ denote the number of components of a graph H, let P be a Hamiltonian path in G, and consider $S \subseteq V(G)$.

Proof 1 (counting components). Successive deletion of vertices from a path increases the number of components of the path by at most one each time, so $c(P - S) \leq 1 + |S|$. Since P is a spanning subgraph of $G - S$ and adding edges cannot increase the number of components, we have $c(G - S) \leq c(P - S) \leq 1 + |S|$.

Proof $1'$ (following the path). P starts somewhere and visits each component of $G - S$. It must exit all but one of these before it first enters the last such component, and these exits must go to distinct vertices of S. Hence $|S| \geq c(G - S) - 1$.

Proof 2 (graph transformation). Let u, v be the endpoints of P. If $u \leftrightarrow v$, then G is Hamiltonian, and then $c(G - S) \leq |S| < |S| + 1$. If uv is not an edge, then $G' = G + uv$ is Hamiltonian, which implies $c(G' - S) \leq |S|$.

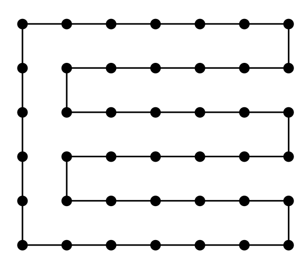
However, $c(G - S) \leq c(G' - S) + 1$, since adding an edge to a graph reduces the number of components by at most 1. Hence again $c(G - S) \leq |S| + 1$.

7.2.5. *Every 5-vertex path on the dodecahedron extends to a Hamiltonian cycle.* The dodecahedron has an automorphism taking a given face to any other face, with any rotation. Thus it suffices to shown that the spanning cycle shown at the beginning of this section in the text contains all types of 5-vertex paths relative to a given face. These are 1) four edges on one face, 2) three edges on one face and an edge extending off it, 3) at most two edges on every face. Since each pair of successive edges lie on a common face, there is essentially only one path of type 3 relative to a given central vertex. Every such path can be mapped to any other because the central vertex can be mapped to any other with an arbitrary rotation of its three incidence edges (mapped by an automorphsim, that is).

7.2.6. *Matchings in Hamiltonian bipartite graphs.*

a) If G is a Hamiltonian bipartite graph, then $G - x - y$ has a complete matching if and only if x and y are on opposite sides of the bipartition of G. Let G be an X, Y-bigraph with a Hamiltonian cycle C. Since C alternates between X and Y, we have $|X| = |Y|$. If two vertices are deleted from one partite set, then the other cannot be saturated by a matching. If we delete $x \in X$ and $y \in Y$, then each of the two paths forming $C - \{x, y\}$ must alternate between colors and have endpoints of opposite colors, since the endpoints are neighbors of x and y. If the vertices on these two paths are $u_1, \ldots, u_{2r}$ and $v_1, \ldots, v_{2s}$ in order, then the edges $u_{2i-1}u_{2i}$ and $v_{2j-1}v_{2j}$ for all i and j together form the desired matching in $G - \{x, y\}$.

b) Defective chessboards (missing two squares) can be covered by dominoes (1 by 2 rectangles) if and only if the two missing squares have opposite colors. By part (a), it suffices to show that the graph G corresponding to the chessboard is Hamiltonian. This is true for every grid ($P_m \square P_n$) with an even number of rows, as illustrated below. Follow the rows back and forth, but reserve one end column to tie together the first and last row and complete the cycle. If the number of rows is even, then this path ends on the same side of the grid in the first and last rows.



Comment: Without using the result on Hamiltonian bipartite graphs, there are several other ways to prove that a chessboard missing two squares of opposite has a tiling by dominoes. Proof 2: prove by induction on n that the property holds for all n by n chessboards with n even. Proof 3: Explicitly construct a matching, given that (i, j) and (k, l) are the two missing squares. Proof 4: Establish the existence of an alternating path joining two unsaturated squares whenever a set of dominoes does not fully cover the defective chessboard. These proofs involve a fair amount of detail and are not as general as the method above.

7.2.7. *A mouse eating cheese.* Model this with a graph G on 27 vertices in which vertices are adjacent if they correspond to adjacent subcubes. We ask whether G has a Hamiltonian path between the vertex corresponding to the center cube and a vertex corresponding to a corner cube. The vertices correspond to the 3-digit vectors with entries 0,1,2. The edges join vectors that differ by 1 in one position. Since they join vertices with opposite parity of coordinate-sum, G is bipartite.

If G has the desired Hamiltonian path, then the graph G' obtained by adding an edge between the corner cube (0,0,0) and the center cube (1,1,1) has a Hamiltonian cycle. These vectors lie in opposite partite sets of G, so G' is also bipartite. Hence the desired path yields a Hamiltonian cycle in a bipartite graph with an odd number of vertices, which is impossible.

7.2.8. *The $4 \times n$ chessboard has no knight's tour.* Let G be the graph having a vertex for each square and an edge for each pair of squares whose positions differ by a knight's move. Every neighbor of a square in the top or bottom row is in the middle two rows, so the top and bottom squares form an independent set. Deleting the $2n$ squares in the middle rows leaves $2n$ components remaining; that is not enough to prohibit the tour.

Instead, note that every neighbor of a white square in the top and bottom rows is a black square in the middle two rows. Therefore, if we delete the n black squares in the middle two rows, the white squares in the top and bottom rows become n isolated vertices, and there remain $2n$ other vertices in the graph, which must form at least one more component. Hence we have found a set of n vertices whose deletion leaves at least $n+1$ components, which means that G cannot be Hamiltonian. (For most n, the graph has a Hamiltonian path.)

7.2.9. *An infinite family of non-Hamiltonian graphs satisfying the necessary condition of Proposition 7.2.3 for Hamiltonian cycles.* It is easy to generalize the first example of a non-Hamiltonian graph satisfying the condition. Begin with a complete graph H with n vertices. Let x, y, z, w be vertices in H. Add vertices a, b, c and edges xa, yb, zc, wa, wb, wc to form G. Every separating set includes w and another vertex for each small component cut off from the clique, so the condition holds. However, visiting a, b, c requires three edges incident to w.

7.2.10. *Spanning cycles in line graphs.*

a) A 2-connected non-Eulerian graph whose line graph is Hamiltonian. The kite has two vertices of odd degree and hence is not Eulerian. Its line graph is $K_1 \vee C_4$, which is Hamiltonian.

b) $L(G)$ is Hamiltonian if and only if G has a closed trail that includes a vertex of every edge. If G is a star, then $L(G)$ is Hamiltonian and G has such a closed trail of length 0. Otherwise, there is no vertex cover of size 1, so a closed trail with a vertex of each edge must be nontrivial.

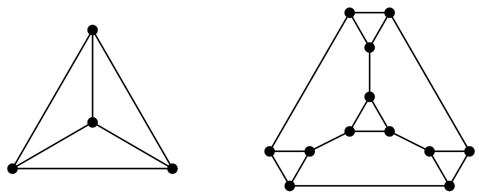
Sufficiency. Let T be such a trail in G, with vertices $v_1, \ldots, v_t$ in order. Consecutive edges on T are incident in G, so $E(T)$ in order becomes a cycle C in $L(G)$. For each edge $e \in E(G) - E(T)$, select an endpoint v of e that occurs in $V(T)$. Although v may occur repeatedly on T, select one particular occurrence of v in T as v_i. Between the vertices of C corresponding to the edges $v_{i-1}v_i$ and v_iv_{i+1} in T, insert the vertices of $L(G)$ for all edges of $E(G) - E(T)$ whose selected vertex occurrence is v_i. Since these edges all share endpoint v_i, the corresponding vertices replace an edge in $L(G)$ with a path. Every vertex of $L(G)$ is in the original cycle C or in exactly one of the paths used to enlarge it, so the result is a spanning cycle of $L(G)$.

Necessity. Given a spanning cycle in $L(G)$, we obtain such a closed trail in G. First we shorten the cycle. If there are three successive vertices e_{i-1}, e_i, e_{i+1} on the remaining cycle in $L(G)$ that correspond to edges in G with a common endpoint, we delete e_i from the cycle. Since e_{i-1} and e_{i+1} have a common endpoint, what remains is still a cycle in $L(G)$. Each deletion preserves the property that the remaining edges include an endpoint of every edge in G.

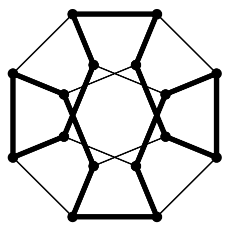
When no more deletions are possible, every three successive vertices in the resulting cycle C in $L(G)$ correspond to edges in G with no common endpoint, but two successive vertices on C are incident edges in G. Orient each such edge in G by letting the tail be the endpoint it shares withits predecessor on C in $L(G)$; the head is the vertex it shares with its successor on C. This expresses the edge set in C as the vertex set of a closed trail in G (and it contains a vertex of every edge in G).

7.2.11. *A 3-regular 3-connected graph whose line graph is not Hamiltonian.* Let G be a non-Hamiltonian 3-regular 3-connected graph, such as the Petersen graph. Form G' by replacing each vertex v of G with a triangle T_v. Each original edge uw becomes an edge joining a vertex of T_u with a vertex of T_w. Observe that G' is 3-regular. Also, if G' has a 2-vertex cut, then deleting the corresponding two or one vertices in G also cuts G. Below we illustrate the application of the transformation to K_4.

Suppose that C is a closed trail in G' that touches every edge. Since edges of T_v are incident only to vertices of T_v, the trail C must enter each T_v. Since only three edges enter T_v, the trail C can enter and leave T_v only once. Hence contracting C back to G by contracting the triangles yields a cycle that visits each vertex once. Since G has no spanning cycle, G' has no such trail.



7.2.12. *The graph below is Hamiltonian.*



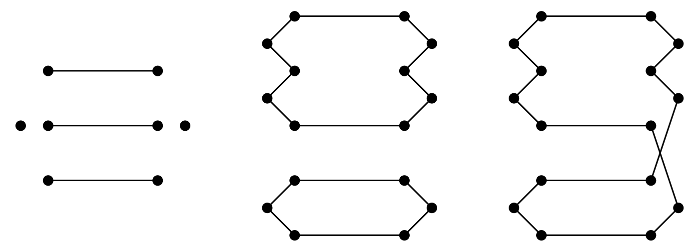
7.2.13. *The 3-regular graph obtained from the Petersen graph by expanding one vertex into a triangle matched to the former neighbors of the deleted vertex is not Hamiltonian.* Since there are only three edges incident to the triangle, it can only be entered once on a cycle. It must visit all vertices of the triangle during one visit to the triangle. Therefore, shrinking the triangle to a single vertex shortens the cycle by two steps and yields a Hamiltonian cycle in the Petersen graph. There is no such cycle, so the original cycle also cannot exist.

7.2.14. *Every uniquely 3-edge-colorable 3-regular graph is Hamiltonian.* Each color class induces a perfect matching. Consider the subgraph H formed by the edges in two of these matchings. It has degree 2 at every vertex, and thus H a 2-factor, i.e. a union of disjoint cycles. The cycles have even length, since the two colors alternate on its edges. If H is not a single (i.e. Hamiltonian) cycle, then we can switch these two colors on one of the cycles to obtain a 3-edge-coloring with a different partition of the edges. Thus **unique** 3-edge-colorability requires that the union of any two color classes is a Hamiltonian cycle.

7.2.15. *C_n^2 is the union of two disjoint Hamiltonian cycles.* This graph consists of n vertices in cycle order, with each adjacent to the nearest two in each direction. Let the points be $v_1, \ldots, v_n$ in order. If we use $(v_1, \ldots, v_n)$ as one cycle, then the remaining edges form a cycle $(v_1, v_3, \ldots., v_{n-2})$ (traveling around twice) if n is odd. If n is even, then the remaining edges form two disjoint cycles of length $n/2$, and we must make a switch. In this case replace the three edges v_{n-1}, v_n, v_1, v_2 in the original cycle by the edges v_{n-1}, v_1, v_n, v_2; the result is still a cycle. Now the remaining edges form the cycle $(v_1, v_3, \ldots, v_{n-1}, v_n, v_{n-2}, \ldots, v_4, v_2)$. All indices change by two on each edge of this cycle except the two edges $v_{n-1}v_n$ and v_2v_1 that were switched out of the original cycle.

7.2.16. *The graph G_k obtained from two disjoint copies of $K_{k,k-2}$ by adding a matching between the two "partite sets" of size k is Hamiltonian if and only if $k \geq 4$.* If $k = 2$, then G_k is disconnected. If $k = 3$, then deleting the two centers of claws leaves three components (on the left below).

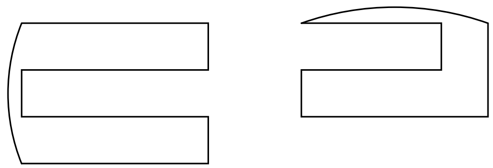
When $k \geq 4$, first take two cycles to cover $V(G_k)$: use two edges of the middle matching in each cycle, and use one fewer vertex from the outside parts than from the inside parts (as in the middle below). Then switch a pair of edges on one side to link the two cycles (as on the right).



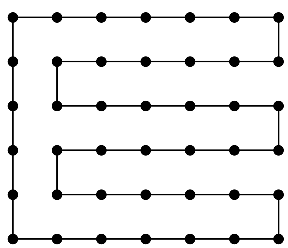
7.2.17. *The Cartesian product of two Hamiltonian graphs is Hamiltonian.* It suffices to show that the product of two cycles is Hamiltonian, because the product of two Hamiltonian graphs has a spanning subgraph of this form. Index the vertices of the first cycle as $1, \ldots, m$ and those of the second cycle as $1, \ldots, n$; the vertices of the product are then $\{(i, j)\colon 1 \leq i \leq m,\ 1 \leq j \leq n\}$. The product is a grid with m rows and n columns plus a wraparound edge in each row and column.

If m is even, then we start in the upper left corner (1,1) and follow rows alternately to the right and left, finishing in the lower left corner $(m, 1)$ after visiting all vertices; the wraparound edge in the first column completes the cycle. If m is odd, then we follow the same zigzag from $(1, 1)$ to traverse the first $n - 1$ columns, ending at $(m, n - 1)$. We then traverse the last column from (m, n) to $(1, n)$ and take the wraparound edge from $(1, n)$ to $(1, 1)$ to complete the cycle.

To show that the hypercube Q_k is Hamiltonian, we use induction on k. Note that Q_1 is K_2, which is not Hamiltonian. Basis step: For $k = 2$ and $k = 3$, we have explicit constructions. Induction step: For $k \ge 4$, we observe that $Q_k \cong Q_2 \square Q_{k-2}$. Since each factor is Q_l for some l with $2 \le l \le k-2$, the induction hypothesis tells us that both factors are Hamiltonian, and then the first part of the problem yields this for the product.



7.2.18. *The product of graphs with Hamiltonian paths has a Hamiltonian cycle unless both factors have odd order.* Since deleting edges never introduces a Hamiltonian path or cycle, it suffices to prove the claim when the two graphs are paths. In this case the product is the grid $P_m \square P_n$. If the factors do not both have odd order, then we may assume that the grid has an even number of rows. Follow the rows back and forth, but reserve one end column to tie together the first and last row and complete the cycle. Since the number of rows is even, the zigzag path ends on the same side of the grid in the first and last rows.



The product of two graphs with Hamiltonian paths fails to have a Hamiltonian cycle if and only if both graphs are bipartite and have odd order, in which case the product has a Hamiltonian path. If both graphs are bipartite and have odd order, then the product is bipartite and has odd order, so it cannot be Hamiltonian. The discussion above handles the case where at least one factor has even order. Hence we may assume that both factors have odd order and at least one is not bipartite. Since paths are bipartite, the Hamiltonian path in one factor must have an chord that completes an odd cycle. It thus suffices to construct a Hamiltonian path when one factor is a path of odd order and the other (the "horizontal" factor in the grid) is a path of odd order plus a single edge that forms such a chord.

Let $P = P_m$ with vertices $v_1, \dots, v_m$ in order, and let $Q = P_n + e$ with vertices $u_1, \dots, u_n$ in order on the path, plus the edge $e = u_r u_s$ where $s - r$ is even and positive. Let $P_{i,j}$ denote the v_i, v_j-path in P, and let $Q_{i,j}$ denote the u_i, u_j-path in Q (along the path P_n in Q). If P, Q are disjoint paths such that the last vertex of P is adjacent to the first vertex of Q, then $P : Q$ denotes the path consisting of P followed by Q. Let $s_1(P_{i,j}, Q_{k,l})$ be the "back-and-forth" Hamiltonian path of $P_{i,j} \square Q_{k,l}$ that follows the rows, switching from one row to the next in the end columns corresponding to u_k and u_l. The path starts at (v_i, u_k). It ends at (v_j, u_k) if $j - i$ is odd and at (v_j, u_l) if $j - i$ is even. Similarly, $s_2(P_{i,j}, Q_{k,l})$ is the Hamiltonian path following using all the column edges, starting at (v_i, u_k) and ending at (v_i, u_l) if $l - k$ is odd and at (v_j, u_l) if $l - k$ is even.

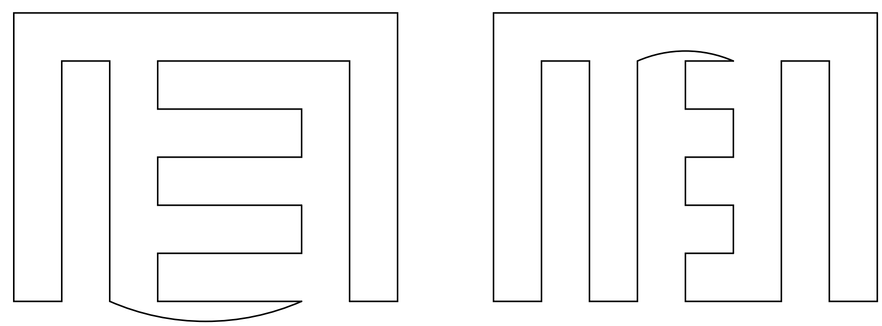
Recall that $u_r u_s$ is the extra edge in the factor H. If r, s are odd (such as when H is an odd cycle), then

$$s_2(P_{2,m}, Q_{1,r}) : s_1(P_{m,2}, Q_{s,r+1}) : s_2(P_{2,m}, Q_{s+1,n}) : (u_1 \square Q_{n,1})$$

is a Hamiltonian cycle of $G \square H$. If r, s are even, then

$$s_2(P_{2,m}, Q_{1,r}) : s_1(P_{2,m}, Q_{s,r+1}) : s_2(P_{m,2}, Q_{s+1,n}) : (u_1 \square Q_{n,1})$$

is a Hamiltonian cycle of $G \square H$. In each case all vertices are listed, and the last vertex of each segment is adjacent to the first vertex of the next.



7.2.19. *Construction of a $(k-1)$-connected k-regular non-Hamiltonian bipartite graph for odd k.* Let H be the graph with vertex set $W \cup X \cup Y \cup Z$, where W, Z have size $(k-1)/2$ and X, Y have size $(k+1)/2$. Add the edges $W \times X$, $X \times Y$, and $Y \times Z$; now the vertices X, Y have degree k. Take k copies of H. Add special vertex sets A and B, each of size $(k-1)/2$. Add an edge from a_i to each of the k copies of w_i, for each i; this gives a_i degree k and increases the degree of w_i to k. Similarly add an edge from b_i to each of the k copies of z_i, for each i. This completes the desired graph G_k. The set $A \cup B$ has size $k-1$, and $G_k - A - B$ has k components, all isomorphic to H. We omit the verification that G_k is $(k-1)$-connected. It is conjectured that every k-connected k-regular bipartite graph is Hamiltonian.

7.2.20. *Hamiltonian cycles in powers of graphs.*

a) If $G - x$ has at least three nontrivial components in which x has exactly one neighbor, then G^2 is not Hamiltonian. Let v_1, v_2, v_3 be the unique neighbor of x in three such components H_1, H_2, H_3 of $G - x$. Let $S = \{x, v_1, v_2, v_3\}$. Since each H_i is non-trivial, $G^2 - S$ has at least three components. Within S, only x and v_i have neighbors in $H_i - v_i$. A spanning cycle of G^2 must enter and leave $H_i - v_i$ via distinct vertices of S; these can only be x and v_i. This forces at least three edges incident to x, one to each H_i, which is impossible in a Hamiltonian cycle.

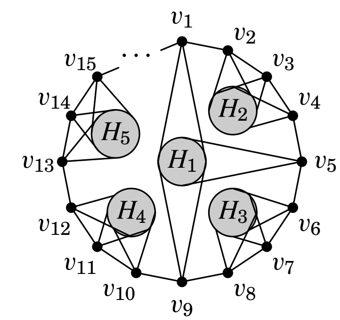
b) The cube of each connected graph (with $n \ge 3$) is Hamiltonian. The cube of a connected graph contains the cube of each of its spanning trees, so it suffices to prove the claim for trees. We use induction on $n(T)$ to prove the stronger result that T^3 has a Hamiltonian cycle such that a specified pair x, y of adjacent vertices in T are consecutive on the cycle. For $n(T) = 3$, this is trivial since T^3 is a clique; suppose $n(T) \ge 4$. The graph $T - xy$ consists of two disjoint trees R and S containing x and y, respectively. By symmetry, we may assume $n(R) \le n(S)$. Choose $z \in N_S(y)$ and, if $n(R) > 1$, choose $w \in N_R(x)$. If each subtree has at least three vertices, then the induction hypothesis provides Hamiltonian cycles of R^3 and S^3 containing the edges xw and yz, respectively. Since T^3 contains both R^3 and S^3, we obtain the desired Hamiltonian cycle of T^3 by replacing the edges xw and yz with xy and wz, which exist because $d_T(w, z) = 3$. If $n(R) = 2$, we replace yz in the cycle through S^3 by y, x, w, z. If $n(R) = 1$, we replace yz by y, x, z.

7.2.21. *Non-Hamiltonian complete k-partite graphs.* If $m < n/2$, then there is a non-Hamiltonian complete k-partite graph with minimum degree m and all partite sets nonempty as long as $m \ge k-1$. Simply make the largest partite set X have size $n - m$, and partition the remaining set S into $k-1$ parts. Now $G - S$ has more than $|S|$ components, which violates the necessary condition for Hamiltonicity.

For the values stated in the text, $m = \frac{n}{2}\frac{k-1}{k}\frac{2l}{2l+1} = (k-1)l$, and thus $m < n/2$ with $n = k(2l+1)$.

7.2.22. *For $k, t \ge 4$, in the class $\mathbf{G}(k,t)$ of connected k-partite graphs in which each partite set has size t and every two parts induce a matching of size t, there is a graph that is not Hamiltonian.* Let $V_1, \ldots, V_k$ be the partite sets. To construct G, start with $C_{3t} \cup tK_{k-3}$ as follows. Form C_{3t} with vertices $v_1, \ldots, v_{3t}$ in order so that $v_i \in V_j$ if and only if $i \equiv j \pmod 3$. For $1 \le i \le t$, add edges to make a clique consisting of the ith vertex of each of $V_4, \ldots, V_k$. Now we have $C_{3t} \cup tK_{k-3}$; let $H_1, \ldots, H_t$ be the copies of K_{k-3}. Each vertex in H_i needs a neighbor in each of V_1, V_2, V_3. Add all edges joining $V(H_1)$ and $\{v_1, v_5, v_9\}$; note that $v_1 \in V_1$, $v_5 \in V_2$, $v_9 \in V_3$. Add all edges joining $V(H_2)$ and $\{v_2, v_3, v_4\}$. Add all edges joining $V(H_3)$ and $\{v_6, v_7, v_8\}$. For $4 \le j \le t$, add all edges joining $V(H_j)$ and $\{v_{3j-2}, v_{3j-1}, v_{3j}\}$.

Since each vertex has exactly one neighbor in each other part, and the graph is connected, the graph belongs to $\mathbf{G}(k,t)$. However, deletion of $\{v_1, v_5, v_8\}$ leaves a graph with four components, and hence the graph is not Hamiltonian.



7.2.23. *The Petersen graph G has toughness* $4/3$. We seek the smallest value of $|S|/c(G-S)$ achieveable by a separating set S, where $c(H)$ counts the components of H. To separate into two components we need $|S| \ge 3$, since G is 3-connected.

To separate G into three components, we claim that $|S| \ge 4$. Deleting any three vertices deletes at most 9 edges, which leaves at least six edges among the remaining seven vertices. If $c(G-S) = 3$, then $G - S$ is a forest with at most four edges unless one component is a 5-cycle, which allows only five edges. Hence deleting three vertices cannot create more than two components. We can separate the graph into three components by deleting an independent set of size 4, so $t \le 4/3$.

To separate G into more than three components, we must leave an independent set T of size 4 with one vertex in each component. Since a vertex neighborhood in G is a dominating set, T has no three vertices with a common neighbor. Hence there are six different vertices that are common neighbors of two vertices in T, and these must all be deleted to leave 4 components. This ratio is $6/4$.

After consider all separating sets, we conclude that $4/3$ is the smallest ratio and hence is the toughness.

7.2.24. *The toughness $t(G)$ of a $K_{1,3}$-free graph is half its connectivity.* We may assume that G is connected, since otherwise the toughness and connectivity are 0. Let $c(H)$ denote the number of components of H.

For every connected graph, the toughness is at most half the connectivity, since a minimum vertex cut separates the graph into at least two components and has size $\kappa(G)$. The inequality $|S| \ge t(G)c(G-S)$ thus yields $t(G) \le |S|/c(G-S) \le |S|/2 = \kappa(G)/2$ when S is a minimum cut.

Now let S be a vertex cut achieving the minimum ratio of $|S|/c(G-S)$; in other words, $t(G) = |S|/c(G-S)$. Let u be a vertex in a component C of $G-S$, and let v be a vertex of S. By Menger's Theorem, there exist $\kappa(G)$ pairwise internally disjoint u,v-paths in G. These paths enter S at distinct vertices, establishing edges to C from $\kappa(G)$ distinct vertices of S. This holds for each component of $G-S$. Since G is $K_{1,3}$-free, each vertex of S has neighbors in at most two components of $G-S$ and hence is incident to at most 2 of the edges we have generated. This yields the inequality $\kappa(G)c(G-S) \le 2|S|$, and hence $t(G) \ge \kappa(G)/2$.

7.2.25. *If G is a simple graph that is not a forest and has girth at least 5, then $\overline{G}$ is Hamiltonian.* Let $H = \overline{G}$. If H satisfies Ore's Condition, then H is Hamiltonian. Otherwise, H has nonadjacent vertices x and y such that $d_H(x)+d_H(y) \le n-1$. Thus $xy \in E(G)$ and $d_G(x)+d_G(y) \ge n-1$. Avoiding cycles of length less than 5 in G yields $N_G(x) \cap N_G(y) = \varnothing$, and also there is no edge from $N(x)$ to $N(y)$.

We have argued that $N_G(x) \cup N_G(y)$ induces a tree with at least $n-1$ vertices. Since G is not a forest, exactly one vertex, z, remains outside this set. Furthermore, girth at least 5 implies that z has exactly one neighbor a in $N_H(x)$ and one neighbor b in $N_G(y)$. No other edge can appear in G. Now H has a spanning cycle that visits the vertices in the order $z, x, b, N_H(y)-\{b\}, N_H(x)-\{a\}, a, y$.

7.2.26. *The maximum number of edges in a non-Hamiltonian n-vertex simple graph is $\binom{n-1}{2}+1$.* The graph consisting of an $(n-1)$-clique plus a single pendant edge has $\binom{n-1}{2}+1$ edges and is not Hamiltonian. To show that this is the maximum size, suppose that G is not Hamiltonian, and let $d_1,\dots,d_n$ be the vertex degrees of G, indexed in nondecreasing order.

Since G must fail Chvátal's Condition, there is some $i < n/2$ such that $d_i \le i$ and $d_{n-i} < n-i$. Let u be the vertex with the ith smallest degree, and let v be the vertex with the $n-i$th smallest degree. Thus $d_G(u)+d_G(v) \le i+(n-i-1) = n-1$. In the complement, we have $d_{\overline{G}}(u)+d_{\overline{G}}(v) = (n-1-d_G(u))+(n-1-d_G(v) \ge 2(n-1)-(n-1) = n-1$.

Since u and v have degree sum at least $n-1$ in $\overline{G}$, and since a simple graph has at most one edge joining them (counted twice in the degree sum), there must be at least $n-2$ edges in $\overline{G}$ incident to $\{u,v\}$. Hence $e(G) \le \binom{n}{2}-(n-2) = \binom{n-1}{2}+1$.

7.2.27. *By induction on n, the maximum number of edges in a non-Hamiltonian n-vertex simple graph is $\binom{n-1}{2}+1$.* The graph consisting of an $(n-1)$-clique plus a single pendant edge has $\binom{n-1}{2}+1$ edges and is not Hamiltonian. For $n=2$, this graph is K_2 and is trivially the largest. For $n=3$, exceeding the bound requires three edges, and the resulting simple graph can only be K_3.

For $n>3$, suppose that $e(G) > \binom{n-1}{2}+1$. Thus $e(\overline{G}) < n-2$, and $\overline{G}$ has a vertex v of degree at most 1. In G, we have $d(v) \ge n-2$. Since $\binom{n-1}{2}-(n-2) = \binom{n-2}{2}$, the induction hypothesis provides a Hamiltonian cycle C in $G-v$. Since v has at most one nonneighbor in $V(G)-\{v\}$ and $n-1 \ge 3$, vertex v has two consecutive neighbors on C. Hence we can enlarge C to include v and obtain a spanning cycle in G.

7.2.28. *Generalization of the edge bound.*

a) If $f(i) = 2i^2-i+(n-i)(n-i-1)$ and $n \ge 6k$, then on the interval $k \le i \le n/2$, the maximum value of $f(i)$ is $f(k)$. The derivative is $6i-2n$, and the second derivative is 6. Since the second derivative is always positive, the maximum occurs only at the endpoints. The minimum is at $i = n/3$ (where the derivative is 0), and the parabola is symmetric around $i = n/3$. Hence to show that $f(k) \ge f(n/2)$ and complete the proof, it suffices to show that k is farther from the axis $n/3$ than $n/2$ is. This is the inequality $n/3-k \ge n/2-n/3$, which is equivalent to the hypothesis $n \ge 6k$.

b) If $\delta(G) = k$ and G has at least $6k$ vertices and has more than $\binom{n-k}{2}+k^2$ edges, then G is Hamiltonian. By Chvátal's Condition, it suffices to show that $d_i > i$ or $d_{n-i} \ge n-i$ for every $i < n/2$, where $d_1 \le \cdots \le d_n$ are the vertex degrees of G. If this condition fails for some i, we have $d_i \le i$; this requires $i \ge k$, since every vertex has degree at least k. Hence we may assume $k \le i < n/2$.

The number of edges is half the degree-sum; hence the hypothesis guarantees a degree-sum greater than $(n-k)(n-k-1)+2k^2$. If Chvátal's Condition fails for some i, we know that $d_1 = k$, the $i-1$ next smallest degrees are at most i, the degrees $d_{i+1},\dots,d_{n-i}$ are at most $n-i-1$, and the i largest degrees are at most $n-1$. This places an upper bound on the degree-sum of $k+i(i-1)+(n-2i)(n-i-1)+i(n-1) = k+2i^2-i+(n-i)(n-i-1)$. We now have $2k^2+(n-k)(n-k-1) < \sum d_i \le k+2i^2-i+(n-i)(n-i-1)$, but this contradicts the conclusion $f(k) \ge f(i)$ from part (a).

7.2.29. *If G is simple with vertex degrees $d_1 \le d_2 \le \cdots \le d_n$ and $\overline{G}$ has vertex degrees $d_1' \le d_2' \le \cdots \le d_n'$, then $d_m \ge d_m'$ for all $m \le n/2$ guarantees that G has a Hamiltonian path.* Chvátal proved (Theorem 7.2.17) that if $d_m \ge m$ or $d_{n-m+1} \ge n-m$ for all $m < (n+1)/2$, then G has a Hamiltonian path.

Consider $m < (n+1)/2$, which is equivalent to $m \le n/2$ when m,n are integers. We have $d_m' = n-1-d_{n+1-m}$. Hence $d_m \ge d_m'$ implies that $d_m \ge n-1-d_{n+1-m}$, or $d_m + d_{n+1-m} \ge n-1$. If $d_m \le m-1$, then $d_{n+1-m} \ge$

$n-m$. Thus $d_m \ge m$ or $d_{n+1-m} \ge n-m$. Since this holds for all m in the desired range, we have proved that Chvátal's Condition for spanning paths is satisfied by G, and we conclude that G has a Hamiltonian path.

If G is isomorphic to $\overline{G}$, then $d_m = d'_m$ for all m, and the condition holds. Thus every self-complementary graph has a Hamiltonian path.

7.2.30. *Chvátal's Theorem implies Ore's Theorem.* It suffices to show that Ore's Condition implies Chvátal's Condition, because then Chvátal's Theorem implies that the graph is Hamiltonian.

Consider $i < n/2$. If $d_i \le i$, then a vertex v with degree d_i has at least $n-1-i$ nonneighbors. By Ore's Condition, each nonneighbor has degree at least $n-i$. Hence at least $n-1-i$ vertices have degree at least $n-i$. Thus $d_{i+2} \ge n-i$. Since $i+2 \le n/2+1 \le n-i$, we have $d_{n-i} \ge n-i$. Thus $d_i > i$ or $d_{n-i} \ge n-i$, and Chvátal's Condition holds.

7.2.31. *If G has at least $\alpha(G)$ vertices of degree $n(G)-1$, then G is Hamiltonian.* Any set whose deletion separates G must include all vertices of degree $n(G)-1$. Hence $\kappa(G)$ is at least the number of vertices of degree $n(G)-1$, and the specified condition implies $\kappa(G) \ge \alpha(G)$. This implies that G is Hamiltonian, by the Chvátal-Erdős Theorem.

7.2.32. Let $d_1 \le \cdots \le d_n$ be the degree sequence of an X,Y-bigraph G with equal-size partite sets. Let G' be the supergraph of G obtained by adding edges so that $G[Y] = K_{n/2}$.

a) G is Hamiltonian if and only if G' is Hamiltonian, and the degree sequence of G' is formed by adding $n/2-1$ to the degrees of vertices in Y and moving them (in order) to the back. Let X,Y be the partite sets. Because $|X| = |Y|$, we can add arbitrary edges within Y without affecting whether G is Hamiltonian; the independence of X forces a Hamiltonian cycle to alternate between the sets anyway. Hence we add a clique on Y to obtain a graph G' that is Hamiltonian if and only if G is Hamiltonian. This raises the degree of each vertex in Y by $n/2-1$.

b) If $d_k > k$ or $d_{n/2} > n/2-k$ whenever $k \le n/4$, then G is Hamiltonian. By part (a), it suffices to show that this condition on G implies that G' satisfies Chvátal's Condition. In G', the vertices of Y are the $n/2$ vertices of largest degree (otherwise, G has a vertex in Y with degree 0 and a vertex in X with degree $n/2$, which is impossible). If there is a value $k < n/2$ such that G' has k vertices of degree at most k and $n-k$ vertices of degree less than $n-k$, then G has k vertices in X with degree at most k and $n/2-k$ vertices in Y with degree less than $n/2-k+1$ (at most $n/2-k$). If $i = \min\{k, n/2-k\}$, then G has i vertices of degree at most i and $i+n/2-i = n/2$ vertices of degree at most $n/2-i$, contradicting the given condition. Thus there is no such k, G' satisfies Chvátal's condition, and G' and G are both Hamiltonian.

7.2.33. *If G has n vertices and $e(G) \ge \binom{n-1}{2}+2$, then G is Hamiltonian; if $e(G) \ge \binom{n-1}{2}+3$, then G is Hamiltonian-connected.* We prove the two statements simultaneously by induction on n. The statements are vacuous for very small graphs. For $n=4$, both conditions can hold; $K_4 - e$ is Hamiltonian and K_4 is Hamiltonian-connected. For the induction step, suppose that $n > 4$. For clarity, we write the conditions as $e(\overline{G}) \le n-4$ for a Hamiltonian-connected graph and $e(\overline{G}) \le n-3$ for a Hamiltonian graph.

If $e(\overline{G}) \le n-4$, then we seek a Hamiltonian x,y-path, where x,y are arbitrary vertices of G. If x is not isolated in $\overline{G}$, then $e(\overline{G-x}) \le n-5$, and the induction hypothesis guarantees that $G-x$ is Hamiltonian-connected. Since at most $n-4$ edges are missing, we can choose $z \in N(x)-\{y\}$ and add xz to a Hamiltonian z,y-path in $G-x$ to obtain a Hamiltonian x,y-path in G. If x is isolated in $\overline{G}$, then $e(\overline{G-x}) \le n-4$, and the induction hypothesis guarantees that $G-x$ is Hamiltonian. We break an edge involving y (say yw) on an arbitrary Hamiltonian cycle in $G-x$ and add the edge wx to obtain the desired Hamiltonian x,y-path in G.

Since a Hamiltonian-connected graph is Hamiltonian (using a Hamiltonian x,y-path when $x \leftrightarrow y$), we may assume for the second statement that $e(\overline{G}) = n-3$. Hence $2 \le \delta(G)$. Since the complement of a matching is Hamiltonian, we may assume that some vertex x has degree at least 2 in $\overline{G}$. Now $e(\overline{G-x}) \le n-5$, and the induction hypothesis guarantees that $G-x$ is Hamiltonian-connected. Since $d_G(x) \ge 2$, we can select $y,z \in N(x)$ and add the path z,x,y to a Hamiltonian y,z-path in $G-x$ to complete a Hamiltonian cycle in G.

7.2.34. *Hamiltonian-connected graphs - necessary condition.*

a) A Hamiltonian-connected graph G with $n \ge 4$ vertices has at least $\lceil 3n/2 \rceil$ edges. It suffices to show that $\delta(G) \ge 3$, because then $e(G) = \sum d(v)/2 \ge 3n/2$; since the number of edges is an integer, this means $e(G) \ge \lceil 3n/2 \rceil$. If a vertex has degree 0 or 1, there is no Hamiltonian path or no Hamiltonian path without it as an endpoint. If x has degree 2, then since there is no Hamiltonian path that has the neighbors of x as the endpoints (when $n \ge 4$), since the two neighbors of x appear immediately next to x in any Hamiltonian path where x is not the endpoint.

b) If m is odd, then $G = C_m \,\square\, K_2$ is Hamiltonian-connected. We phrase the cases for general odd m but illustrate with $C_7 \,\square\, K_2$. Express $V(G)$ as $U \cup W$, where $U = \{u_0, \dots, u_{m-1}\}$ and $W = \{w_0, \dots, w_{m-1}\}$; thus $G[U] = G[W] = C_m$, and the remaining edges are $\{u_i w_i \colon 0 \le i \le m-1\}$. We construct a Hamiltonian y,z-path for each pair $y,z \in V(G)$. Since G is vertex-transitive, we may assume that $y = u_0$. By up/down symmetry in the indices, we may assume that $z = w_{2j}$ when $z \in W$ and $z = u_{2j+1}$ when $z \in U$. (Note: There are many other ways to describe the cases.)

Case 1: *A u_0, w_{2j}-path.* Begin the path by zig-zagging: $u_0, w_0, w_1, u_1, \ldots$. The step is from U to W on even indices and from W to U on odd indices, thus finishing at u_{2j-1} after w_{2j-1}. Now finish the path by traversing U from u_{2j-1} to u_{m-1} and W from w_{m-1} to w_{2j}.

Case 2: *A u_0, u_{2j+1}-path.* Begin in the same way, stopping the zig-zag at w_{2j}. Now finish the path by traversing W from w_{2j} to w_{m-1} and U from u_{m-1} to u_{2j+1}.

7.2.35. *Hamiltonian-connected graphs - sufficient condition.*

a) A simple n-vertex graph G is Hamiltonian-connected if $\delta(G) > n/2$. We must guarantee a Hamiltonian path from each vertex to every other; let u, v be an arbitrary pair of vertices in G. Let G' be the graph obtained from G by adding a vertex w and adding the edges wu, wv. Then G has a Hamiltonian u, v-path if and only if G' has a Hamiltonian cycle. We prove that G' has a Hamiltonian cycle.

A graph is Hamiltonian if and only if its closure is Hamiltonian. The closure of G' contains a clique induced by the vertices of G, because $d_G(x) + d_G(y) \ge n(G) + 1 = n(G')$ when x and y are nonadjacent vertices of G. After adding all the edges on $V(G)$, the degrees are high enough that the edges to w will also be added. Thus the closure of G' is a clique and G' is Hamiltonian, which yields the spanning u, v-path in G.

b) An n-vertex graph with minimum degree $n/2$ that is not Hamiltonian-connected. Let G_n consist of two cliques of order $n/2 + 1$ sharing an edge xy. The minimum degree is $n/2$, and because $\{x, y\}$ is a separating 2-set, there is no Hamiltonian path with endpoints x, y.

Another example is $K_{n/2,n/2}$. Since a spanning path must alternate between the partite sets and the total number of vertices is even, there is no spanning x, y-path when x and y lie in the same partite set.

7.2.36. *Las Vergnas' Condition.* The condition, which implies that the n-closure is complete, is the existence of a vertex ordering $v_1, \ldots, v_n$ for which there is no nonadjacent pair v_i, v_j such that $i < j$, $d(v_i) \le i$, $d(v_j) < j$, $d(v_i) + d(v_j) < n$, and $i + j \ge n$.

a) Chvátal's Condition implies Las Vergnas' Condition. Consider a vertex ordering with $d(v_i) = d_i$ and $d_1 \le \cdots \le d_n$. If Las Vergnas' condition fails, then every ordering (including this one) has a bad pair (i, j) of indices. Badness requires $i < j$, $i + j \ge n$, $d_i \le i$, $d_j < j$, and $d_i + d_j < n$. Given such a j, choose a minimal i satisfying these properties.

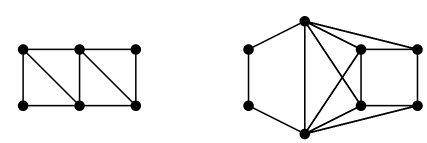
Since $i < j$ and $i + j \ge n$, we have $j > n/2$. If $i + j = n$, then Chvátal's Condition yields $d_i > i$ or $d_j \ge j$, a contradiction. If $i + j > n$, then $d_i = i$, since otherwise $d_{i-1} \le d_i \le i - 1$, and the properties would hold also for the pair $(i - 1, j)$. If $i \ge n/2$, we now have $d_i + d_j \ge n$, again a contradiction. Thus Chvátal's Condition yields $d_{n-i} \ge n - i$. Now the final contradiction:

$$n = i + (n - i) \le d_i + d_{n-i} \le d_i + d_j < n.$$

b) Las Vergnas' Condition on small graphs. The smaller graph below has degree sequence 223344. Chvátal's Condition fails at $i = n/2 - 1 = 2$, since $d_2 = 2$ and $d_4 < 4$. The Hamiltonian closure is complete, because each 2-valent vertex receives an edge to a non-neighbor of degree 4, and then minimum degree 3 allows every edge to be added.

To verify Las Vergnas' Condition, place the vertices in increasing order of degree, but choose v_2 and v_4 to be adjacent vertices of degrees 2 and 3. A violation the condition requires a pair of nonadjacent vertices with degree sum less than 6 and index sum at least 6. Such degrees must be 2,3 or 2,2. The index sum for the latter pair is 3. The index sum for the former pair is at least 6 only for v_2 and v_4, but these vertices are adjacent.

With degrees 22334455, the larger graph fails Chvátal's Condition for $i = n/2 - 1 = 3$. The closure raises the degrees to 22444466 and then to 33555577, and then all remaining edges can be added. Suppose that Las Vergnas' condition holds, with $v_1, \ldots, v_8$ a suitable ordering. The 2-valent vertices are independent of the 3-valent vertices and 4-valent vertices. Since some 2-vertex has index at least 2, no 3-vertex or 4-vertex has index at least 6. With both 3-vertices and both 4-vertices among the first 5, some 2-vertex has index at least 6. Now no 3-vertex has index at least 3, and no 4-vertex has index at least 4. This forces four vertices into the first three positions, which is impossible.



7.2.37. *Lu's Theorem implies the Chvátal–Erdős Theorem.* Lu proved that if $t(S) \ge \alpha(G)/n(G)$ whenever $\varnothing \ne S \subset V(G)$, where $t(S) = \frac{|\overline{S} \cap N(S)|}{|\overline{S}|}$, then G is Hamiltonian. To show that this implies the Chvátal–Erdős Theorem, it suffices to show that the condition $\kappa(G) \ge \alpha(G)$ implies Lu's Condition.

Let $k = \kappa(G)$. If $|S| \ge n(G) - k$, then $t(S) = 1$. If $|S| < n(G) - k$, then $t(S) \ge k/(n(G) - |S|)$. Since $n(G) > n(G) - |S|$, this yields $t(S)n(G) > k = \kappa(G)$. Hence $\kappa(G) \ge \alpha(G)$ implies $\theta(G)n(G) \ge \alpha(G)$.

7.2.38. *A connected graph G with $\delta(G) = k \ge 2$ and $n(G) > 2k$ has a path of length at least $2k$.*

a) The vertices of a maximal path P in G form a cycle in some order if the path has at most $2k$ vertices. Let u, v be the endpoints of P, and let $H = G[V(P)]$. Since P is maximal, $N(u)$ and $N(v)$ are contained in $V(H)$. Hence $d_H(u) + d_H(v) \ge 2k \ge n(H)$, and $H + uv$ is Hamiltonian. By Ore's Theorem, H is Hamiltonian; that is, the vertices of P form a cycle.

b) G has a path with at least $2k+1$ vertices. Choose a longest path P in G. If P has at most $2k$ vertices, then part (a) guarantees a cycle through $V(P)$. Since G is connected, there is an edge from $V(P)$ to $V(G) - V(P)$. Together with the vertices of P in the order of the cycle, this gives a longer path, contradicting the choice of P.

c) Quadratic algorithm for finding a Hamiltonian cycle if $d(u) + d(v) \ge n(G)$ whenever $u \nleftrightarrow v$. Find a maximal path P (greedily, in linear time). If the endpoints u, v are adjacent, then there is a cycle C through $V(P)$. Otherwise, the condition $d(u) + d(v) \ge n(G) \ge n(P)$ forces a neighbor of u following a neighbor of v by the usual switch argument; again we have a cycle C through $V(P)$. By following P, we find C in linear time.

The condition $d(u) + d(v) \ge n(G)$ also forces diameter at most 2. If $V(C) \ne V(G)$, we select a vertex not on C. Either it has a neighbor on C, or it has a neighbor with a neighbor on C. Thus we find an edge from $V(C)$ to $V(G) - V(C)$ in linear time. This gives us a path longer than P, which we extend greedily through the new vertex. We repeat the process.

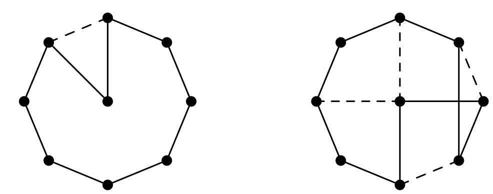
Each iteration takes only linear time, and the length of P increases fewer than n times, so in quadratic time we find a spanning cycle of G.

7.2.39. ($\bullet$) Prove that if a simple graph G has degree sequence $d_1 \le \cdots \le d_n$ and $d_1 + d_2 < n$, then G has a path of length at least $d_1 + d_2 + 1$ unless G is the join of $n - (d_1 + 1)$ isolated vertices with a graph on $d_1 + 1$ vertices or $G = pK_{d_1} \vee K_1$ for some $p \ge 3$. (Ore [1967b])

7.2.40. *Every $2k$-regular simple graph G on $4k+1$ vertices is Hamiltonian (using Dirac's theorem that a 2-connected simple graph has a cycle of length at least 2δ).* To apply Dirac's theorem, we first must show that G is 2-connected. Suppose G has a vertex x whose removal leaves a disconnected graph (this includes the case where G is not connected). Let H_1 be the smallest component of $G - x$, and let H_2 be another component. H_1 has at most $2k$ vertices. If any vertex in H_1 was not joined to x in G, then it still has degree $2k$ in $G - x$. This is impossible, since H_1 has at most $2k$ vertices and G is simple. So, H_1 must have exactly $2k$ vertices; all joined to x. This means that H_2 also has at most $2k$ vertices. The same argument requires that every vertex of H_2 have x as a neighbor in G, but this assigns $4k$ neighbors to x. This contradiction means there could not have been such

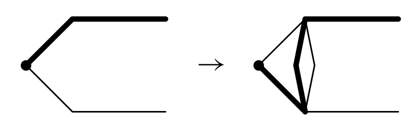
an x, and G is 2-connected.

Dirac's theorem now implies that G has a cycle C of length at least $4k$. If G is not Hamiltonian, let x be the vertex not included in C, and let X and Y denote the neighbors and non-neighbors of x. If X has two adjacent vertices on C, then we can visit x between them and augment C to a Hamiltonian cycle (see first figure below). Since X has half the vertices on C, C therefore alternates between X and Y. Now, if any two vertices of Y are adjacent, then it is possible to form a Hamiltonian cycle as indicated in the second figure below. On the other hand, if the only neighbors of vertices in $Y \cup \{x\}$ are the vertices in X, they must each neighbor every vertex in X (since there are only $2k$ of them), and thus every vertex in Y has $2k + 1$ neighbors. Since this contradicts $2k$-regularity, one of the possibilities mentioned above, in which G is Hamiltonian, must occur. (Note: the fact that an $(n-1)/2$-regular graph is Hamiltonian if $n \equiv 1 \pmod 4$ is just a slight improvement over minimum degree $n/2$. It has in fact been proved that an $n/3$-regular graph is Hamiltonian).



7.2.41. *Scott Smith's Conjecture (for $k \le 4$ only).*

a) Let G be a 4-regular graph with l vertices that is the union of two cycles, and suppose that $l \le 3$. If G' is a 4-regular graph with $l + 2$ vertices obtained from G by subdividing one edge from each of the cycles forming G and adding a double edge between the two new vertices, then G' is also the union of two spanning cycles. Since $l \le 3$, the two subdivided edges in G share an endpoint. Hence the new edges can be traversed as detours when following the subdivided edges on the old cycles, as shown below.



b) For $2 \le k \le 4$, any two longest cycles C and D in a k-connected graph H have at least k common vertices. Suppose that C and D have l common vertices, where $l < k$. Let $S = V(C) \cap V(D)$. Since $k > l$, there is a path P from $V(C) - S$ to $V(D) - S$ in H. Discard all edges not in $C \cup D \cup P$. In the remaining graph, replace threads (maximal paths whose internal vertices have degree 2) with single edges. Now $C \cup D \cup P$ is a subdivision of the resulting graph G'. Also, let G be the graph obtained from $C \cup D$ by the same replacement operation, so G is a subdivision of $C \cup D$. Note that G has l vertices. Now G' is obtained from G by the operation in part (a). By part (a), G' is the union of two cycles. These cycles correspond to cycles in H whose union is $C \cup D \cup P$. The total lengths of these two cycle exceeds that of C and D together. This contradicts the hypothesis that C and D were longest cycles, so C and D must have k common vertices.

7.2.42. *The Eulerian circuit graph is Hamiltonian when $\Delta(G) = 4$.* For convenience, we use *tour* here to mean Eulerian circuit. Let G be a loopless Eulerian multigraph, and let V' be the set of tours of G. We treat tours as equivalent if they have the same pairs of consecutive edges (hence a tour and its reversal are equivalent). Two tours are adjacent in G' if and only if one can be obtained from the other by reversing the direction of a proper closed subtour, which is the portion between some two visits to one vertex. Since every tour passes through a vertex v exactly $d(v)/2$ times, each vertex of V' thus has $\sum_{v\in V(G)} \binom{d(v)/2}{2}$ neighbors in G', and the Eulerian circuit graph is regular. (Its degree is generally too small to apply general results about spanning cycles.)

Using induction on the number of 4-valent vertices, we prove that G' is Hamiltonian when $\Delta(G) = 4$ and G' has at least three vertices. The graph G' is l-regular, where l is the number of 4-valent vertices in G (there is one switch available at each 4-valent vertex). If $l = 0$, then $G' = K_1$; if $l = 1$, then $G' = K_2$. These graphs have spanning paths. If G has two 4-valent vertices x and y, then G' is a 3-cycle when G has four x,y-paths, and G' is a 4-cycle when G has two x,y-paths.

To facilitate the induction step, we prove the stronger statement that if $l \ge 2$, then G' has a spanning cycle through any specified edge t_1t_2, where t_1 and t_2 are adjacent tours. We have verified this for $l = 2$. For the induction step, consider $l \ge 3$. Let v be the vertex where the reversal occurs to obtain t_2 from t_1. Since $\Delta(G) = 4$, we have $d(v) = 4$, with incident edges $e_0, e_1, e_2, e_3 \in E(G)$.

Let V_i' be the subset of $V(G')$ consisting of tours in which the visit through v that uses e_0 also traverses e_i, for $i \in \{1, 2, 3\}$. Each vertex of G' lies in exactly one of these sets; call this the *v-partition* of G'. Let $G_i' = G'[V_i']$. The induced subgraph G_i' is isomorphic to the Eulerian circuit graph of the graph G_i obtained from G by splitting v into two 2-valent vertices x, x', where the edges incident to x are $\{e_0, e_i\}$, and those incident to x' are the other two edges at v. For any tour, the tour adjacent to it by the reversal at v lies in a different set in the v-partition. Reversal at any other vertex does not change the pairing at v and thus reaches another tour in the same block of the v-partition. Therefore, the edges of G' that join two sets in $\{V_1', V_2', V_3'\}$ form a perfect matching of G' and correspond to reversals at v. Call these the *cross-edges*.

If v is a cut-vertex, then because all vertex degrees are even, v has two edges to each component of $G-v$, say e_0 and e_3 to one component and e_1 and e_2 to the other. In this case, G_3 is empty and $G_1 \cong G_2$, with corresponding vertices joined by an edge. That is $G' = G_1 \square K_2$. Since G_1 is Hamiltonian or is a single edge, G' has a spanning cycle through any cross-edge (see Exercise 7.2.17).

If v is not a cut-vertex, then each set in the v-partition is nonempty. The reason is that $G - v$ is connected, and hence the graph obtained from $G - v$ by adding vertices x and x' whose neighbors are the endpoints of $\{e_0, e_i\}$ and the other neighbors of v, respectively, is connected. This is precisely the graph G_i; being even and connected, it is Eulerian. (The sets V_i need not have the same size, as shown by letting G be the 4-regular graph consisting of K_4 with an extra copy of two disjoint edges, where the sizes of the V_i are 16,16,6.)

By the induction hypothesis, each G_i has a Hamiltonian cycle through any specified edge (or a path through the single edge, if it has two vertices). Thus it suffices to find a cycle C in G' that contains t_1t_2 and alternates between cross-edges and non-cross-edges, using exactly one edge within each G_i (consecutive cross-edges are acceptable if $G_i = K_1$). Using the cross-edges on C plus a Hamiltonian path of each G_i joining its vertices on C yields a Hamiltonian cycle of G containing t_1t_2

Let t_1t_2 be the specified edge, using a reversal at v. We may assume that $t_1 \in G_1$ and $t_2 \in G_2$. The vertex v cuts t_1 into two segments. Since v is not a cut-vertex, the two segments share another vertex u, which therefore has degree (at least) 4. The desired cycle C is now obtained by alternating reversals at v and u.

To list the tours of C explicitly, break t_1 into four successive trails with endpoints v and u; that is, express t_1 as $[v, Q, u, R, v, S, u, T]$, in the sense that A starts at v and ends at u, etc. We may further assume that Q starts with e_1, R ends with e_2, and T ends with e_0, so that $t_1 \in G_1$ and $t_2 \in G_2$. Let $\overline{Q}, \overline{R}, \overline{S}, \overline{T}$ denote the reversals of these trails. For the six successive tours on C, we have

$$\begin{aligned} t_1 &= [v, Q, u, R, v, S, u, T] \in G_1 & t_4 &= [v, S, u, R, v, Q, u, T] \in G_3 \\ t_2 &= [v, \overline{R}, u, \overline{Q}, v, S, u, T] \in G_2 & t_5 &= [v, S, u, \overline{Q}, v, \overline{R}, u, T] \in G_3 \\ t_3 &= [v, \overline{R}, u, \overline{S}, v, Q, u, T] \in G_2 & t_6 &= [v, Q, u, \overline{S}, v, \overline{R}, u, T] \in G_1 \end{aligned}$$

With this approach, the construction of the desired Hamiltonian cycle is easy. The approach also works for the general case without limits on $\Delta(G)$. For the general problem, Zhang and Guo [1986] use three cases like this when $d(v) = 6$ and two cases when $d(v) = 2t > 6$.

7.2.43. *For a graph G, the Eulerian circuit graph G' of Exercise 7.2.42 is $(\sum_{v\in V(G)}\binom{d(v)/2}{2})$-regular, which is not enough to apply general results on Hamiltonicity of regular graphs.* The formula for the degree is obtained in the first paragraph of the solution to Exercise 7.2.42. For a given Eulerian orientation, Theorem 2.2.28 computes the number of Eulerian circuits as $c\prod_v(d(v)/2-1)!$, where c is the number of in-trees or out-trees from any vertex. Already this number is very much bigger than the degree, and in addition there are many Eulerian orientations. Summing over all the orientations and dividing by 2 counts the vertices in G'. Hence $n(G')$ is hugely bigger than the degree, not bounded by a factor of 2 or 3 times the degree, which would be needed to apply general sufficiency conditions for Hamiltonian cycle. This explains why a specialized structural argument is needed in Exercise 7.2.42.

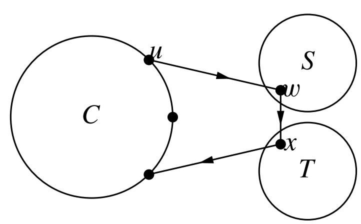
7.2.44. *Every tournament has a Hamiltonian path.*

Proof 1. If a directed path P of maximum length omits x, then $u\to x\to v$, where u and v are the origin and terminus of P. Considering the vertices of P in order, there must therefore be a consecutive pair y,z on P such that $y\to x\to z$. This detour absorbs x to form a longer path. Hence a path of maximum length in a tournament omits no vertex.

Proof 2. The result follows immediately from the Gallai-Roy Theorem, since $\chi(K_n)=n$ and every tournament is an orientation of K_n.

7.2.45. *Strong tournaments are Hamiltonian.* We prove first that a vertex on a k-cycle is also on a $(k+1)$-cycle, if $k<n$. suppose C is a k-cycle containing u. If some vertex w not on C has both a predecessor and a successor on C, then there is a successive pair v_i, v_{i+1} on C such that $v_i\to w$ and $w\to v_{i+1}$, and we can detour between them to pick up w and obtain a longer cycle through u.

Hence we may assume that every vertex off C has no successors on C or no predecessors on C; let these sets of vertices be S and T, respectively. Since there is a vertex not on C and the tournament is strong, there must be an edge wx from S to T. We can leave C at u and detour through wx, skipping the successor of u on C, to obtain a cycle of length $k+1$ through u.

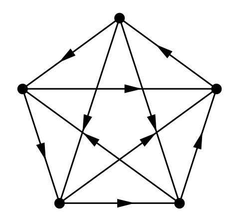


If a tournament is strong, then for every edge uv there is also a v,u-path, which together with uv completes a cycle through u. Successive application of the statement above turns this into a spanning cycle. (In fact, by considering chords we can first get down to a 3-cycle, and then we obtain a cycle of every length through u).

7.2.46. *If G is a 7-vertex tournament in which every vertex has outdegree 3, then G has two disjoint cycles.* If G is not strong, then G has a cut $[S,\overline{S}]$ with every vertex of S pointing to every vertex of $\overline{S}$. Since outdegrees in S are 3, $|\overline{S}|\le 3$, but now vertices of $\overline{S}$ don't have enough successors.

Hence G is strong. By Exercise 7.2.45, G has a 3-cycle C. Let $H=G-V(C)$. If H has a cycle, we are done. Otherwise, H is a 4-vertex transitive tournament: vertices $v_0,\ldots,v_3$ with $v_i\to v_j$ when $i<j$. Outdegree 3 implies that v_i has i successors in $V(C)$, for each i. Let u be the successor of v_1 in $V(C)$; we have a 3-cycle with vertices (v_0,v_1,u). Since v_2 has two successors in C, we can choose w as the predecessor of v_2 in $V(C)$. Now we obtain a second 3-cycle with vertices (v_2,v_3,w).

7.2.47. (+) Prove that every tournament has a Hamiltonian path with the edge between beginning and end directed from beginning to end, except the cyclic tournament on three vertices and the tournament T_5 on five vertices drawn below. (Grünbaum, in Harary [1969, p211])



(Hint: this can be proved by induction, which requires a bit of care for invoking the induction hypothesis to prove the claim for six vertices. In all cases, find the desired configuration or $G=T_5$.)

7.2.48. *Sharpness of Ghouila-Houri's Theorem.* We construct for each even n a n-vertex digraph D that is not Hamiltonian even though it satisfies "at most one copy of each ordered pair is an edge" and $\min\{\delta^-(D),\delta^+(D)\}\ge n/2$. Take two sets A and B of size $n/2$. Add edges $A\,\square\,A$ and $B\,\square\,B$ (hence there is a loop at each vertex and opposed edges joining each pair in one set), and add a matching from A to B. Each vertex has indegree and outdegree $n/2$ within its own set, but the full digraph is not strongly connected.

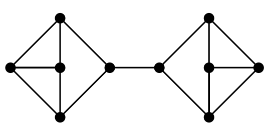
7.2.49. *Ghouilà-Houri's Theorem implies Dirac's Theorem for Hamiltonian cycles.* Suppose that a simple graph G satisfies Dirac's Condition $\delta(G)\ge n(G)/2$. From G we form a digraph D be replacing each edge with a pair of oppositely directed edges having the same endpoints. Thus

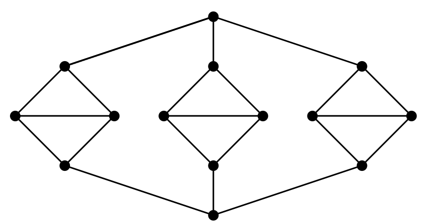
$d_D^+(x) = d_D^-(x) = d_G(x)$ for all $x \in V(G)$. Since $n(D) = n(G)$, we obtain $\min\{\delta^+(D), \delta^-(D)\} = \delta(G) \ge n(G)/2 = n(D)/2$. Hence Ghouilà-Houri's Theorem implies that D is Hamiltonian. Since a Hamiltonian cycle C in D does not use two oppositely directed edges from G, the edges of G giving rise to the edges in C also form a Hamiltonian cycle in G.

7.3. PLANARITY, COLORING, & CYCLES

7.3.1. *Every Hamiltonian 3-regular graph has a Tait coloring.* A 3-regular graph has even order, so two colors can alternate along a Hamiltonian cycle C. Deleting $E(C)$ leaves a 1-factor to receive the third color.

7.3.2. *Examples of 3-regular simple graphs:* a) planar but not 3-edge-colorable. b) 2-connected but not 3-edge-colorable. c) planar with connectivity 2, but not Hamiltonian. For part (b), the Petersen graph is an example. For (a) and (c), suitable graphs appear below. Regular graphs with cut-vertices are not 1-factorable, and graphs having 2-cuts that leave 3 components are not Hamiltonian.





7.3.3. *Every maximal plane graph other than K_4 is 3-face-colorable.* With fewer than four vertices, the maximal plane graphs have fewer than three faces. For larger graphs, every face is a triangle, so the dual is 3-regular. Since the dual is planar, it does not contain K_5. Hence the dual is not a complete graph, and by Brooks' Theorem it is 3-colorable. This becomes a proper 3-coloring of the original graph.

7.3.4. *Every Hamiltonian plane graph G is 4-face-colorable.* It suffices to show that the faces inside C can be properly 2-colored, since the same argument applies to the faces outside C using two other colors. View the union of C and the edges embedded inside C as an outerplane graph H; all the vertices are on the outer face.

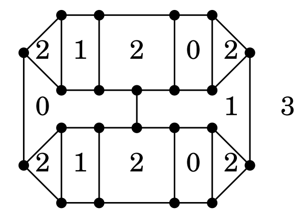
Proof 1. In the dual H^*, the bounded faces in H become vertices. We claim that the subgraph of H^* induced by these vertices is a tree T^*. If

they induce a cycle, then that cycle lies inside C in the embedding of G and encloses a face of H^*, which in turn contains a vertex of H. This is a vertex of G that does not lie on the outer face of H, which contradicts C being a spanning cycle.

Proof 2. We properly 2-color the faces inside C using induction on the number of edges inside. With no such edges, H has one bounded face and is 1-colorable. Otherwise, let e be an inside edge whose endpoints are as close together as possible on C. By the choice of e, there is a face whose boundary consists of e and edges of C. This face F is adjacent to only one other, F'. Deleting e merges F into F' in a smaller graph H'. By the induction hypothesis, H' has a proper 2-face-coloring f'. To obtain the proper 2-face-coloring f of H, let f give the same color as f' for each face other than F and give F the opposite color from $f(F')$.

7.3.5. *A 2-edge-connected plane graph is 2-face-colorable if and only if it is Eulerian.* Let G be a 2-edge-connected plane graph; note that $(G^*)^* = G$. We have G 2-face-colorable if and only if G^* is bipartite, which by $(G^*)^* = G$ and Theorem 6.1.16 is equivalent to G being Eulerian.

7.3.6. *The graph below is 3-edge-colorable.* By Tait's Theorem, it suffices to show that the graph is 4-face-colorable.



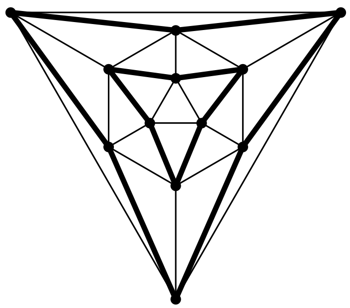
7.3.7. Let G be a plane triangulation.

a) The dual G^* has a 2-factor. The dual of a plane triangulation is 3-regular and has no cut-edge (since G has no loop). Hence the dual has a 1-factor, by Petersen's Theorem (Corollary 3.3.8). Deleting the 1-factor leaves a 2-factor.

b) The vertices of G can be 2-colored so that every face has vertices of both colors. Given the 2-factor F of G^* resulting from part (a), we can 2-color the faces of the dual by given each face the parity of the number of cycles in F that contain it. This assigns colors to the vertices of G, which correspond to the faces of G^*.

Each face of G correspond to a vertex v of G^*, with degree 3. The 2-factor F uses two edges at v, lying on one cycle of F. Hence each face of G is entered by one cycle of F. This cycle cuts one of the vertices of F from the other two, and hence the face has vertices of both colors.

7.3.8. *The icosahedron is Class 1.* The graph is 5-regular; we describe a proper 5-edge-coloring. Show in bold is a 2-factor consisting of even cycles; on this we use two colors. For the remaining three colors, we color by the angle in the picture. Color 0 goes on the six edges that are vertical or horizontal. Colors 1 and 2 go on the edges obtained by rotating this 1-factor by 120 or 240 degrees in the picture.



7.3.9. *Every proper 4-coloring of the icosahedron uses each color exactly 3 times.* The icosahedron has 12 vertices; it suffices to show that it has no independent set of size 4. In the figure above, an independent set takes at most one vertex from the inner triangle, one vertex from the outer triangle, and at most three from the 6-cycle C between them. If it takes three vertices from C, then they alternate on C and include neighbors of all other vertices Two opposite vertices on C also kill off the rest. Two vertices at distance 2 along C kill off one triangle but leave one vertex on the other triangle that can be added.

7.3.10. *By Whitney's result that every 4-connected planar triangulation is Hamiltonian, the Four Color Problem reduces to showing that every Hamiltonian planar graph is 4-colorable.* The Four Color Problem reduces to showing that triangulations are 4-colorable. Let S be a minimal separating set in a triangulation G; we show first that $|S| \geq 3$. Each vertex x of S has a neighbor in each component of $G - S$. Since there is no edge joining two components of $G - S$ and every face is a triangle, in the embedding of G edges must emerge from x between edges to different components of $G - S$. These edges go to other vertices of S. Hence $G[S]$ has minimum degree at least two, and $|S| \geq 3$.

If $|S| = 3$, then $\delta(G[S]) \geq 2$ implies that S is a clique. Hence a proper 4-coloring of each S-lobe of G uses distinct colors on S, and we can permute the names of the colors to agree on S. This yields a proper 4-coloring of G. Hence a minimal planar triangulation that is not 4-colorable must be 4-connected. Since every such graph is Hamiltonian, it suffices to show that Hamiltonian planar graphs are 4-colorable.

7.3.11. *Highly connected planar graphs.* The icosahedron is 5-connected. The symmetry of the solid icosahedron is such that two vertices at distance d in the graph can be mapped into any other pair at distance d by rotating the solid. Hence it suffices to consider one pair at distance d, for each d, and show that they are connected by five pairwise internally disjoint paths. This can be done on any drawing of the graph.

Since every planar graph has a vertex of degree at most 5, there is no 6-connected planar graph.

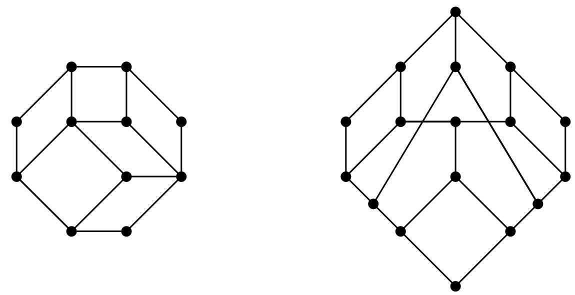
7.3.12. *A plane triangulation has a vertex partition into two sets inducing forests if and only if the dual is Hamiltonian.* Every plane triangulation F is connected, so $(F^*)^* = F$. Let $G = F^*$.

Let G be a Hamiltonian plane graph. A spanning cycle C is embedded as a closed curve, and the subgraph H of G consisting of C and all edges drawn inside C is outerplanar. In the dual of an outerplane graph H, every cycle contains the vertex for the outer face, since every cycle in H^* encloses a vertex, and thus a cycle in H^* not including the vertex for the outer face in H would yield a vertex of H not on the outer face. We conclude that in G^*, the vertices for faces of H induce a forest. The same argument applies to the graph consisting of C and the edges of G drawn outside G.

Conversely, let F be a plane triangulation with such a vertex partition. Since F is connected, there exist edges joining components in the union of these two forests. We add edges joining components, possibly changing the vertex partition while doing this, until we obtain a vertex partition into two sets $S, \overline{S}$ inducing trees.

Adding any edge from S to $\overline{S}$ yields a spanning tree of G, so $[S, \overline{S}]$ is a bond. Hence the duals of the edges in $[S, \overline{S}]$ form a cycle. We claim that this is a spanning cycle in the dual. It suffices to show that $\left|[S, \overline{S}]\right| = f$. Since F has $3n - 6$ edges and we use $n - 2$ edges in the two trees, we have $2n - 4$ edges from S to $\overline{S}$. By Euler's Formula, this is indeed the number of faces in a triangulation.

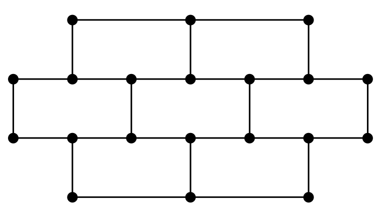
7.3.13. *Grinberg's Theorem.* Neither of the graphs below is Hamiltonian. Grinberg's Theorem requires $\sum(i-2)(\phi_i - \phi_i') = 0$, where ϕ_i and ϕ_i' are the number of i-faces inside and outside the Hamiltonian cycle. The plane graph on the left has six 4-faces and one 8-face. Since $2(\phi_4 - \phi_4')$ must be a multiple of 4 and $6(\pm 1)$ cannot be a multiple of 4, there is no way these can sum to 0. Similarly, redrawing the graph on the right yields a plane graph with three 4-faces and six 6-faces. This time $2(\phi_4 - \phi_4')$ cannot be a multiple of 4, but $4(\phi_6 - \phi_6')$ must be; again they cannot sum to 0.



7.3.14. *A non-Hamiltonian graph.* In any spanning cycle of the graph below, both edges incident to a vertex of degree 2 must appear. Applying this to the vertices of degree 2 on the outside face generates a non-spanning cycle that must appear.

Irrelevance and relevance of Grinberg's Theorem. This plane graph has four 5-faces, three 6-faces, and one 14-face. It is possible to choose nonnegative integers f_i' and f_i'' such that $f_5'+f_5''=4$, $f_6'+f_6''=3$, $f_{14}'+f_{14}''=1$, and $\sum(i-2)(f_i'-f_i'')=0$. This is achieved by $f_5'=f_5''=2$, $f_6'=3$, $f_{14}''=1$, and $f_6''=f_{14}'=0$. Hence the graph does not violate the numerical conditions of Grinberg's Theorem.

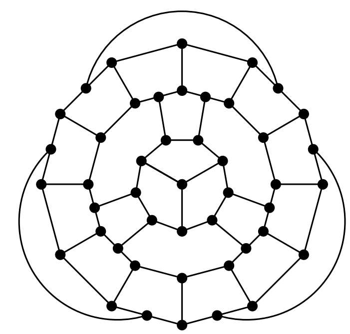
On the other hand, since the four long horizontal edges in the drawing are incident to vertices of degree 2 and therefore must appear in any Hamiltonian cycle, subdividing them once each does not affect whether the graph is Hamiltonian. The new plane graph has seven 6-faces and one 18-face. Since the difference of two numbers summing to 7 is odd, Grinberg's Condition now requires an odd multiple of 4 to equal an even multiple of 4, which is impossible. Hence the graph is not Hamiltonian.



7.3.15. *Proof of Grinberg's Theorem from Euler's Formula.* Let C be a Hamiltonian cycle in a plane graph G, and let f_i' be the number of faces of length i inside C. It suffices to prove that $\sum_i(i-2)f_i'=n-2$, since the same argument applies to the regions outside the cycle. We apply Euler's Formula to the outerplanar graph G' formed by C and the chords inside it.

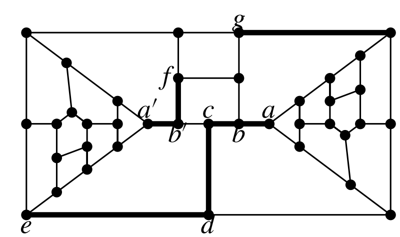
We can rewrite the desired formula as $2=n-\sum_i if_i'+2\sum_i f_i'$. Note that $\sum_i if_i'$ counts every internal edge of G' twice and every edge on the cycle once. Thus $\sum_i if_i'=2e-n$. Also, $\sum_i f_i'=f-1$, the total number of bounded faces in G'. Thus we want to prove that $4=2n-2e+2f$, which follows immediately from Euler's Formula.

7.3.16. *The Grinberg graph is not Hamiltonian.* In the plane graph below, all faces have length 5, except for three of length 8 and the one unbounded face of length 9. If it is Hamiltonian and f_i', f_i'' denote the number of faces of length i inside and outside the cycle, respectively, then Grinberg's Condition requires that $3(f_5'-f_5'')+6(f_8'-f_8'')+7(f_9'-f_9'')=0$. This can happen only when $7(f_9'-f_9'')$ is divisible by 3, which is impossible since there is exactly one face of length 9.

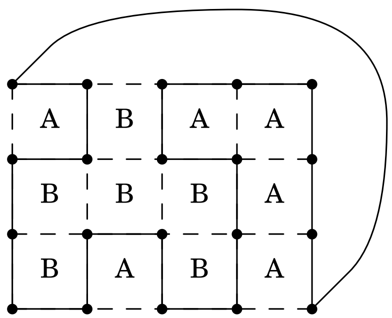


7.3.17. *The smallest known 3-regular planar non-Hamiltonian graph.* The triangular portion on both ends is the subgraph of the Tutte graph called H. Since it has three entrance points here, it must be traversed by a spanning path connecting the entrance points. Example 7.3.6 in the text shows that no such path exists joining the top and bottom entrances.

Hence edges $a'b'$ and ab must be used. By symmetry, we may assume that bc is used. If cb' is used, then completion of a cycle will miss d or the portion on the top. Hence cd is used. Since each copy of H can be visited only once, de must be used. Now the cycle must traverse the left copy of H, emerge at b', and turn up to f. On the other end, the cycle exits the right copy of H at g. Now the cycle cannot be completed without missing one of the common neighbors of f and g.

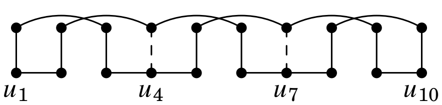


7.3.18. *A Hamiltonian path between opposite corners of a grid splits the squares of the grid into two sets of equal size.* Suppose Q is a Hamiltonian path from the upper-leftmost vertex to the lower-rightmost vertex of $P_m \square P_n$. Adding an edge through the unbounded face from the upper-leftmost vertex to the lower-rightmost vertex completes a Hamiltonian cycle. Each face containing the added edge has length $m+n-1$, and they are on opposite sides of the cycle. By Grinberg's Theorem, then, the number of 4-faces inside the cycle must equal the number of 4-faces outside the cycle. One of these measures the area of the regions escaping to the top and right, and the other measures the area of the regions escaping to the bottom and left.



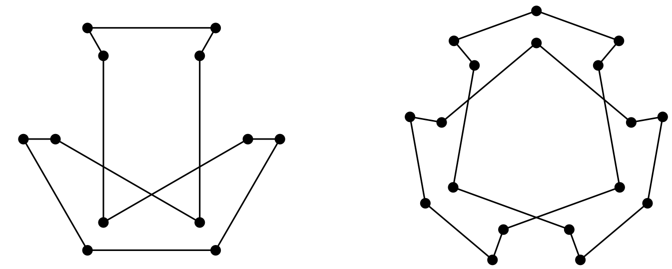
7.3.19. The *generalized Petersen graph* $P(n,k)$ is the graph with vertices $\{u_1,\ldots,u_n\}$ and $\{v_1,\ldots,v_n\}$ and edges $\{u_iu_{i+1}\}$, $\{u_iv_i\}$, and $\{v_iv_{i+k}\}$, where addition is modulo n. The usual Petersen graph is $P(5,2)$ with $\chi'=4$.

a) If $k\equiv 1 \pmod 3$ with $r\ge 1$, then we can cover $\bigcup_{i=j+1}^{j+k}\{u_i,v_i\}$ (k consecutive pairs) using a single cycle involving the edge $v_{j+k}u_{j+k}$. For $r=1$, the remainder of the cycle is the path u_{j+4}, u_{j+3}, v_{j+3}, v_{j+1}, u_{j+1}, u_{j+2}, v_{j+2}, v_{j+4}. To obtain the cycle for $r+1$ from the cycle for r, we replace the edge $v_{j+k}u_{j+k}$ with a path through the six new vertices: v_{j+k}, v_{j+k+2}, u_{j+k+2}, u_{j+k+3}, v_{j+k+3}, v_{j+k+1}, u_{j+k+1}, u_{j+k}. This proves the claim by induction on r.



b) $\chi'(P(n,2))=3$ if $n\ge 6$. The construction of part (a) produces a spanning cycle if $k\equiv 1 \pmod 3$ and $k\ge 4$; since there are $2k$ vertices, this is a 2-factor using even cycles. If $k\equiv 2 \pmod 3$ and $k\ge 8$, then we can combine a cycle on 4 pairs with a cycle on the remaining $k-4$ pairs, since $k-4\equiv 1 \pmod 3$ and is at least 4. If $k\equiv 3 \pmod 3$ and $k\ge 12$, then we can combine two cycles on 4 pairs each with a cycle on the remaining $k-8$ pairs, since $k-8\equiv 1 \pmod 3$ and is at least 4. This resolves all

cases except $k=6$ and $k=9$, for which we present explicit spanning cycles below. (The Petersen graph $P(n,2)$ is not 3-edge-colorable.)



Alternatively, there is an explicit coloring when $k\equiv 0 \pmod 3$. Let $c(e)$ denote the color on edge e. Treating the vertex indices modulo n and the colors as $\mathbb{Z}_3$, we let $c(u_iv_i)\equiv i \pmod 3$, $c(u_iu_{i+1})\equiv (i-1) \pmod 3$, and $c(v_iv_{i+2})\equiv (i+1) \pmod 3$.

7.3.20. *If a 3-regular graph is the union of two cycles, then it is 3-edge-colorable.* (Note: The statement is not true with "three" in place of "two", since the Petersen graph can be expressed as the union of three cycles.) At each vertex, each incident edge is in one of the cycles, so one of the edges must be in both. If C and C' are the two cycles, then the desired 1-factors are $E(0)-E(C')$, $E(C')-E(C)$, and $E(C)\cap E(C')$.

7.3.21. *The flower snarks: G_k consists of three "parallel" k-cycles with vertex sets $\{x_i\}$, $\{y_i\}$, $\{z_i\}$ and vertices $w_1,\ldots,w_k$ such that $N(w_i)=\{x_i,y_i,z_i\}$ for each i. H_k is obtained from G_k by replacing $\{x_kx_1, y_ky_1\}$ with $\{x_ky_1, y_kx_1\}$.*

a) G_k is Type 1. A proper 3-edge-coloring has distinct color pairs on the cycle edges at x_i, at y_i, and at z_i, because the three edges to w_i have distinct colors. Thus, if the x,y,z-edges from $i-1$ to i are colored a,b,c, then those from i to $i+1$ can be colored b,c,a or c,a,b, respectively. We travel around the k-cycles, always stepping the cyclic permutation forward. Upon reaching the last set of edges, the forward rotation or the backward rotation is compatible with both the previous triple and the first triple.

b) H_k is Type 2 when k is odd. Again we must have distinct color pairs on the cycle edges at x_i, y_i, z_i. The x,y,z-edges from $i-1$ to i cannot have the same color, because the color pairs at i could not then be distinct. Hence they are a permutation or one color is omitted. If one color is omitted, then the fact that each color appears twice among the three pairs implies that the omitted color appears twice between i and $i+1$ (and the color that was twice is now omitted). Since this argument applies in both directions (and between k and 1), all triples are permutations or all triples omit one color.

If all triples are permutations, then the permutation must maintain the same parity. In particular, placing a cyclic permutation of c,b,a next to a, b, c produces an incident pair with the same color. When the edges $x_k y_1$, $y_k x_1$, $z_k z_1$ are reached, the x, y-switch in edges switches the parity of the permutation. Thus edge-coloring using permutations on these triples cannot be compatible all the way around.

If all triples omit one color, then by the remarks above one color appears in every triple, and the other two colors alternate between appearing twice and not appearing. The alternation of appearances of these two colors implies that k must be even. When k is odd, H_k is not 3-edge-colorable. (When k is even, this discussion leads to a 3-edge-coloring of H_k.)

7.3.22. *Every edge cut of $K_k \square C_t$ that does not isolate a vertex has at least $2k$ edges, unless $k = 2$ and $t = 3$.* Edge cuts that isolate vertices have only $k+1$ edges, since $K_k \square C_t$ is $(k+1)$-regular. The graph consists of t copies of K_k (the "cliques") arranged in a ring, with corresponding vertices from the cliques forming a cycle. Two successive cliques are joined by a matching.

When a clique is split into sets of size l and $k-l$ by the edge cut, with $l \le k-l$, we call it an l-split. Such a clique induces $l(k-l)$ edges of the cut, which increases with l. Suppose that there is an l-split clique Q and another clique Q' that is unsplit. On the paths from Q to Q' in both directions around the cycle, we cut at least $2l$ edges. Hence in this case we cut at least $l(k+2-l)$ edges. For $l \ge 2$ this is at least $2k$.

Hence every clique is split or all splits are 1-splits. Since splitting a clique cuts at least $k-1$ edges and $t \ge 3$, the former case cuts at least $3k-3$ edges and suffices unless $t = 3$ and $k = 2$. This is the exceptional case, and indeed $K_2 \square C_3$ has a nontrivial cut of size 3.

Therefore, all split cliques are 1-splits and some clique is unsplit. If two cliques are 1-split, then we can find an unsplit clique preceding a split clique and a second split clique followed by an unsplit clique. In addition to the $k-1$ edges from each split clique, we have at least one edge in the cut to the neighboring unsplit clique, for a total of at least $2k$ edges.

Hence at most one clique Q is split, and if so it is a 1-split. Because the cut does not isolate a vertex, some other clique Q' is entirely on the same side with the singleton from Q. We cut $k-1$ edges within Q and some edge along each path in each direction from the large part of Q to Q'. Hence we cut at least $3k-3$ edges, which suffices unless $k = 2$ and no other edges are cut. In this case, the "large" part of Q is isolated by the cut.

Finally, if no clique is split, then having a nonempty cut requires some clique on one side and another on the other side, and then we cut at least $2k$ edges of the paths along the cycles as we go back and forth from one side to the other. This case achieves equality for all k and t.

7.3.23. *Applying Isaacs' dot product operation (Definition 7.3.12) to two snarks yields a third snark.* Let G_1 and G_2 be snarks, with disjoint edges uv and wx from G_1 and adjacent vertices y and z from G_2 deleted to perform the dot product operation. Let G be the resulting graph, adding the edges ua, vb, wc, and xd.

Since G_1 and G_2 are 3-regular, by construction G is 3-regular; the four vertices left with degree 2 in each subgraph receive new neighbors.

Since G_2 has girth at least 5, the vertices in G_2 receiving neighbors in G_1 form an independent set. Hence any cycle in G involving the new edges must use at least two of them plus at least one edge of G_1 and at least two of G_2. Other cycles lie in G_1 or G_2. Hence G has girth at least 5.

Any edge cut of G that separates $V(G_1)$ or separates $V(G_2)$ has as many edges as a corresponding edge cut of G_1 or G_2, and the only edge cut that cuts neither of those sets has size 4. Hence G is 3-edge-connected and cyclically 4-edge-connected.

Finally, if G has a proper 3-edge-coloring f, then G_1 or G_2 is 3-edge-colorable. Being 3-regular, G_1 and G_2 have even order. Since each color class is a perfect matching in G, it appears an even number of times in $\{ua, vb, wc, xd\}$. Call this property "parity". If $f(ua) = f(vb)$, then $f(wc) = f(xd)$, by parity, and assigning $f(ua)$ to uv and $f(wc)$ to wx yields a proper 3-edge-coloring of G_1. If $f(ua) \ne f(vb)$, then by parity and symmetry we may assume that $f(wc) = f(ua)$, and hence parity yields $f(xd) = f(vb)$. Now assign the color $f(ua)$ to ay and cz, the color $f(vb)$ to by and dz, and the third color to yz; this completes a proper 3-edge-coloring of G_2.

7.3.24. *If G_1 has a nowhere-zero k_1-flow and G_2 has a nowhere-zero k_2-flow, then $G_1 \cup G_2$ has a nowhere-zero k_1k_2-flow.* Let D be an orientation of $G = G_1 \cup G_2$, extend the flows on G_i by giving weight 0 to edges of $E(G) - E(G_i)$ them, and change signs of weights as needed so that both extended flows have orientation D. Thus G has a k_1-flow (D, f_1) and k_2-flow (D, f_2) such that (D, f_i) is nonzero on the edges of G_i, for each i

Let $f = f_1 + k_1 f_2$. By Proposition 7.3.16, (D, f) is a flow. For $e \in E(G_1)$, $f(e)$ is nonzero because $|f_1(e)| < k_1$. For $e \in E(G_2) - E(G_1)$, $f(e)$ is nonzero because $f_2(e)$ is nonzero. Furthermore, $|f(e)| \le (k_1 - 1) + k_1(k_2 - 1) = k_1k_2 - 1$. Thus (D, f) is a nowhere-zero k_1k_2-flow on G.

7.3.25. *Every spanning tree of a connected graph G contains a parity subgraph of G.* (A **parity subgraph** of G is a spanning subgraph H such that $d_H(v) \equiv d_G(v) (\pmod 2)$ for all $v \in V(G)$.) Let T be a spanning tree of G.

Proof 1 (induction on $k = e(G) - n(G) + 1$): We have $k = 0$ if and only if $G = T$, in which case G itself is the desired subgraph H. For the induction step, consider $k > 0$, and let $e = xy$ be an edge outside T. Since T is a spanning subgraph of $G' = G - e$, the induction hypothesis yields a parity subgraph H' of G' contained in T. Vertex degrees are the same in G and G' except for x and y. Form $E(H)$ by taking the symmetric difference of H' with the unique x, y-path in T. This changes the parity of the degree only at x and y, as desired.

Proof 2 (construction): Let $U = \{v_1, \dots, v_{2l}\}$ be the set of vertices in G with odd degree. Let P_i be the unique v_{2i-1}, v_{2i}-path in T. Let H_0 be the spanning subgraph of G with no edges, and let $H_i = H_{i-1} \triangle P_i$ for $1 \le i \le l$. (Equivalently, $H = H_l$ has precisely those edges appearing in an odd number of $P_1, \dots, P_l$.) The processing of P_i changes the degree parity only at v_{2i-1} and v_{2i}. Thus H_l has odd degree at precisely the vertices of U.

Proof 3 (induction on $n(G)$): We prove the statement more generally for multigraphs. When $n(G) = 1$, the vertex has even degree and the 1-vertex spanning tree is a parity subgraph. For $n(G) > 1$, select a leaf x of T, and let y be its neighbor in T. Form G' from $G - x$ by adding a matching of size $\lfloor d_G(x)/2 \rfloor$ on $N_G(x)$; this may introduce multiple edges. When $d_G(x)$ is odd, let y be the vertex omitted from the matching. By construction, $d_{G'}(v) \equiv d_G(v) \pmod 2$ for all $v \in V(G')$, except for y when $d_G(x)$ is odd. Also $T - x$ is a spanning tree of G'. By the induction hypothesis, G' has a parity subgraph H' contained in $T - x$. If $d_G(x)$ is even, then we add x to H' as an isolated vertex to obtain the desired parity sugraph H. If $d_G(x)$ is odd, then we also add the edge yx.

7.3.26. *For $k \ge 3$, a smallest nontrivial 2-edge-connected graph G having no nowhere-zero k-flow must be simple, 2-connected, and 3-edge-connected.* We may assume that G is connected. Loops never contribute to net flow out of a vertex, so their presence does not affect the existence of nowhere-zero k-flows. If G has a vertex v of degree 2, then G has a nowhere-zero k-flow if and only if the graph obtained by contracting an edge incident to v has a nowhere-zero k-flow. Thus we may assume that $\delta(G) \ge 3$.

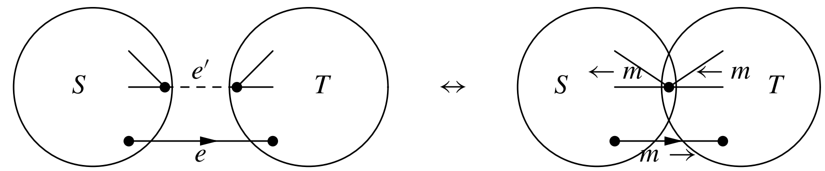
Since G has no cut-edge, each block of G is 2-edge-connected. If G has a cut-vertex, G has no nowhere-zero k-flow only if some block of G has no nowhere-zero k-flow. Thus We may assume that G is 2-connected.

Suppose that $e, e' \in E(G)$ have the same endpoints. If $G - e'$ is not 2-edge-connected, then $\{e, e'\}$ is a block in G, and we can apply the preceding paragraph. Thus we may assume that $G - e'$ has a nowhere-zero k-flow (D, f). We obtain such a flow for G by shifting $f(e)$ up or down by 1 and letting $f(e') = 1$, oriented with or against e depending on whether we shifted $f(e)$ down or up. The shift is possible because $k \ge 3$. Thus we may assume that G is simple.

It remains only to consider a nontrivial 2-edge cut $\{e, e'\}$ (we have eliminated the case where e, e' share a vertex of degree 2). The bridgeless graphs are those where every two vertices lie in a common circuit, and contracting an edge of such a graph with at least three vertices does not destroy this property. Thus we may assume that $G \cdot e'$ has a nowhere-zero k-flow

(D, f). Let S, T be the vertex sets of the components of $G - \{e, e'\}$, and let w be the vertex of $G \cdot e'$ obtained by contracting e'. We may assume that e is oriented from S to T in D.

Let $m = f(e)$. Because $f^*(S \cup w) = 0$, the edges between w and T contribute $-m$ to $f^*(w)$. Similarly, the edges between S and w contribute m to $f^*(w)$. Thus we let $f(e') = m$, oriented from T to S, to obtain a nowhere-zero k-flow on G.



7.3.27. *Every Hamiltonian graph G has a nowhere-zero 4-flow.* Since G has a nowhere-zero 4-flow if and only if it is the union of 2 even subgraphs (Theorem 7.3.25), we express G in this way. The Hamiltonian cycle C is one such subgraph. Let P be a spanning path obtained by omitting one edge of C. For each $e \in G - E(C)$, let $C(e)$ be the cycle created by adding e to P. Each edge outside C appears in exactly one of these cycles. Let C' be the spanning subgraph whose edge set consists of all edges appearing in an odd number of the cycles $\{C(e) \colon e \in E(G) - E(C)\}$. Since C' is a binary sum of even graphs, it is an even graph. It also contains $E(G) - E(C)$.

7.3.28. *Every bridgeless graph G with a Hamiltonian path has a nowhere-zero 5-flow.* If G is Hamiltonian, then G is 4-flowable (Exercise 7.3.27). Otherwise, let G' be the graph obtained from G by adding the edge e joining the endpoints of a spanning path in G.

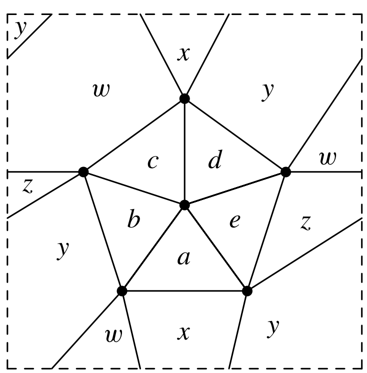
We claim that G' has a nowhere-zero 4-flow with weight 1 on e. Let C be a spanning cycle in G' through e, with vertices $v_1, \dots, v_n$ in order starting and ending at the endpoints of e. The remaining edges are chords of C; let there be m of them. Let $u_1, \dots, u_{2m}$ be the endpoings of chords of C, in order on C, listed with multiplicity (a vertex may be an endpoint of many chords. Let C' be the subgraph of G consisting of the chords of C together with the u_{2i-1}, u_{2i}-path on C not containing e, for $1 \le i \le m$. The parity of the number of chords incident to v_j is the same as the parity of the number of edges of C incident to v_j in C'.

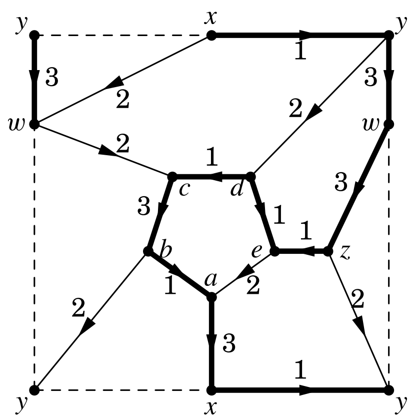
Hence C' and C are both even subgraphs and have positive 2-flows. Let (D, f') and (D, f) be the extension of these to G that are 0 outside C' and C, respectively. Since $C' \cup C = G'$, $(D, f + 2f')$ is a positive 4-flow on G' with weight 1 on e.

Let x be the tail and y the head of e under D. Let S be the set of vertices reachable from x under D without using e. If $S \ne V(D)$, then $[S, \overline{S}] = \{e\}$, with total flow 1. Since the net flow out of S is 0, exactly one unit of flow returns. Since flow comes in integer units, there is only one edge e' in $[\overline{S}, S]$. This makes e' a cut-edge of $G - e$, which has been forbidden.

We conclude that $S = V(D)$, and hence there is a x, y-path P not using e. Increasing the weights by 1 on P and decreasing the weight to 0 on e yields a nowhere-zero 5-flow on G.

7.3.29. *The dual of K_6 on the torus.* Below we show an embedding of K_6 and its surface dual in separate pictures for clarity. The vertices on the right correspond to the faces on the left; note that vertex degrees on the right correspond to face lengths on the left. The heavy edges on the right show that the dual is Hamiltonian. As in Exercise 7.3.27, we combine a constant flow of 1 along the spanning cycle with a constant flow of 2 on an even subgraph containing the remaining edges to obtain a nowhere-zero 4-flow; the resulting flow is shown below. Every nowhere-zero 4-flow is also a nowhere-zero 5-flow.





7.3.30. *A graph G is the union of r even subgraphs if and only if G has a nowhere-zero 2^r-flow.* *Necessity*. Let $G_1, \ldots, G_r$ be even subgraphs with union G. Given an orientation D of G that restricts to D_i on G_i, for each i, let (D_i, f_i) be a nowhere-zero 2-flows on G_i. Extend f_i to $E(G)$ by letting $f_i(e) = 0$ for $e \in E(G) - E(G_i)$. By Proposition 7.3.16, linear combinations of flows with the same orientation are flows. Hence (D, f) is a flow on G, where $f = \sum_{i=1}^{r} 2^{i-1} f_i$. Since $0 < \sum_{i=1}^{j} 2^{i-1} < 2^j$, always $0 < |f(e)| < 2^r$.

Sufficiency. Proposition 7.3.19 observes the case $r = 1$. We proceed by induction on r, modeling the induction step on the proof in Theorem 7.3.25.

Let (D, f) be a positive 2^r-flow on G. Let $E_1 = \{e \in E(G)\colon f(e)$ is odd$\}$. By Lemma 7.3.23, E_1 forms an even subgraph of G. Thus there is a nowhere-zero 2-flow (D_1, f_1) on E_1, where D_1 agrees with D. Extend f_1 to $E(G)$ by letting $f_1(e) = 0$ for $e \in E(G) - E_1$; now (D, f_1) is a 2-flow on G.

Define f_2 on $E(G)$ by $f_2 = (f - f_1)/2$. By Proposition 7.3.16, (D, f_2) is a flow on G. It is an integer flow, since $f(e) - f_1(e)$ is always even. Since $1 \le f(e) \le 2^r - 1$ and $-1 \le f_1(e) \le 1$, we have $0 \le f_2(e) \le 2^{r-1}$. Equality may hold in either bound. Let $E_2 = \{e \in E(G)\colon 1 \le f_2(e) \le 2^{r-1} - 1\}$. Let G_2 be the spanning subgraph of G with edge set E_2.

If $f_2(e) \ne 2^{r-1}$ for all e, then (D, f_2) restricts to a nowhere-zero 2^{r-1}-flow on G_2. Otherwise, we lose inflow or outflow that is a multiple of 2^{r-1} at each vertex. Since the total of net outflows is 0, at least one has positive outflow and at least one has negative outflow. Furthermore, the set of vertices reachable in D from vertices with positive outflow contains a vertex with negative outflow, because otherwise the set of reachable vertices induces a subgraph with positive total of net outflows.

Let P be a x, y-path in D, where x has with positive net outflow and y has negative net outflow under (D, f); both outflows are multiples of 2^{r-1}. Obtain D' from D by flipping the orientation of each edge in P, and obtain f_2' from f_2 by letting $f_2'(e) = 2^{r-1} - f_2(e)$ for each $e \in E(P)$. Now f_2' is a positive weight on D', and net outflow is unchanged at all vertices except x and y. Furthermore, the total of the absolute values of the net outflows declines by $2 \cdot 2^{r-1}$.

Repeating this process yields a positive 2^{r-1}-flow on G_2. Since $f_1(e) \ne 0$ if $e \in E(G) - E(G_2)$, we have expressed G as the union of subgraphs G_1 and G_2, where G_1 with edge set E_1 is an even subgraph and G_2, by the induction hypothesis, is a union of $r - 1$ even subgraphs.

7.3.31. $(*)$ Let G be a graph having a cycle double cover forming 2^r even subgraphs. Prove that G has a nowhere-zero 2^r-flow. (Jaeger [1988]) *Comment: The short proof of this exercise uses group-valued flows and the flow polynomial of a graph. The arguments are not long, but the concepts are beyond the scope of the brief treatment here, so this exercise will be deleted.*

7.3.32. *A graph has a nowhere-zero 3-flow if and only if it has a modular 3-orientation.*

Necessity. Given that G is 3-flowable, let (D', f') be a positive 3-flow. Let $E_2 = \{e \in E(G)\colon f(e) = 2\}$. Let (D, f) be defined by switching the orientation on edges of E_2 to obtain D from D' and changing the weights on those edges from 2 to 1 to obtain f from f'. The change at edge of E_2 changes the net outflow at each endpoint by 3. Since all weights under f equal 1, we have net outflow at v equal to $d_D^+(v) - d_D - (v)$. Since the net outflow is a multiple of 3, D is a modular 3-orientation.

Sufficiency. Given a modular 3-orientation D, let $f(e) = 1$ for all $e \in E(G)$. Under (D, f), the net outflow at each vertex is a multiple of 3. An argument as in Exercise 7.3.30 now converts this to a nowhere-zero 3-flow on G. If each vertex has net outflow 0, then (D, f) is such a flow.

Otherwise, since the total of the net outflows is 0, the net outflow is a positive multiple of 3 at some vertex and a negative multiple of 3 at some other vertex, with at least one of each type. Furthermore, the set of vertices reachable in D from vertices with positive outflow contains a vertex with negative outflow, because otherwise the set of reachable vertices induces a subgraph with positive total of net outflows.

Let P be a x, y-path in D, where x has with positive net outflow and y has negative net outflow under (D, f); both outflows are multiples of 3. Obtain D' from D by flipping the orientation of each edge in P, and obtain f' from f by letting $f'(e) = 3 - f(e)$ for each $e \in E(P)$. Now f' is a positive weight on D', and net outflow is unchanged at all vertices except x and y. Furthermore, the total of the absolute values of the net outflows declines by $2 \cdot 3$. Repeating this process produces a positive 3-flow on G.

7.3.33. *If G is a bridgeless graph, D is an orientation of G, and $a, b \in \mathbb{N}$, then the following statements are equivalent.*

A) $\frac{a}{b} \le \frac{|[S,\overline{S}]|}{|[\overline{S},S]|} \le \frac{b}{a}$ for every nonempty proper vertex subset S.

B) G has an integer flow using D and weights in the interval $[a, b]$.

C) G has a real-valued flow using D and weights in the interval $[a, b]$.

B $\Rightarrow$ C. Every integer flow is a real-valued flow.

C $\Rightarrow$ A. Let (D, f) be such a flow. Weights in the interval $[a, b]$ are positive. Comparing total inflow and outflow, we have $|[\overline{S}, S]|\,a \le f^-(S) = f^+(S) \le |[S, \overline{S}]|\,b$, which yields the first inequality. Similarly, $|[S, \overline{S}]|\,a \le f^+(S) = f^-(S) \le |[\overline{S}, S]|\,b$, which yields the second.

A $\Rightarrow$ B. The all-0 flow is a nonnegative integer b-flow on G. Let (D, f) be a nonnegative integer b-flow on G that maximizes m, the minimum weight used, and within that minizimes the number of edges with weight m. Let e^* be an edge with weight m, directed as uv.

If there is a v, u-path P in G that travels forward along edges in D with weight less than b or backward along edges in D with weight more than $m + 1$, then we can increase $f(e^*)$ by 1, increase weight by 1 on forward edges of P, and decrease weight by 1 on backward edges of P to obtain another flow that contradicts the choice of (D, f).

Let S be the set of all vertices reachable from v by paths in G whose weights satisfy these constraints. We may assume that $u \notin S$, so $[S, \overline{S}]$ is a nontrivial edge cut. We have $f(e) = b$ for $e \in [S, \overline{S}]$ and $f(e) \in \{m, m+1\}$ for $e \in [\overline{S}, S]$. Since the net flow across any cut is 0, we have

$$b\,|[S,\overline{S}]| = \textstyle\sum_{e\in[S,\overline{S}]} f(e) = \sum_{e\in[\overline{S},S]} f(e) \le (m+2)\,|[\overline{S},S]| < a\,|[\overline{S},S]|\,.$$

Thus $|[S,\overline{S}]| / |[\overline{S},S]| < a/b$, which contradicts the hypothesis. We conclude that such switches can be made until the desired positive b-flow is obtained.

7.3.34. *Cycle double covers for special graphs.* Let C_m have vertices $v_1, \dots, v_m$ in order.

$C_m \vee K_1$. Use each triangle of the form xv_iv_{i+1}, where x is the central vertex and indices are taken modulo m, and use the cycle through $v_1, \dots, v_m$.

$C_m \vee 2K_1$. Use all $2m$ triangles.

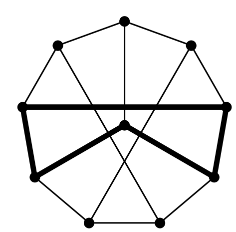
$C_m \vee K_2$. Let x and y be the two vertices outside the m-cycle. Use all triangles of the form xv_iv_{i+1} and yv_iv_{i+1}. We have not yet touched the edges xy and v_mv_1, and the edges xv_1, yv_1, xv_m and yv_m have been used only once. To finish the job, add the cycle through x, y, v_m, v_1 and the cycle through x, y, v_1, v_m.

7.3.35. *For every 3-regular simple graph with 6 vertices, the cycle double covers with fewest cycles consist of three 6-cycles.* There are only two such graphs, $K_{3,3}$ and $C_3 \square K_2$, which follows readily from the two cases of whether the graph contains a triangle or not. The total length of the cycles in the cover is 18, which can be achieved with as few as three cycles only if three 6-cycles are used.

In $K_{3,3}$, a 6-cycle C leaves 3 disjoint edges uncovered. There are two ways to pass a 6-cycle through these three edges, each using alternate edges on C. Hence every 6-cycle in $K_{3,3}$ lies in exactly one smallest cycle double cover.

If $C_3 \square K_2$, a 6-cycle C must visit both triangles, so it uses exactly two of the three edges joining the triangles. Hence it also uses exactly two edges on each triangle. Again, the remaining three edges are disjoint and must appear in both of the other 6-cycles. Again the two ways to complete a 6-cycle through these edges use alternate edges on C and complete a cycle double cover, so again each 6-cycle appears in one such cover.

7.3.36. *Cycle double cover of the Petersen graph using not all 5-cycles.* Consider the drawing of the Petersen graph with a 9-cycle on the "outside". Use the 9-cycle and the 6-cycle formed from the three pairwise-crossing edges and three edges on the outside 9-cycle. Finally, add three 5-cycles; each consists of one of the three crossing edges, two edges incident to the central vertex, and two edges on the outside 9-cycle, as shown below.



7.3.37. *Cycle covers in the Petersen graph.* Let G be the Peterson graph.

Every two 6-cycles in G share at least two edges. Since there are 10 vertices, two 6-cycles have at least two common vertices. In a 3-regular graph, two cycles have a common edge at each common vertex. Hence we have two common edges unless we have exactly two common vertices as the endpoints of one shared edge. In this case, the symmetric difference of the two 6-cycles is a 10-cycle, which does not exist in G.

G has no CDC consisting of five 6-cycles. One 6-cycle in such a CDC would share two edges with each of the other four 6-cycles. Since it has only 6 edges, this produces an edge covered three times.

G has no CDC consisting of even cycles. There is no 10-cycle or 4-cycle, and the total length is 30. Hence the possibilities for cycle lengths are five 6-cycles or $(8, 8, 8, 6)$. We have forbidden the former.

The latter would be a CDC using four even subgraphs. The symmetric difference of one with the other three (see Exercise 7.3.39) yields a CDC consisting of three even subgraphs. Since the graph is 3-regular, the full graph is the union of any two of the even subgraphs in a CDC by three even subgraphs. By Theorem 7.3.25, the graph then has a nowhere-zero 4-flow and hence a proper 3-edge-coloring. Since this does not exist for the Petersen graph, it has no CDC consisting of four cycles.

7.3.38. *Orientable CDC and even subgraphs.*

a) If a graph G has a nonnegative k-flow (D, f), then f can be expressed as $\sum_{i=1}^{k-1} f_i$, where each (D, f_i) is a nonnegative 2-flow on G.

Proof 1 (networks, Menger's Theorem, and induction on k). The case $k = 2$ is Proposition 7.3.19; consider $k > 2$. Let $E_0 = \{e \in E(D)\colon f(e) = 0\}$ and $E' = \{e \in E(D)\colon f(e) = k-1\}$, and let $r = |E'|$. Construct a network N from $D - E' - E_0$ by adding vertices s and t with edges sy and xt of weight $k-1$ for each edge $xy \in E'$. View f as both flow and capacity. Adding an edge ts of weight $r(k-1)$ would turn f into a circulation, so f on N is a flow of value $r(k-1)$.

Hence every s,t-cut in N has capacity at least $r(k-1)$. Since every edge has weight at most $k-1$, every s,t-cut has at least r distinct edges. By Menger's Theorem, N has r pairwise edge-disjoint s,t-paths. These combine with E' to form an even subgraph E_{k-1} of D, containing E' and contained in $D - E_0$. Let (D, f_{k-1}) be the nowhere-zero 2-flow that is nonzero on E_{k-1}.

Reducing weights by 1 on E_{k-1} yields a nonnegative $(k-1)$-flow (D, f') on G. By the induction hypothesis, there are 2-flows (D, f_i) for $1 \le i \le k-1$ such that $f' = \sum_{i=1}^{k-2} f_i$. Now $f = \sum_{i=1}^{k-1} f_i$, as desired.

Proof 2 (manipulation of flows and induction on k). Define E_0 and E' as above. When (D, f) is restricted to $D - E'$, the net outflow at each vertex is a multiple of $k-1$. The argument in Exercise 7.3.30 and Exercise 7.3.32 that switches orientation along paths from vertices with positive outflow to vertices with negative outflow produces a positive $(k-1)$-flow (D', f') on $G - E' - E_0$. If D' and D are opposite on an edge e, then $f'(e) = k-1-f(e)$. Switching these edges back yields a nowhere-zero $(k-1)$-flow (D, f_1) on $G - E' - E_0$ with $f_1(e) \equiv f(e) \pmod{k-1}$ for all e.

Since f is nonnegative, $(D, f - f_1)$ is an integer k-flow on G in which every edge has weight $k-1$ or 0. Let $f_2 = (f - f_1)/(k-1)$. Now $(D, f - f_2)$ is a nonnegative $(k-1)$-flow on G, and (D, f_2) is a nonnegative 2-flow on G. By the induction hypothesis, there are $k-2$ 2-flows with orientation D that sum to $f - f_2$, and adding f_2 to them yields f.

b) A graph G has a positive k-flow (D, f) if and only if D is the union of $k-1$ even digraphs such that each edge e in D appears in exactly $f(e)$ of them. Necessity follows from part (a). Since the $k-1$ guaranteed 2-flows on G have the same orientation D and are all nonnegative, they yield the desired even digraphs $D_1, \dots, D_{k-1}$ by letting $E(D_i) = \{e \in E(D)\colon f_i(e) = 1\}$. For sufficiency, the even digraphs convert to 2-flows that sum to (D, f).

c) A graph G has a nowhere-zero 3-flow if and only if it has an orientable cycle double cover forming three even subgraphs.

Necessity. If G has a nowhere-zero 3-flow, then G has a positive 3-flow (D, f), and part (b) yields two even digraphs such that each edge e appears in $f(e)$ of them. Reversing the orientation on one of them yields two even digraphs D_1 and D_2 that are oppositely oriented on the edges with $f(e) = 2$. At a given vertex v, those edges contribute the same indegree and outdegree. Therefore, $D_1 \cup D_2$ has the same number of edges entering and leaving v among the edges with $f(e) = 1$. Let D_3 be the reverse of this subgraph of $D_1 \cup D_2$. Now D_3 is a third even subgraph completing an oriented cycle double cover with D_1 and D_2.

Sufficiency. Let D_1, D_2, D_3 be the subgraphs in the given oriented CDC. Let D_1' be the reversal of D_1. Now $D_1' \cup D_3$ is an orientation of G; call it D. Let $f(e) = 2$ if $e \in D_1' \cap D_3$; otherwise, $f(e) = 1$. By part (b), (D, f) is a positive 3-flow on G.

Direct proof that (D, f) is a positive 3-flow without using part (b). In D, each edge is oriented the same way as the higher-indexed of the two among $\{D_1, D_2, D_3\}$ that contains it, and $f(e)$ is the difference of those indices. For $1 \le i < j \le 3$, let $a_{i,j}$ be the number of edges in G that enter v in D_j and leave it in D_i, and let $b_{i,j}$ be the number that leave it in D_j and enter it in D_i. The net outflow at v is $b_{1,2} + 2b_{1,3} + b_{2,3} - a_{1,2} - 2a_{1,3} - a_{2,3}$. The computations below show that this quantity is 0, since D_1, D_2, D_3 are even digraphs. Thus (D, f) is a positive 3-flow on G.

$$b_{1,3} + b_{2,3} = d^+_{D_3}(v) = d^-_{D_3}(v) = a_{1,3} + a_{2,3}$$
$$b_{1,2} + b_{1,3} = d^-_{D_1}(v) = d^+_{D_1}(v) = a_{1,2} + a_{1,3}$$

7.3.39. *If a graph G has a CDC formed from four even subgraphs, then G also has a CDC formed from three even subgraphs.* Let E_0, E_1, E_2, E_3 be the edge sets of the four even subgraphs. Form instead the sets $E_0 \triangle E_1$, $E_0 \triangle E_2$, and $E_0 \triangle E_3$. These form even subgraphs, since the symmetric difference of two even subgraphs is an even subgraph.

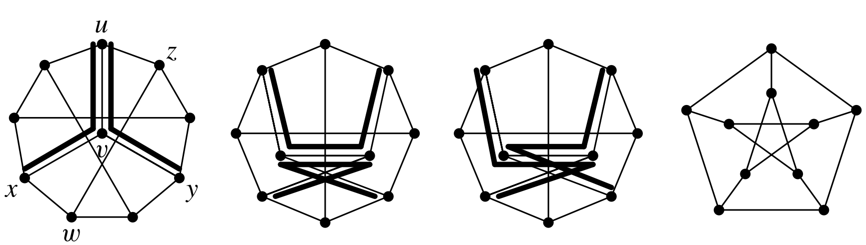
If previously an edge belonged to E_i and E_j but not E_0, then now in belongs to $E_0 \triangle E_i$ and $E_0 \triangle E_j$ but not to $E_0 \triangle E_k$. If it belonged to E_0 and E_k, then the same statement holds. Hence the new family is a CDC by three even subgraphs.

7.3.40. *The solution to the Chinese Postman Problem in the Petersen graph has length 20, but the least total length of cycles covering the Petersen graph is 21.* Since the Petersen graph is regular of odd degree, at least one edge must be added incident to each vertex to obtain an Eulerian spanning supergraph. Hence at least 5 edges must be added, and the total length of an Eulerian circuit in a supergraph is at least 20. Since the graph has a 1-factor, a solution of length 20 exists.

Cycles covering the graph together solve the Chinese Postman Problem, so their total length must be at least 20. If equality holds, then the lengths must be one of $(9,6,5)$, $(8,6,6)$, $(5,5,5,5)$, since the cycle lengths are in $\{5,6,8,9\}$.

If a 9-cycle is used, then one vertex is missed, and the other two cycles must visit it. Let v be the vertex missed by the 9-cycle, let u and x be the neighbors of v on the 6-cycle, and let u and y be the neighbors of v on the 5-cycle. Now the 6-cycle must be completed using a u,x-path of length 4, and the 5-cycle must be completed using a u,y-path of length 3. There are two choices for each of these paths, but all choices miss one particular edge, which we have labeled zw in the drawing on the left below.

If an 8-cycle is used, then two adjacent vertices are missed. The two 6-cycles must both visit these vertices, and girth 5 requires that all 6-cycles through two adjacent vertices use the edge joining them. The remaining edges in the 6-cycles can be distributed in several ways, but in each way some edge remains uncovered.



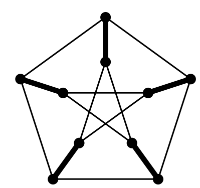
If four 5-cycles are used, then draw the graph so that one of them is the outer 5-cycle. A 5-cycle uses 0 or 2 of the edges crossing from the outer

5-cycle to the inner one. In order to cover these cross edges, the remaining three 5-cycles must use two cross edges each. Such a 5-cycle uses one or two edges on the central cycle; two if the cross edges reach consecutive vertices on the outer cycle, one if they do not.

In order to cover the inner 5-cycle, at least two of the three crossing 5-cycles must have two edges on it. If they share a crossing edge, then the remaining 5-cycle covers the two remaining cross edges, cover no new edges on the inner 5-cycle, and an edge of the inner cycle is left uncovered. If they do not share a crossing edge, then their union avoids a vertex of the inner 5-cycle. Hence the remaining 5-cycle is left to cover three edges at one vertex, which it cannot do.

7.3.41. *Given a perfect matching in the Petersen graph, there is no list of cycles that together cover every edge of M exactly twice and all other edges exactly once.* Suppose that such a list of cycles exists, where M is the bold matching in the drawing below. If a cycle in the cover uses two edges of M, then it uses at least three edges not in M. If a cycle uses four edges of M, then it uses at least four edges not in M. Thus every cycle uses at least as many edges not in M as edges in M.

Together, the cycles must cover edges of M 10 times and edges not in M 10 times. Thus each cycle used must cover the same number of edges of each type. Such cycles have four edges of M and four edges not in M. Thus the total length is a multiple of 8, which is impossible since it is 20.



7.3.42. *If an optimal solution to the Chinese Postman Problem on a graph G decomposes into cycles, then G has a cycle cover of total length at most $e(G)+n(G)-1$.* By Exercise 7.3.25, every spanning tree of G contains a parity subgraph, and this has at most $n(G)-1$ edges. Taking all edges of G and adding one extra copy of each edge in a parity subgraph yields an Eulerian supergraph with at most $e(G)+n(G)-1$ edges. Hence the optimal solution to the Chinese Postman Problem has at most this size, and the hypothesis guarantees a cycle cover of at most this size.

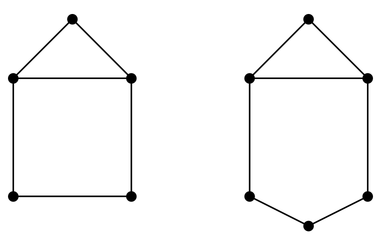
The minimum length of a cycle cover of $K_{3,t}$ is $4t$. Two cycles must visit each vertex of the larger part, and each such visit requires two edges, so the total length of the cycles is at least $4t$. This total length is achievable for $t \ge 2$ using cycles of lengths 4 and 6.

8.ADDITIONAL TOPICS

8.1. PERFECT GRAPHS

8.1.1. *Clique number and chromatic number of $\overline{C}_{2k+1}$ equal k and $k+1$ when $k \ge 2$.* A clique in $\overline{G}$ is an independent set in G. The largest independent set in C_{2k+1} has size k (using alternate vertices along the cycle). A proper coloring of $\overline{G}$ is a covering of $V(G)$ by cliques in G. The only cliques on a chordless cycle are two consecutive vertices, so we need at least $k+1$ of them to cover $V(C_{2k+1})$. When $k=1$, we can use a single triangle, and $\chi(\overline{C}_3)=1$.

8.1.2. *The smallest imperfect graph G such that $\chi(G)=\omega(G)$.* The only graph with at most five vertices that is not chordal or bipartite is the "house" H on the left below. This is perfect, since its proper subgraphs are chordal or bipartite and for the full graph $\chi(H)=3=\omega(H)$. Hence we need at least 6 vertices. The graph H' on the right below is imperfect, since it has a chordless cycle, but $\chi(H')=3=\omega(H')$.



8.1.3. *Cographs.*

a) G is P_4-free if and only if G can be reduced to the empty graph by iteratively taking complements within components. If G is not P_4-free, then a 4-set inducing P_4 also induces P_4 after complementation of its component, so these vertices can never be separated.

For the converse, we use induction on $n(G)$. By the induction hypothesis, it suffices to show that if G is P_4-free and connected, then $\overline{G}$ is

disconnected. Since G is P_4-free, $\text{diam}\,(G)=2$. If G has a cut-vertex x, then $\text{diam}\,(G)=2$ implies that every pair of vertices from distinct components of $G-x$ have x as a common neighbor. In this case $x \leftrightarrow V(G)-x$ and x is isolated in $\overline{G}$. Hence we may assume that $G-x$ is connected and P_4-free for all $x \in V(G)$. The induction hypothesis implies that $\overline{G}-x$, which is the same as $\overline{G-x}$, is disconnected. If deletion of an arbitrary vertex of $\overline{G}$ leaves a disconnected subgraph, then $\overline{G}$ has no spanning tree and is itself disconnected.

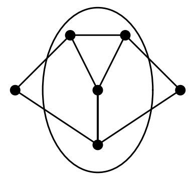
b) Every P_4-free graph is perfect.

Proof 1. If the graphs in a hereditary family **G** are not all perfect, then **G** contains a p-critical graph. By the Perfect Graph Theorem, a minimal imperfect graph and its complement are connected. Hence part (a) implies that there is no p-critical P_4-free graph, and all such graphs are perfect.

Proof 2. Since the class is hereditary, it suffices to prove that G has a clique consisting of a vertex of each color in the greedy coloring with respect to an arbitrary vertex ordering. If k colors are used, let Q be the largest clique of the form $\{u_{i+1},\dots,u_k\}$ such that u_j has color j. If $i=0$, we are done. Otherwise, there is no vertex of color i adjacent to all of Q, but by the greedy coloring algorithm every vertex of Q is adjacent to some (earlier) vertex of color i. Let v be a vertex of color i adjacent to the maximum number of vertices in Q, and choose $x \in Q-N(v)$. Let y be a vertex of color i adjacent to x. By the choice of v, there is a vertex $w \in Q$ such that $w \not\leftrightarrow y$. Since also $y \not\leftrightarrow v$, we have P_4 induced by v,w,x,y. This idea appears in Chvátal's proof that obstruction-free orderings are perfect orderings; this more specialized situation allows a simpler discussion.

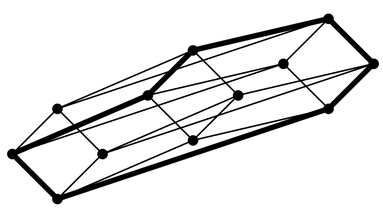
8.1.4. *Clique identification preserves perfection.* Let G_1 and G_2 be two induced subgraphs of G that are perfect and share only a clique S. If a proper induced subgraph of G is not an induced subgraph of G_1 or G_2, then it is formed by pasting together induced subgraphs of G_1 and G_2 at a subset of S. Therefore, we need only verify $\chi(G)=\omega(G)$ to complete an inductive proof that G is perfect. Since there are no edges between G_1-S and G_2-S, we have $\omega(G)=\max\{\omega(G_1),\omega(G_2)\}$. For $\chi(G)$, consider optimal colorings of G_1 and G_2. Since S is a clique, each uses different colors on the vertices of S. By permuting the labels of the colors, we can thus make the colorings agree on S to get a coloring of G with $\max\{\chi(G_1),\chi(G_2)\}$ colors. Since $\chi(G_1)=\omega(G_1)$ and $\chi(G_2)=\omega(G_2)$ by hypothesis, we have $\chi(G)=\max\{\chi(G_1),\chi(G_2)\}=\max\{\omega(G_1),\omega(G_2)\}=\omega(G)$.

8.1.5. *Identification at star-cutsets does not preserve perfection.* The union of two houses sharing a paw has an induced 5-cycle, as shown below. The star-cutset C is circled.



8.1.6. *If G is a cartesian product of complete graphs, then $\alpha(G) = \theta(G)$.* We have $G = K_{n_1} \square \cdots \square K_{n_r}$, indexed so that $n_1 \le \cdots \le n_t$. Always $\theta(G) \ge \alpha(G)$, so it suffices to exhibit a stable set and a clique covering of the same size. Using copies of the largest clique, we have $\theta(G) \le \prod_{i=1}^{r-1} n_i$. To obtain a stable set of this size, recall that H is m-colorable if and only if $\alpha(H \square K_m) = n(H)$ (Exercise 5.1.31). We apply this with $H = \prod_{i=1}^{r-1} K_{n_i}$; it suffices to show that H is n_r-colorable. Here we use another coloring result: $\chi(F_1 \square F_2) = \max\{\chi(F_1), \chi(F_2)\}$ (Proposition 5.1.11). Iterated, this yields $\chi(H) = n_{r-1} \le n_r$, and the proof is complete.

$K_2 \square K_2 \square K_3$ is not perfect. This graph has chromatic number and clique number both equal to 3 but has an induced 7-cycle, as shown below.



8.1.7. *The only color-critical 4-chromatic graph with six vertices is $C_5 \vee K_1$.* Let G be a 4-chromatic graph with six vertices. If G is perfect, then G contains K_4. If it is also 4-critical, then it cannot contain anything but K_4. Therefore, we may assume that G is imperfect. A 6-vertex graph is imperfect if and only if it has an induced 5-cycle. The 5-cycle is 3-colorable, with the third color used only once, anywhere. Therefore, the only way for G to be 4-chromatic it for it to be $C_5 \vee K_1$, which is in fact 4-critical.

8.1.8. (+) Prove that G is an odd cycle if and only if $\alpha(G) = (n(G) - 1)/2$ and $\alpha(G - u - v) = \alpha(G)$ for all $u, v \in V(G)$. (Melnikov–Vizing [1971], Greenwell [1978])

8.1.9. Let $v_1, \ldots, v_n$ be a simplicial elimination ordering of G.

a) Applying the greedy coloring algorithm to the construction ordering $v_n, \ldots, v_1$ yields an optimal coloring, and $\omega(G) = 1 + \max \sum_{x \in V(G)} |Q(x)|$, where $Q(v_i) = \{v_j \in N(v_i)\colon j > i\}$. Since $\{x\} \cup Q(x)$ is a clique, we have $\chi(G) \ge \omega(G) \ge 1 + \max |Q(x)|$. The greedy algorithm with this ordering yields $\chi(G) \le 1 + \max |Q(x)|$, since x has $\{Q(x)\}$ earlier neighbors, so equality holds throughout.

Then the stable set $\{y_1, \ldots, y_k\}$ obtained greedily from the elimination ordering is a maximum stable set, and the sets $\{y_i\} \cup Q(y_i)$ form a minimum clique covering. "Obtained greedily" means set $y_1 = v_1$, discard what remains of $Q(y_1)$ from the remainder of the ordering, and iterate. Let $S = \{y_1, \ldots, y_k\}$. When a vertex is included, it has no edge to an earlier chosen vertex, so S is stable. Furthermore, the vertices discarded when y_i is chosen form a subset of $Q(y_i)$, and $\{x\} \cup Q(x)$ is always a clique. Since every vertex in the list is chosen or discarded, $\{\{y_i\} \cup Q(y_i)\}$ forms a clique cover with the same size as S, so this is a minimum clique cover and S is a maximum stable set.

8.1.10. (•) Add a test to the MCS algorithm to check whether the resulting ordering is a simplicial elimination ordering. (Tarjan–Yannakakis [1984])

8.1.11. *The intersection graph G of a family of subtrees of a tree has no chordless cycle.* Suppose that G has a chordless cycle $[v_1, \ldots, v_k]$, with $T_1, \ldots, T_k$ being the corresponding subtrees of the host tree T. For each i, the edge $v_i v_{i+1}$ (indices modulo k) yields a vertex $w_i \in V(T_i) \cap V(T_{i+1})$. The subtree T_i has a unique w_{i-1}, w_i-path in T. Let x_i be the last common vertex of the w_i, w_{i+1}-path and the w_i, w_{i-1}-path. The x_{i-1}, x_i-path in T is contained in T_i, but its internal vertices belong to no other trees in the list. Therefore, the union of the x_{i-1}, x_i-paths in T, over all i, is a closed walk with no repeated vertices, which contradicts T having no cycles.

8.1.12. *Every graph is the intersection graph of a family of subtrees of some graph.* Given a graph G, let G' be the graph obtained by subdividing each edge of G. Associate with $v \in V(G)$ the star in G' formed by the edges incident to v. The stars for u and v intersect in G' if and only if u and v are adjacent in G.

8.1.13. *Every chordal graph has an intersection representation by subtrees of a host tree with maximum degree 3.* A chordal graph has an intersection representation by subtrees of a host tree T. If T has a vertex x of degree exceeding 3, then form T' as follows. In T, subdivide each edge incident to x (introducing a set S of new vertices), delete x, and add edges forming a path with vertex set S; this yields T'. Modify the subtrees that contained x by replacing x with S. The pairs of subtrees that intersect remain the same, and T' has one less vertex of degree exceeding 3 than T does. Iterating this process produces the desired representation.

8.1.14. *If Q is a maximal clique in a connected chordal graph G and $x \in V(G)$, then Q has two vertices whose distances from x are different.*

Proof 1 (chordless cycles). If $x \in Q$, then we take x and some other vertex of Q. Otherwise, suppose that all vertices of Q have the same distance from x. For $v \in Q$, let $R(v)$ be the set of vertices just before v on shortest x, v-paths. For $v, v' \in Q$, we claim that $R(v)$ and $R(v')$ are ordered by inclusion. Otherwise, there exist $u \in R(v)$ and $R(v')$ such that $u \not\leftrightarrow v'$ and $u' \not\leftrightarrow v$. Now let P and P' be a shortest x, u-path and a shortest x, u'-path. The subgraph induced by $V(P) \cup V(P')$ is connected; let D be a shortest u, u' path in this subgraph. The length of D is at least 1. No internal vertices of D are adjacent to v or v', since this would yield shorter paths to Q. Therefore, adding the edges $uv, vv', v'u'$ to D completes a chordless cycle C of length at least 4. This is a contradiction.

We conclude that $R(v)$ and $R(v')$ are ordered by inclusion. Since this is true for all pairs of vertices in Q, there is a vertex $w \in Q$ such that $R(w)$ is contained in $R(v)$ for all $v \in Q - \{w\}$. Thus there is a vertex, in $R(w)$, that is adjacent to all of Q. This contradicts Q being a maximal clique. We conclude that Q has two vertices with different distance from x.

Proof 2 (clique trees). Consider a clique tree representation of G in a smallest host tree T. By Lemma 8.1.16, the vertices of T correspond to the maximal cliques in G. Let q be the vertex corresponding to Q. Since $x \notin Q$, the tree T_x assigned to x cannot contain q. Let P be the path in T from q to T_x, with q' being the neighbor of q on P. Let Q' be the maximal clique of G corresponding to q', and choose $v \in Q - Q'$. Since G is connected, some shortest path (of subtrees) links T_x to q. Hence some vertex w belongs to $Q \cap Q'$. Thus P encounters T_w before T_v when followed from T_x. This implies that $d_G(x, w) < d_G(x, v)$.

8.1.15. *Intersection graphs of subtrees of graphs.* A *fraternal orientation* of a graph is an orientation such that any pair of vertices with a common successor are adjacent.

a) A simple graph G is chordal if and only if it has an acyclic fraternal orientation. If G is chordal, then G has a simplicial elimination ordering $v_1, \ldots, v_n$. With respect to this ordering, orient the edge v_iv_j from v_j to v_i if $i < j$. Then $v_j \to v_i$ and $v_k \to v_i$, implies that v_j, v_k are remaining neighbors of v_i when v_i is deleted, so the simplicial property of the ordering guarantees $v_j \leftrightarrow v_k$ in G. If G is not chordal, let C be a chordless cycle in G. Let F be an arbitrary acyclic orientation of G. Along C there must be a successive triple u, v, w such that $u \to v$ and $w \to v$ in F, which means F is not fraternal.

b) Example of a graph with no fraternal orientation. Let G be the graph consisting of two 4-cycles sharing a vertex v, and suppose G has a fraternal orientation. Since $N(v)$ is independent, v has at most one edge oriented inward, so we may choose one of the 4-cycles to have both edges incident to v oriented out from v. If u is the remaining vertex of that 4-cycle, we cannot have both edges involving u oriented into u, but $u \not\leftrightarrow v$ forbids either edge to be oriented out from u.

c) G is the intersection graph of a rootable family of trees if and only if G has a fraternal orientation (a family of subtrees is *rootable* if the trees can be assigned roots so that a pair of them intersects if and only if at least one of the two roots belongs to both subtrees). Suppose G is the intersection graph of a rooted family of trees in a graph, with $f(v)$ the tree assigned to $v \in V(G)$. If $xy \in E(G)$, orient the edge xy toward the vertex in $\{x, y\}$ whose tree has root lying in both of $\{f(x), f(y)\}$ (choose the orientation arbitrarily if both roots satisfy this). If $u \to v$ and $w \to v$, then the root of $f(v)$ lies in both $f(u)$ and $f(w)$; hence $f(u)$ and $f(w)$ intersect and u, w are adjacent.

Conversely, let G be a fraternally oriented graph. For each vertex v, let $f(v)$ be the substar of G consisting of all edges of G oriented out from v, and root it at v. For any edge xy oriented as $x \to y$, we have the root of $f(y)$ in $f(x)$. To complete the proof that G is the intersection graph of $\{f(v)\}$, we show that $xy \notin E(G)$ implies $f(x) \cap f(y) = \varnothing$. Nonadjacency of x, y implies that neither of $f(x), f(y)$ contains the root of the other, and hence $f(x) \cap f(y) \neq \varnothing$ requires that x, y have a common successor. This contradicts the assumption that the orientation is fraternal.

8.1.16. *A simple graph G is a forest if and only if every pairwise intersecting family of paths in G has a common vertex.* If G contains a cycle, then the three paths on the cycle joining any two of three specified vertices on the cycle form a pairwise intersecting family of paths with no common vertex.

Conversely, if G is a forest, then a pairwise intersecting family of paths lies in a single component of G and hence is a pairwise intersecting family of subtrees of a tree. By Lemma 8.1.10, the paths have a common vertex.

8.1.17. *For a graph G, the following are equivalent (and define split graphs).*

A) $V(G) = S \cup Q$, where S and Q induce a stable set and clique in G.

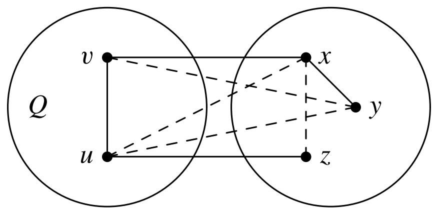
B) G and $\overline{G}$ are chordal.

C) G has no induced C_4, $2K_2$, or C_5.

A $\Rightarrow$ B. A cycle in C cannot visit vertices of S in succession, so a cycle of length at least 4 has two nonconsecutive vertices in Q; they are adjacent.

B $\Rightarrow$ C. C_4 and C_5 are chordless cycles. The vertices of a $2K_2$ in G would induce a chordless 4-cycle in $\overline{G}$.

C $\Rightarrow$ A. Let Q be a maximum clique minimizing $e(G - Q)$. Suppose that $G - Q$ has an edge xy. Since G is C_4-free, $N(x) \cap Q$ and $N(y) \cap Q$ are ordered by inclusion; we may assume that $N(y) \cap Q \subseteq N(x) \cap Q$. Since G is $2K_2$-free, x cannot have two nonneighbors in Q. Since Q is a maximum clique, x has a nonneighbor u in Q and y another nonneighbor v in Q. The edges (solid) and non-edges (dashed) within $\{x, y, u, v\}$ are shown below.

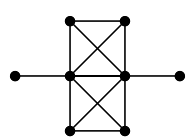


Since $Q - u + x$ is a clique, the choice of Q implies that u as many neighbors as x outside Q. Since $y \in N(x) - N(u)$, there exists $z \notin Q$ such that $z \in N(u) - N(x)$. Now $G[x, y, z, u] \neq 2K_2$ requires $y \leftrightarrow z$, after which $G[x, y, z, v] \neq C_4$ requires $v \not\leftrightarrow z$ and $G[u, v, x, y, z] \neq C_5$ requires $v \leftrightarrow z$. The contradiction implies that $G - Q$ has no edges.

8.1.18. *If $d_1 \geq \cdots \geq d_n$ is the degree sequence of a simple graph G, and m is the largest value of k such that $d_k \geq k - 1$, then G is a split graph if and only if $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$.* The Erdős-Gallai condition says that $d_1 \geq \cdots \geq d_n$ are the vertex degrees of such a graph if and only if $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i r$ for $1 \leq k \leq n$; call this the *k-th E-G condition*. *Split graphs* are those with a vertex partition $V(G) = Q \cup S$ such that Q is a clique and S is a stable set.

Necessity. If G is a split graph with maximum clique Q, then the $|Q|$ vertices of Q have degree at least $|Q| - 1$, and other vertices have degree at most $|Q| - 1$. Hence $m = |Q|$, and the vertices of Q are $|Q|$ vertices of highest degree. Counting their degree by the edges inside Q and out shows that the mth E-G condition holds with equality.

Sufficiency. Conversely, assume equality in the mth E-G condition. Whenever the kth E-G condition holds with equality, the k vertices of highest degree must form a clique. If in addition the contribution in $\sum_{i=k+1}^{n} \min\{k, d_i\}$ is always d_i, then all edges from the remaining vertices go to the k highest-degree vertices, and G is a split graph. When $k = m$, $d_i < i - 1 \leq k$ for all $i > k$, so this does indeed hold. (Note: The graph below is not a split graph, but satisfies the E-G condition with equality at $k = 2 \neq 4 = m$.)



8.1.19. *The trees that are split graphs are stars or double-stars. No nonisomorphic trees that are split graphs have the same degree list. Nonisomorphic split graphs with the same degree list arise by finding nonisomorphic bipartite graphs with the same degrees within partite sets and placing a complete graph on one partite set.* The only cliques in a tree T are single vertices or two adjacent vertices. If T is a split graph, then the remaining vertices are adjacent only to these. Hence T is a star or double-star. Such a tree is determined by the degrees of the vertices with degree exceeding 1.

For explicit nonisomorphic split graphs, note that C_8 and $2C_4$ have the same degree list, and adding edges to make one of the partite sets into a clique maintains that property. The graphs are not isomorphic.

8.1.20. *G is a k-tree (obtained from a k-clique by successively adding simplicial vertices of degree k) if and only if G 1) is connected, 2) has a k-clique but no $k+2$-clique, and 3) has only k-cliques as minimal vertex separators.*

Necessity. By induction on $n(G)$. The conditions hold for the unique k-tree obtained by adding two vertices to the initial k-clique. Each subsequent vertex addition maintains connectedness. Let v be the last vertex added. Any new clique must contain new edges and thus must contain v, but v belongs only to a $k+1$-clique. For (3), suppose S is a minimal vertex separator of G. If S is a minimal vertex separator of $G - v$, we apply the induction hypothesis. A vertex of a minimal vertex separator S has neighbors in two components of $G - S$; hence no minimal vertex separator of a graph contains a simplicial vertex. Hence $v \notin S$. If v is not isolated in $G - S$, then S is a minimal vertex separator in $G - v$, because $N(v)$ is a clique. We conclude that v is isolated in $G - S$, this requires $S = N(v)$, which induces a k-clique.

Alternative proof of (3), without induction. For property (3), the construction procedure implies that G is chordal, which implies that every minimal vertex separator induces a clique. If S is a minimal x, y-separator, let x be the first of $\{x, y\}$ in the construction ordering. Among the component of $G - S$ containing y, let z be the first vertex in the construction ordering. When z is added, the only vertices that z can have as neighbors lie in S. Hence $k \leq |S| \leq k + 1$. Suppose $|S| = k + 1$ and H is a component of $G - S$. Every vertex of S has a neighbor in H, but no vertex of H is adjacent to all of S. Hence we can choose $u, v \in S$ and $x, y \in V(H)$ such that $u \leftrightarrow x$, $v \leftrightarrow y$, $u \not\leftrightarrow x$, $v \not\leftrightarrow y$. Adding a shortest x, y-path in H yields a chordless cycle; hence $|S| = k$.

Sufficiency. (3) implies that G is a chordal graph. Let v be the first vertex in a simplicial elimination ordering. Since v is isolated in $G - N(v)$, $N(v)$ contains a minimal vertex separator. Hence (3) implies $|N(v)| \geq k$. Since by (2) G has no $k+2$-clique, we have $|N(v)| \leq k$. Hence $d(v) = k$. To complete the proof, we must show that deleting a simplicial vertex of degree k does not destroy the conditions, so we can complete a "k-valent" simplicial elimination sequence by applying induction. Deleting a simplicial vertex does not disconnect a graph or create a $k+2$-clique, and if G is not a clique, then (3) implies that $G-v$ retains a k-clique. To prove that (3) is preserved, if a minimal vertex separator of $G-v$ is a minimal vertex separator of G, then it induces a k-clique.

We claim that every minimal x,y-separator of an induced subgraph of a graph is contained in a minimal x,y-separator of the full graph. If so, then a minimal x,y-separator of $G-v$ that is not a k-clique must be part of a minimal x,y-separator of G that contains v, which is impossible since no simplicial vertex belongs to a minimal vertex separator. To prove the claim, suppose S is a minimal x,y-separator in an induced subgraph H of G, so $S\cup(V(G)-V(H))$ separates x and y in G. Hence this set contains a minimal x,y-separator of G, but such a separator must include all of S, else we retain an x,y-path from H.

8.1.21. *An n-vertex chordal graph with no $(k+2)$-clique has at most $kn-\binom{k+1}{2}$ edges, with equality if and only if it is a k-tree.* This is the special case of Exercise 8.1.23 obtained by setting $r=k+1$.

8.1.22. *The number of k-trees with vertex set $[n]$ is $\binom{n}{k}[k(n-k)+1]^{n-k-2}$.* We show that the number of rooted k-trees with vertex set $[n]$ that have a fixed set of k vertices as a root clique is $[k(n-k)+1]^{n-k-1}$. To obtain the formula from this, note that every k tree has $1+k(n-k)$ k-cliques, beginning with a root and adding k each time a new vertex is grown from an old k-clique. On the other hand, there are $\binom{n}{k}$ ways to pick a set of k vertices to form a root clique; hence we multiply by $\binom{n}{k}$ and divide by $[k(n-k)+1]$ to obtain the final formula. Note that when $n=k$ there is only one k-tree, which agrees with the formula, so henceforth we may assume $n>k$.

To count the k-trees with label set $[n]$ and a fixed root $R\subseteq[n]$, we put them in 1-1 correspondence with lists of length $n-k-1$ chosen from a fixed alphabet of size $1+k(n-k)$. The alphabet consists of 0, which refers to the root, together with pairs (v,i) such that $v\in([n]-R)$ and $i\in[k]$. Since $n>k$, every vertex belongs to a $k+1$-clique; when we deal with *rooted* k-trees, the *leaves* are the vertices not in the root that belong to only one $k+1$-clique. Given a k-tree with root R, we form a list by iteratively deleting the leaf u with the least label and recording an appropriate member of the alphabet. If $N(u)=R$, we record 0. If $N(u)\neq R$, we want to record some other code in the alphabet that will enable us to recover the k-clique to which u is joined in growing the current tree from the root.

In growing the current tree from the root, there is a unique list of vertex additions that leads from the root to u (ignoring other additions not needed to reach u). When $N(u)\neq R$, there is a last non-root vertex v before u on this list; let this be the vertex part of the code recorded. When we add u, $N(u)$ consists of v together with all but one vertex of the k-clique to which v was connected when added. Let the index part of the code recorded be the *position* among this list of k of the vertex not in $N(u)$. After $n-k-1$ iterations, there remains one non-root vertex joined to the root.

This defines a unique list from each k-tree. To reconstruct from any list on these labels the unique k-tree that generates it, and thereby show that the map is a bijection, there are two phases. In the first phase, at each iteration select the least non-root label u that has not yet been marked finished. If the current code is 0, create edges from u to R. If the code is a vertex-index pair, create an edge from u to the vertex v that is the vertex part of the code. Mark u finished. After $n-k-1$ iterations, there remains one unfinished non-root vertex; join it to R.

The first phase produces a “skeleton” describing possible ways to grow the k-tree from the root. If we shrink the root to a single node, this is in fact a rooted tree that, for each non-root vertex, describes by its path to the root the list of vertices that must be added before it is added. The second phase fleshes out this skeleton. Moving outward from the root as the construction procedure would, we iteratively “expand” a non-root vertex u such that every previous vertex on the path to the root has already been expanded; this expansion creates the other edges formed when the k-tree is grown to u. Let u be a vertex whose deletion generated a non-root code (v,i). When we expand u, the vertex v is the last vertex on the path to it from the root and has already been expanded, which means that we know the set of vertices S to which v was joined when the tree grew to it. The code i tells us which element of S should not be joined to u. This two-phase procedure generates a unique k-tree from every list, and the k-tree generated from a list τ yields τ under the first procedure, so this is a bijection.

8.1.23. *An n-vertex chordal graph G with clique number r has at most $\binom{r}{j}+\binom{r-1}{j-1}(n-r)$ cliques of order j, with equality (for all j simultaneously) if and only if G is an $r-1$-tree.* We use induction on n. The formula holds for $n=r$. For $n>r$, let v be the first vertex to be deleted in a simplicial elimination order. Since v has at most $r-1$ neighbors, it is involved in at most $\binom{r-1}{j-1}$ cliques of order j. The j-cliques not containing v are bounded by the induction hypothesis. Furthermore, equality holds if and only if it holds for $G-v$ and adding v adds $\binom{r-1}{j-1}$ cliques of order j, which by the inductive hypothesis implies that G is an $r-1$-tree.

8.1.24. *Pairwise intersecting real intervals have a common point.* Let a be the rightmost left endpoint among these intervals, and let b be the leftmost right endpoint. If some right endpoint occurs before some left endpoint, then those two intervals do not intersect. Hence $a\leq b$. For every interval, its left endpoint is at most a, and its right endpoint is at least b. Hence every interval in the family contains the interval $[a, b]$, which we have shown is nonempty.

8.1.25. *A tree is an interval graph if and only if it is a caterpillar.* We prove the following equivalent for a tree G.
A) G is an interval graph.
B) G is a caterpillar.
C) G does not contain the tree Y formed by subdividing each edge of a claw.

B $\Rightarrow$ A. Create an interval for each vertex on the spine of the caterpillar, such that each interval intersects its the intervals for its neighbors on the spine and no others. This leaves part of each interval intersecting no other. Place small intervals for the leaf neighbors of each vertex x of the spine in the "displayed" area of the interval for x.

C $\Rightarrow$ B. A longest path P contains an endpoint of every edge. If some edge is missed, then there is an edge with neither endpoint on P but having a neighbor x on P (since the tree is connected). Since P is a longest path, P continues at least two edges in each direction from x. Now the tree contains Y, consisting of these six edges within distince 2 of x.

A $\Rightarrow$ C. If G contains Y but is an interval graph, then in an interval representation of G the intervals for the leaves of Y are pairwise disjoint. Name the leaves x, y, z in the order of the corresponding intervals, from left to right. The union of the intervals for the x, z-path in G must cover the gap between the intervals for x and z in the representation. Since this gap contains the interval for y, we obtain a contradiction, because y has no neighbor on this path.

8.1.26. *Every interval graph is a chordal graph and is the complement of a comparability graph.* If it is not a chordal graph, then it has a chordless cycle. A chordless cycle has no interval representation, because the two paths along the cycle between the vertices corresponding to the leftmost and rightmost intervals among these vertices must occupy all the space between them on the line, which produces chords between the two paths when the intersections are taken. Hence the full graph has no interval representation.

Given an interval representation of a graph G, orienting $\overline{G}$ by $x \to y$ if the interval for x is completely to the right of the interval for y expresses $\overline{G}$ as a comparability graph.

8.1.27. *A graph G has an interval representation if and only if the clique-vertex incidence matrix of G has the consecutive 1s property.*

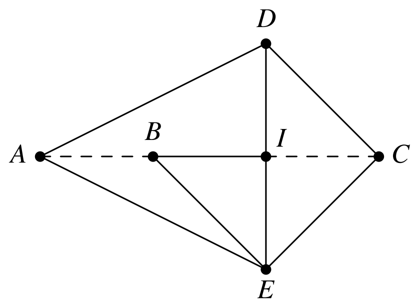
Necessity. From an interval representation, we obtain a natural ordering of the maximal cliques. By the Helly property (Exercise 8.1.24) the intervals corresponding to the vertices of a maximal clique have a common point. These points are different for distinct maximal cliques, because the interval for a vertex nonadjacent to some vertex of a maximal clique must be disjoint from the interval for that vertex. Therefore, we can place the cliques in a linear order by the order of the chosen points. Using this ordering on the clique-vertex incidence matrix exhibits the consecutive 1s property, because the interval for a vertex extends from the first chosen point for a clique containing it to the last. The vertex belongs to all maximal cliques whose chosen cliques are between these, and it belongs to no other maximal cliques, since intervals have no gaps.

Sufficiency. Let M be the clique-vertex incidence matrix of G, and suppose that M has the consecutive 1s property. We construct an interval representation. Permute the rows of M so the 1s are consecutive in the columns. On a line, select points in order left to right corresponding to the rows of M. For each column of M (vertex of G), specify an interval from the point for the first 1 in it to the point for the last 1 in it. This defines one interval for each vertex because the 1s are consecutive. It yields an interval representation of G because vertices are adjacent if and only if there is a maximal clique that contains both of them.

8.1.28. *A graph is an interval graph if and only if it has a vertex ordering $v_1, \ldots, v_n$ such that the neighborhood of each v_k among the lower-indexed vertices is a terminal segment $v_i, \ldots, v_{k-1}$.* Given an interval representation f, index the vertices in order of the right endpoints of the corresponding intervals. If $v_k \leftrightarrow v_i$ with $i < k$, then $f(v_k)$ extends as far to the left and the right endpoint of $f(v_i)$, so it contains the right endpoints of $f(v_i), \ldots, f(v_{k-1})$ and is adjacent to all those vertices.

8.1.29. *Interval graphs have no asteroidal triples.* Consider an interval representation for a graph G. Suppose that G has an asteroidal triple $\{x, y, z\}$; that is, three vertices such that connecting any two of them there is a path avoiding the neighborhood of the third. Rename these vertices x, y, z in the order of the corresponding intervals in the representation, from left to right. The union of the intervals for the x, z-path in G must cover the gap between the intervals for x and z in the representation. Since this gap contains the interval for y, we obtain a contradiction, because y has no neighbor on this path. (Comment: Interval graphs are precisely the chordal graphs that have no asteroidal triples.)

8.1.30. *The lying professor.* The intersection graph of the professor's presences in the library is an interval graph. The claimed sightings yield the graph below, where dotted edges are those confirmed from both sides and therefore presumed true. The graph contains two chordless 4-cycles, $DABI$ and $DAEC$. It is not possible to turn this into an interval graph by adding a single edge, and there is no reason to think a suspect would lie by leaving out other possible suspects. Therefore the most reasonable conclusion is that someone lied by trying to cast suspicion on someone else. The only single edge that can be removed to turn this into an interval graph (by destroying both chordless 4-cycles) is the edge due to Desmond's claim of seeing ("Honest") Abe. Hence we conclude that Desmond is the probable thief.



8.1.31. *G is a unit interval graph if and only if the matrix $A(G) + I$ has the consecutive ones property.* For necessity, take a unit interval representation and number the vertices in increasing order of left endpoint. Think of the interval for v_i as representing a loop at v_i. Then the fact that all intervals have the same length puts the right endpoints in the same order, and makes the vertices adjacent to v_i a consecutively-numbered sequence including v_i. In other words, with this ordering, the ones in each column of $A(G) + I$ appear consecutively.

If $A(G) + I$ has the consecutive ones property, then the 1's in each column of $A(G)$ are consecutive and include the diagonal. Let m be the number of distinct rows (or columns, since $A(G) + I$ is symmetric) of G. Construct a unit interval representation f by induction on m, using copies of m distinct intervals. If k columns are the same as the first, delete vertices $v_1, \ldots, v_k$. The remaining graph has the consecutivity property and $m - 1$ distinct rows, so by induction it has a unit representation. If the highest-indexed vertex adjacent to $v_1, \ldots, v_k$ is the jth type of row in $A(G) + I$, assign an interval to $v_1, \ldots, v_k$ that meets the first $j-1$ classes of intervals in f. Note that v_{k+1}, etc., also are adjacent to that high-indexed vertex, by the consecutivity property in $A(G) + I$.

8.1.32. (+) Prove that G is a proper interval graph (representable by intervals such that none properly contains another) if and only if the clique-vertex incidence matrix of G has the consecutive 1s property for both rows and columns. (Fishburn [1985]

8.1.33. *Every P_4-free graph is a Meyniel graph.* Let C be an odd cycle of length at least 5 in a P_4-free graph. Deleting one endpoint of the chord if

C has one chord (or deleting an arbitrary vertex if C has no chord) leaves an induced path with at least four vertices. This cannot occur in a P_4-free graph, so C has at least two chords.

8.1.34. *Every odd cycle of length at least 5 in a chordal graph has two noncrossing chords.* Since the graph is chordal, such a cycle C has a chord xy. This chord forms a cycle C' of length at least 4 with one of the x, y-paths along C. A chord of C' is also a chord of C, and its endpoints are on one of the x, y-paths in C, so the two chords are noncrossing.

8.1.35. *If C is an odd cycle in a graph with no induced odd cycle, then $V(C)$ has three pairwise-adjacent vertices such that paths joining them in C all have odd length.* We use induction on the length of C; the statement is trivial when C is a triangle. When C is longer, we know that it has a chord xy. One of the x, y-paths along C has even length; with xy it forms an odd cycle. By the induction hypothesis, this cycle C' has three has three pairwise-adjacent vertices such that paths joining them in C' all have odd length. Two of these paths are along C, and one uses the edge xy. Replacing xy with the other x, y-path along C in this path yields a path of odd length, since the lengths of xy and the path replacing it are both odd. Therefore, the triple provided by the induction hypothesis for C' has the desired properties for C.

8.1.36. *The conditions below are equivalent.*
A) Every odd cycle of length at least 5 has a crossing pair of chords.
B) For every pair $x, y \in V(G)$, chordless x, y-paths are all even or all odd.

B $\Rightarrow$ A. Two vertices on an odd cycle C are connected by paths of different parity along C, so by the parity condition at least one of the paths has a chord. Applying the same argument to points separated on C by the endpoints of such a chord yields another chord of C.

If the two chords are non-crossing, then one or both combines with part of C to form a smaller odd cycle of length at least 5, unless C itself has length 5. Two crossing chords in the smaller cycle would be crossing chords in C. Hence we may assume that C has length exactly 5 and has two non-crossing chords. Now applying the parity condition to a vertex pair on C that does not induce one of these chords yields a third chord that crosses at least one of the other two.

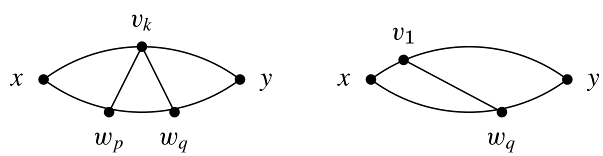
A $\Rightarrow$ B. Suppose that G has a vertex pair connected by chordless paths of opposite parity. Choose the pair $\{x, y\}$ and chordless x, y-paths P_1 and P_2 of even and odd length so that the sum of the lengths of P_1 and P_2 is as small as possible; call this "minimality". If P_1 and P_2 have a common vertex z, then z splits P_1 into two paths whose lengths have the same parity (and length at least two). Also z splits P_2 into two paths of opposite parity (and length at least two). The resulting x, z- or z, y-portions of P_1 and P_2

contradict minimality. Hence we may assume that $P_1 \cup P_2$ is a cycle C. Since P_2 cannot be a chord of P_1, the length of C is odd and at least 5.

We prove that C has no crossing chords. All chords join $P_1 - \{x, y\}$ and $P_2 - \{x, y\}$. Let P_1 and P_2 have vertices $x, v_1, \ldots, v_s, y$ and $x, w_1, \ldots, w_t, y$ in order, respectively (s is odd and t is even). Let w_p and w_q be the first and last neighbors of v_k in $P_2 - \{x, y\}$, if v_k has any such neighbors.

Suppose first that $2 \le k \le s-1$, so v_k partitions P_1 into chordless paths with the same parity as k. The parity of p is opposite to k, else two x, v_k-paths contradict minimality. Similarly, the parity of $t+1-q$ is opposite to k, which makes it the same as the parity of p. If $q > p$, then $q - p$ is even, else w_p, v_k, w_q and the w_p, w_q-portion of P_2 contradict minimality. We have now partitioned P_2 into three subpaths, of which the middle path has even length and the two extreme paths have the same parity; this is impossible and implies that v_k belongs to no chords.

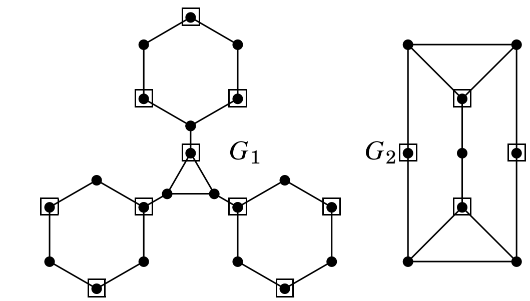
Now consider $k = 1$. As before, the v_1, y-paths yield $t + 1 - q$ even, and when $q > 1$ the x, w_q-paths yield q even. This is impossible, since $t + 1$ is odd. We conclude that $v_1 w_1$ is the only possible chord involving v_1. Similarly, $v_s w_t$ is the only possible chord involving v_s. We have proved that $v_1 w_1$ and $v_s w_t$ are the only possible chords of C, and they do not cross; this contradicts the hypothesis.



8.1.37. *Every perfectly orderable graph is strongly perfect.* Let G be a perfectly orderable graph and L an admissible ordering of G. I.e., G has no induced P_4 such that in L each endpoint appears before its neighbor. Let S be the greedy stable set with respect to L, i.e., place the first vertex of L in S, delete its neighbors, and iterate this step with the remaining vertices. Note that S is the set receiving color 1 under the greedy coloring for L.

We show that S meets every maximal clique. If S misses a maximal clique Q, then each vertex of Q must be deleted from the ordering due to having a prior neighbor that is in S. If all vertices of Q share a prior neighbor in S, then Q is not maximal. Hence we can choose $x, y \in Q$ and $u, v \in S$ such that $u \leftrightarrow x$, $v \leftrightarrow y$, but $u \not\leftrightarrow y$, $v \not\leftrightarrow x$. Since $x \leftrightarrow y$ and $u \not\leftrightarrow v$, these vertices induce P_4; since u comes before x and v before y, they induce an obstruction, contradicting the assumption that L is admissible.

8.1.38. *The graphs below are strongly perfect.* In each case, the marked stable set intersects all maximal cliques, but strong perfection also requires this for all induced subgraphs.



For G_1, an induced subgraph that omits a vertex of the central triangle is bipartite. Every bipartite graph is strongly perfect, because we can form a stable set intersecting all maximal cliques by taking one partite set from each nontrivial component plus all isolated vertices, and the family is hereditary. This takes care of all induced subgraphs of G_1 except those that retain the central triangle. For such a subgraph H, deleting an edge of the triangle yields a bipartite graph H' in which the three central vertices are in the same component. From this component of H', we choose the partite set containing only one vertex of the triangle in H; from others we take either partite set. The resulting set is stable in H' and intersects all maximal cliques in H', and it has the same properties in H.

For G_2, suppose that some induced subgraph H has a maximal clique Q avoiding the marked stable set S. This requires H to omit a vertex of S on a triangle. We may assume that Q is the lower horizontal edge. Now a "rotation" of S around the triangles intersects all maximal cliques in H unless H omits both of the top vertices. Now $H \subseteq P_4 + P_2$, but every disjoint union of paths has the desired property.

The graphs above are not perfectly orderable. A perfectly orderable graph has an orientation (associated with a perfect ordering) such that no induced P_4 has its pendant edges oriented outward. We show that these graphs have no such orientation; suppose that one exists.

For G_1, two of the cut-edges must be oriented in toward the triangle. Let yz be the oriented edge joining them, with xy being the entering cut-edge at its tail. The edges in a matching of size 3 on the 6-cycle containing z must be consistently oriented along the cycle, but one choice of this orientation conflicts with xy, and the other choice conflicts with yz.

For G_2, in the top half of the drawing, two of the three vertical edges must be oriented upward to avoid completing an obstruction with the top triangle. By symmetry, we may assume that these are the left and right vertical edges, but now either orientation of the horizonal edge on the bottom completes an obstruction with one of them.

8.1.39. *The graphs in Exercise 8.1.38 are a Meyniel graph but are not perfectly orderable.* The graphs have no chordless odd cycle (of length at least 5), so they vacuously satisfy the definition of a Meyniel graph. The task of showing they are not perfectly orderable is done in Exercise 8.1.38.

The graph $\overline{P}_5$ is perfectly orderable but is not a Meyniel graph. The graph $\overline{P}_5$ is the "house", a 5-cycle with one chord, so the cycle does not have the requisite two chords. There are two induces 4-vertex paths (each containing one endpoint of the cycle. If the cycle is numbered v_1, v_2, v_3, v_4, v_5 in order so that the chord is v_1v_4, then both copies of P_4 have one endpoint at v_5, so the associated orientation directs that pendant edge in toward the center, and there is no obstruction. Hence this is a perfect ordering.

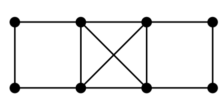
8.1.40. *Every chordal graph is weakly chordal.* If a graph has no chordless cycle, then it has no chordless cycle of length at least five. Suppose $v_1, \ldots, v_k$ in order induce in $\overline{G}$ a chordless cycle, meaning that G contains an antihole on these vertices. If $k = 5$, then $v_1v_3v_5v_2v_4$ is a chordless 5-cycle in G. If $k \geq 6$, then $v_1v_4v_2v_5$ is a chordless 4-cycle in G.

The graph H below is weakly chordal. Any cycle with more than four vertices has at least three in the central clique Q and hence has a chord. In $\overline{H}$, we need only forbid induced C_k for $k \geq 6$, since $\overline{C}_5 = C_5$. Note that $\overline{H}$ has 16 edges (too many for C_8), of which 3 are incident to each vertex of Q and 5 to each of the other vertices. Hence every 7-vertex subgraph has at least 11 edges. The 6-vertex induced subgraphs of $\overline{H}$ with only 6 edges are those where the deleted vertices are neighboring vertices of degree 2 in H (deleting 10 edges from $\overline{H}$), but such a subgraph of $\overline{H}$ is a 4-cycle with two pendant edges.

H is not strongly perfect.

Proof 1. Since $V(H)$ is covered by three disjoint cliques, $\alpha(H) \leq 3$. However, each vertex appears in two maximal cliques, so three vertices cannot meet all 7 maximal cliques.

Proof 2. There are 7 maximal cliques in H: one 4-clique and 6 edges. In a chordless path of three edges, a stable set meeting every maximal clique must contain at least two vertices, including at least one endpoint. Hence if a stable set S meets every maximal clique, the paths on the left and right force S to contain two vertices of the central clique.



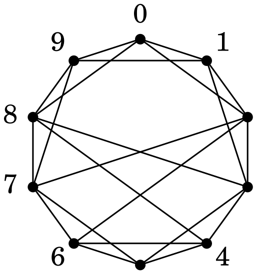
8.1.41. *SPGC $\Rightarrow$ Skew Partition Conjecture $\Rightarrow$ Star-Cutset Lemma.* The *Skew Partition Conjecture* states that no p-critical graph has a skew partition (a *skew partition* of G is a partition of $V(G)$ into nonempty sets X and Y such that $G[X]$ is disconnected and $\overline{G}[Y]$ is disconnected).

The SPGC states that every p-critical graph is an odd cycle or the complement of an odd cycle. Since a skew partition of G is also a skew partition of $\overline{G}$, we obtain the Skew Partition Conjecture from the SPGC by showing that an odd cycle has no skew partition. A skew partition requires X to use more than one segment along the cycle, but then the subgraph of the complement induced by the remaining vertices is connected.

To prove that the Skew Partition Conjecture implies the Star-Cutset Lemma, which states that no p-critical graph has a star-cutset, it suffices to show that a graph with a star-cutset has a skew partition. If C is a star-cutset in G, let $X = V(G) - C$ and $Y = C$. Now $G[X]$ and $\overline{G}[Y]$ are both disconnected, since the dominating vertex in C becomes an isolated] vertex in $\overline{G}[Y]$.

8.1.42. *The graph below is* 3, 3*-partitionable.* Due to the horizontal symmetry through the vertical axis, we need only check six classes of vertices to show that each $V(G - x)$ partitions into three 3-cliques and into three stable 3-sets. This is easy but tedious and seems to require a picture for each vertex.

Alternatively, by Theorem 8.1.39, since $\alpha(G) = \omega(G) = 3$, it suffices to show that (1) each vertex belongs to three 3-cliques and to 3 stable 3-sets, and (2) G has 10 3-cliques and 10 stable 3-sets, paired so that each intersects every set of the other type except its mate. We show this giving a matrix that lists the 3-cliques and stable 3-sets in the rows and columns and has the elements of $Q_i \cap S_j$ as the entries. Each vertex appears in three of the row labels and three of the column labels. However, the matrix does not contain a proof that there are no other cliques or stable sets of size 3. Curiously, the maximum cliques and stable sets are the same as in C_{10}^3 except for a switch of membership in two cliques and two stable sets, underlined below.



$$\begin{array}{cc} & \begin{array}{cccccccccc} 369 & 470 & 925 & \underline{8}16 & 703 & 49\underline{2} & 581 & 036 & 147 & 258 \end{array} \\ \begin{array}{c} 012 \\ 123 \\ \underline{8}34 \\ 345 \\ 456 \\ 567 \\ 67\underline{2} \\ 789 \\ 890 \\ 901 \end{array} & \begin{pmatrix} \varnothing & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 2 \\ 3 & \varnothing & 2 & 1 & 3 & 2 & 1 & 3 & 1 & 2 \\ 3 & 4 & \varnothing & 8 & 3 & 4 & 8 & 3 & 4 & 8 \\ 3 & 4 & 5 & \varnothing & 3 & 4 & 5 & 3 & 4 & 5 \\ 6 & 4 & 5 & 6 & \varnothing & 4 & 5 & 6 & 4 & 5 \\ 6 & 7 & 5 & 6 & 7 & \varnothing & 5 & 6 & 7 & 5 \\ 6 & 7 & 2 & 6 & 7 & 2 & \varnothing & 6 & 7 & 2 \\ 9 & 7 & 9 & 8 & 7 & 9 & 8 & \varnothing & 7 & 8 \\ 9 & 0 & 9 & 8 & 0 & 9 & 8 & 0 & \varnothing & 8 \\ 9 & 0 & 9 & 1 & 0 & 9 & 1 & 0 & 1 & \varnothing \end{pmatrix} \end{array}$$

8.1.43. *If x and v are nonadjacent vertices in a partitionable graph G, then every maximum clique containing x consists of one vertex from each stable set that is the mate of a clique containing v.* (The complementary assertion is that if x and v are adjacent vertices, then every maximum stable set containing x consists of one vertex from each clique that is the mate of a stable set containing v.)

By Theorem 8.1.41, the unique minimum coloring of $G - v$ consists of the $\omega(G)$ stable sets that are mates of the maximum cliques containing v. Since x and v are nonadjacent, a maximum clique containing x omits v and hence must contain exactly one vertex from each stable set in this coloring.

8.1.44. *No p-critical graph has antitwins.* Antitwins are a pair of vertices such that every vertex outside them is adjacent to exactly one of them. Consider a p-critical graph G, and let $\omega = \omega(G)$ and $\alpha = \alpha(G)$.

We first prove that a p-critical graph with antitwins $\{x, y\}$ has a clique of size $\omega - 1$ in $N(x)$ that doesn't extend into $N(y)$. Recall that $\omega(G - S) = \omega(G)$ for any stable set S in a p-critical G (reminder of proof - since $G - S$ is perfect, smaller clique-number would give a smaller coloring, extending to an ω-coloring of G by replacing S). Since G is partitionable, $G - x$ has a unique coloring by ω stable sets of size α; let S be the stable set containing y in this coloring, and let Q be an ω-clique in $G - S$. Since $G - x - S$ is $\omega - 1$-colorable (S is a color in the ω-coloring of $G - x$), Q must contain x. Since $G - x - S$ has no ω-clique, $Q' = Q - x \subset N(x)$ is the desired clique.

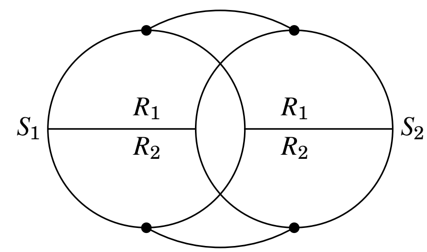
Reversing the roles of x and y yields a similar $\omega - 1$-clique in $N(y)$. Since the complement of a p-critical graph is p-critical, we also can apply the same argument to obtain $\alpha - 1$-cliques in $N_{\overline{G}}(x)$ and $N_{\overline{G}}(y)$ that translates into the desired $\alpha - 1$-stable sets in $N(y)$ and $N(x)$. Let S' be the resulting stable set of size $\alpha - 1$ in $N(y)$ that doesn't extend in $N(x) \cup N(y)$.

Choose u to be the vertex of Q' with the minimum number of neighbors in S'; u must have at least one neighbor v in S', else S' extends to u. Similarly, v must have a non-neighbor z in Q'. Since $v \in N(u) - N(z)$ and z has at least as many neighbors as u in S', z must have a neighbor w in S' that is not adjacent to u. Now y, v, u, z, w induce a chordless 5-cycle in G. This misses x, so G is not p-critical. (Note: For the special non-circulant partitionable graph pictured in the text, which is not p-critical, the top and bottom vertices are antitwins.)

8.1.45. *Stable sets and even pairs in partitionable graphs.*

a) If S_1, S_2 are maximum stable sets in a partitionable graph G, then $G[S_1 \triangle S_2]$ is connected. Let $S = S_1 \triangle S_2$. Let R_1 be the vertex set of a component of $G[S]$, and let $R_2 = S - R_1$. The sets $T_1 = (S_1 - R_1) \cup (S_2 - R_2)$ and $T_2 = (S_1 - R_2) \cup (S_2 - R_1)$ are stable sets with the same union and intersection as S_1 and S_2 (see figure). Hence $|T_1| + |T_2| = 2\alpha(G)$, which implies that $|T_1| = |T_2| = \alpha(G)$ since each has size at most $\alpha(G)$.

Since the rows of the incidence matrix between maximum stable sets and vertices are linearly independent, we cannot have two pairs of stable sets with the same union and intersection. Either $T_1 = S_1$ and $T_2 = S_2$, which yields the contradiction $R_1 = \varnothing$, or $T_1 = S_2$ and $T_2 = S_1$, in which case $R_1 = S$ and $G[S]$ is connected.



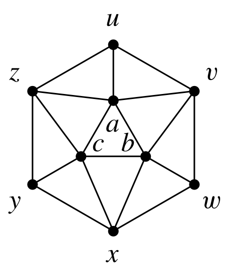
b) No partitionable graph (and hence no p-critical graph) has an even pair. Let x, y be any two vertices in a partitionable graph G. Let S be a maximum stable set containing x in $G - y$, and let T be a maximum stable set containing y in $G - x$. Let $H = G[S \triangle T]$. By part (a), H is connected. Since S and T are stable sets, H is bipartite, with partite sets $S - T$ and $T - S$. By construction, $x \in S - T$ and $y \in T - S$, so every x, y-path in H has odd length. Since H is an induced subgraph of G, a shortest x, y-path in H is a chordless x, y-path in G. Hence x, y is not an even pair in G.

8.1.46. *If G is partitionable, and S_1, S_2 are stable sets in the optimal coloring of $G - x$, then $G[S_1 \cup S_2 \cup \{x\}]$ is 2-connected.* Since S_1, S_2 are maximal, x is adjacent to a vertex of each. Since S_1, S_2 are disjoint, $S_1 \oplus S_2 = S_1 \cup S_2$. Thus part (a) of the preceding problem implies that $H = G_{S_1 \cup S_2 \cup x}$ is connected and that x cannot be a cut-vertex of H. If H has a cut-vertex, we may assume it is $s \in S_1$. Let G_1 be a component of $H - s$ not containing x, and let G_2 be the rest of $H - s$, with $V_i = V(G_i)$.

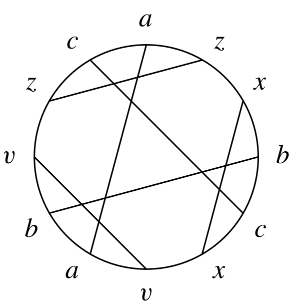
Recall (*): whenever v, x are nonadjacent vertices of a partitionable graph G, any maximum clique containing v omits x and therefore consists of one vertex from each stable set in the unique minimum coloring of $G - x$. Since x has no neighbor in V_1, we can apply (*) to any $v \in V_1$. I.e., each clique in $\Theta(G - x)$ that contains a vertex of G_1 must contain exactly one vertex of each of S_1, S_2. Both these vertices must be in G_1, else we introduce an edge between G_1 and G_2. Thus V_1 has an equal number of vertices from S_1 and S_2, both equal to the number of cliques in $\Theta(G - x)$ that meet G_1.

Choose $u \in S_2 \cap V_1$. To any clique Q of $\Theta(G - u)$, we can apply (*) again to guarantee that Q contains one vertex each of S_1, S_2. In particular, for each $v \in S_1 \cap V_1$, there is a clique of $\Theta(G - u)$ containing it, and this yields a vertex $v' \in S_2 \cap V_1 - u$ adjacent to it. Since S_1 is stable, these cliques of $\Theta(G-u)$ are disjoint, and so the vertices $\{v'\}$ are distinct. This implies $|S_1 \cap V_1| \le |S_2 \cap V_1 - u| < |S_2 \cap V_1|$, contradicting the result of the previous paragraph.

8.1.47. *The graph G below is a circular-arc graph but not a circle graph.* To represent G as a circular-arc graph, we let the arcs for the inner cycle and the outer cycle in the drawing each cover the circle. More precisely, consider a circle of circumference 9, with points on the circle described by numbers modulo 9. Assign arcs as in the middle table below to form a circular-arc representation.



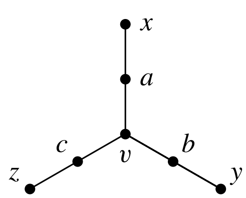
a: $[0,3]$	b: $[3,6]$	c: $[6,0]$
u: $[1,2]$	w: $[4,5]$	y: $[7,8]$
v: $[2,4]$	x: $[5,7]$	z: $[8,1]$

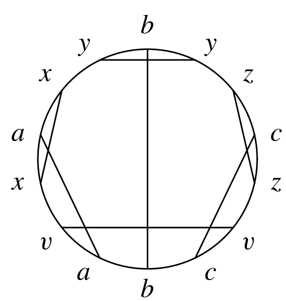


To show that G is not a circle graph, suppose that G has an intersection representation by chords in a circle. The chords for $\{a,b,c\}$ are pairwise intersecting, so their endpoints occur in the order a,b,c,a,b,c on the circle.

The chord for v cannot cross the chord for c, so to intersect the chords for a and b the endpoints for v must precede an a and follow the subsequent b, yielding a,b,c,v,a,b,v,c. We make the analogous argument for x and for z. However, $\{v,x,z\}$ is independent, so the endpoints of chords for any two of them cannot alternate. This means that when we add the endpoints for x and z to satisfy the constraints, we must obtain a,z,x,b,c,x,v,a,b,v,z,c, as shown above. Now we cannot add the chord for u to cross the chords for $\{z,a,v\}$ without crossing the chord for b or c.

The graph H below is a circle graph but not a circular-arc graph. A circle representation is shown in the middle below.





To show that H has no circular-arc representation, note that the arcs for $\{a,b,c\}$ must be pairwise disjoint. Since the arc for v must intersect all three, it must contain one of them completely; by symmetry, we may let it be a. Now the arc for x cannot intersect the arc for a without intersecting the arc for v.

8.1.48. *Paw-free graphs satisfy the SPGC.* The "paw" is the graph obtained from the claw $K_{1,3}$ by adding an edge joining two leaves. We must prove that every paw-free graph having no odd hole and no odd antihole is perfect. It suffices to prove that every paw-free graph G having no odd hole is a Meyniel graph, meaning that odd cycles of length at least 5 have at least two chords. Let C be an odd cycle of length at least 5 in G. Since G has no odd hole, C has a chord xy. This forms two cycles with the x,y-paths on C; one is odd. If the odd one has length at least 5, we obtain another chord of C. Otherwise, it has length 3. Since the subgraph induced by these three vertices and the next vertex on C must not be a paw, it contains an additional chord of C.

8.1.49. *Sets S and T of sizes $a+2$ and $w+2$ that intersect every maximum clique and every maximum stable set, respectively, in the cycle-power C_{aw+1}^{w-1}.* (This completes the proof of Theorem 8.1.51.)

Let $S=\{v_{aw},v_1,v_w,v_{w+2}\}\cup\{v_{iw+1}\colon 2\le i\le a-1\}$. The maximum cliques in C_{aw+1}^{w-1} are the sets of w vertices with consecutive indices. The first four indices listed for S are separated successively by 2, $w-1$, and 2, respectively. The next step is $w-1$, and the subsequent gaps are w until the final step of $w-1$ that returns to the beginning. Since the set never skips as many as w consecutive indices, it intersects all maximum cliques.

Let $T=\{v_{(a-1)w+1},v_{aw},v_1,v_w\}\cup\{v_{w+i}\colon 2\le i\le w-1\}$. The maximum stable sets in C_{aw+1}^{w-1} are the sets of a vertices whose indices increase successively by w (cyclically) starting from some point. In particular, a set of w successive vertices intersects all but one maximum stable set. The set T has $w-1$ of w successive indices from w through $2w-1$. The stable set skipping this interval starts at v_{2w} and contains v_{aw}, so it intersects T. The remaining stable sets are those containing v_{w+1}. These all contain v_1 except the one that starts at $w+1$, but this stable set intersects T at $(a-1)w+1$.

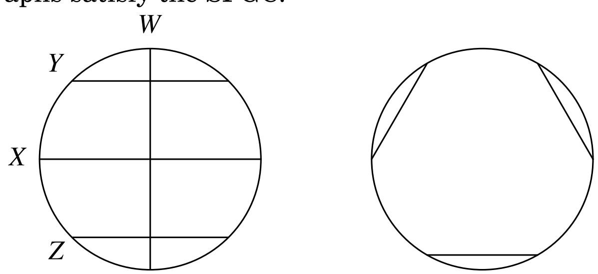
8.1.50. *SPGC for circle graphs.*

a) If x is a vertex in a partitionable graph G, then $G-N[x]$ is connected. If $G-N[x]$ is disconnected, then $N[x]$ is a star-cutset. It thus suffices to show that partitionable graphs have no star-cutsets. Since $\chi(G-x)=\omega$ for each $x\in V(G)$, every proper induced subgraph of G is $\omega(G)$-colorable.

Because $G - x$ has a partition into $\alpha(G)$ disjoint maximum cliques, a stable set intersecting all maximum cliques must be a maximum stable set. However, every maximum stable set misses its mate, so no stable set intersects every maximum clique. These are the hypotheses of the Star-Cutset Lemma Lemma, so G has no star-cutset.

b) Partitionable circle graphs are claw-free. Three pairwise-disjoint chords Y, X, Z of a circle can be intersected by a single chord W only if the endpoints occur as shown below. Suppose that a circle graph G has a claw induced by central vertex w and stable set $\{y, x, z\}$. If x is the vertex corresponding to the middle chord among $\{y, x, z\}$ in the circle representation of G, we have $G - N[x]$ disconnected, since every y, z-path in G must contain a vertex whose chord intersect the chord for x in the representation. By part (a), this cannot occur in a partitionable circle graph.

c) Circle graphs satisfy the SPGC. By part (b), partitionable circle graphs are claw-free. By Corollary 8.1.53, claw-free graphs satisfy the SPGC. Thus every p-critical circle graph is an odd hole or an odd antihole, and circle graphs satisfy the SPGC.



8.2. MATROIDS

8.2.1. *The family of independent vertex sets of a graph need not be the family of independent sets of a matroid.* In the star $K_{1,n}$, let the leaves have weight 1 and the remaining vertex have weight 2. The resulting maximum weighted stable set has weight n, but the greedy algorithm stops with a stable set of weight 2.

8.2.2. *The family of stable sets of a graph G is the family of independent sets of a matroid on its vertex set if and only if every component of G is a complete graph.* If some component of G is not complete, then G has P_3 as an induced subgraph. The stable set of size 1 consisting of the middle of this path cannot be augmented from the stable set of size 2 consisting of its endpoints, so the augmentation inequality fails.

Conversely, if every component is complete, then the hereditary system is a partition matroid, with the stable sets being those sets of vertices having at most one vertex in each component.

8.2.3. *Every partition matroid is a transversal matroid* A partition matroid on E is defined by sets $E_1, \dots, E_k$ partitioning E such that a subset of E is independent if and only if it contains at most one element of each E_i. This is the same as the transversal matroid on E arising from the $E, [k]$-bigraph whose ith component is the star with center i and leaf set E_i, for $1 \le i \le k$.

8.2.4. *Greedy algorithm with arbitrary real weights.* Since $\varnothing$ is always an independent set and has weight 0, a maximum weighted independent set contains no elements of negative weight. Hence it suffices to run the usual greedy algorithm on the restriction of the matroid obtained by discarding the elements of negative weight. This is accomplished simply by stopping the greedy algorithm when all the elements of nonnegative weight have been considered.

8.2.5. *The family of matchings in a graph G is the family of independent sets of a matroid on $E(G)$ if and only if every component of G is a star or a triangle.* The family of matchings in G is the family of stable sets in $L(G)$. By Exercise 8.2.2, the characterization is that every component of $L(G)$ is a complete graph. A component of $L(G)$ is a complete graph if and only if the corresponding component of G is a star or a triangle.

8.2.6. *The cycle matroid of a multigraph G is a uniform matroid if and only if G is a forest, a cycle, a multiple edge, or a collection of loops (plus possible isolated vertices in each case). The uniform matroid $\mathbf{U}_{k,n}$ is a cycle matroid if and only if $k \in \{0, 1, n-1, n\}$.* With G having n edges, the cycle matroids in the cases listed are the uniform matroids $\mathbf{U}_{n,n}, \mathbf{U}_{n-1,n}, \mathbf{U}_{1,n}, \mathbf{U}_{0,n}$, respectively.

For the converse, note that a matroid is non-uniform if and only if it has a dependent set and an independent set of the same size. Any multigraph G that is not a forest or a cycle (plus isolated vertices) has a cycle that does not contain all the edges. The edge set of a smallest cycle C in G is a circuit in $M(G)$. If its size exceeds 2, then deleting an edge of C and replacing it by any edge not in C cannot yield another cycle; hence it yields an independent set of the same size as C in the cycle matroid.

8.2.7. *The cycle matroid of a multigraph G is a partition matroid if and only if the blocks of G are sets of parallel edges. Every partition matroid is graphic.* A matroid M is a partition matroid with blocks $E_1, \dots, E_k$ if and only if the circuits of M are all sets of size 2 contained in single blocks.

This is also the cycle matroid of the graph whose blocks are sets of parallel edges of sizes $|E_1|, \ldots, |E_k|$.

In order for $M(G)$ to be a partition matroid, the circuits must have size 2. Hence G has no loops and no cycles of length greater than 2. The latter occurs if and only if the simple graph obtained by discarding extra copies of multiple edges is a forest. Hence G must be as claimed.

8.2.8. *Vectorial matroids satisfy the induced circuit property: adding an element to a linearly independent set of vectors creates at most one minimal dependent set.* Let I be an independent set of vectors in a vector space, and let e be a vector. Let C_1 and C_2 be minimal dependent sets of vectors in $I \cup \{e\}$. The definition of dependence for sets of vectors is the existence of an equation of dependence. The coefficent on e in such an equation of dependence is nonzero, since I is independent; indeed, since these are minimal dependent sets, all the coefficients are nonzero. Hence in each equation we can solve for e expressing e as a linear combination of $C_1 - \{e\}$ and of $C_2 - \{e\}$. Setting these expressions equal yields an equation of dependence for I if C_1 and C_2 are different, since the coefficients in the original equations are nonzero. Hence $C_1 = C_2$, and $I \cup \{e\}$ contains only one minimal dependent set.

8.2.9. *Circuits of a partition matroid.* By definition, sets of elements are independent if they have at most one element from each block of the partition. Hence a set is a circuit if and only if it consists of two elements from one block of the partition. If distinct circuits have a common element e, then they have the form $\{e, x\}$ and $\{e, y\}$, where e, x, and y all lie in a common block. Hence $\{x, y\}$ is also a circuit, and weak elimination holds.

8.2.10. *Direct verification of submodularity for rank functions of cycle matroids.* Given a graph G, let $k(X)$ denote the number of components of the spanning subgraph G_X with edge set X. Let H be the bipartite graph whose partite sets are the sets of components in G_X and G_Y, with vertices adjacent if the corresponding subgraphs share a vertex.

a) H has $k(X) + k(Y)$ vertices and $k(X \cup Y)$ components, and $k(X \cap Y) \ge e(H)$. By construction, the sizes of the partite sets are $k(X)$ and $k(Y)$. Components of G_X and G_Y that share a vertex lie within a single component in $G_{X \cup Y}$. Hence $k(X \cup Y)$ is the number of components of H.

Every edge of H has the form $C_X C_Y$, where C_X and C_Y are components of G_X and G_Y, respectively. Let $S = V(C_X) \cap V(C_Y)$; since $C_X C_Y$ is an edge, $S \ne \varnothing$. Every vertex outside S is outside $V(C_X)$ or outside $V(C_Y)$. Hence $X \cap Y$ has no edge leaving S, and $G_{X \cap Y}[S]$ is a nonempty union of components of $G_{X\cap Y}$. Thus $k(X \cap Y) \ge e(H)$, since we generate at least one component of $G_{X \cap Y}$ for each edge of H (maybe more than one, such as when G is a 4-cycle and X and Y decompose G into two copies of P_3).

b) For the cycle matroid $M(G)$, the submodularity property $r(X \cap Y) + r(X \cup Y) \le r(X) + r(Y)$ holds. In the cycle matroid, $r(X) = n(G) - k(X)$, so it suffices to show that k is supermodular.

A graph with n vertices and c components has at least $n - c$ edges. Since H has $k(X) + k(Y)$ vertices and $k(X \cup Y)$ components, we conclude that $e(H) \ge k(X) + k(Y) - k(X \cup Y)$. By part (a), also $k(X \cap Y) \ge e(H)$. Hence $k(X \cap Y) + k(X \cup Y) \ge k(X) + k(Y)$, as desired.

8.2.11. *Submodularity of rank functions of transversal matroids, using matching theory.* A transversal matroid on a set E is induced by a family $A_1, \ldots, A_m$ of subsets of E by letting the independent sets be the systems of distinct representatives of subfamilies. Equivalently, the independent sets are the subsets of E that can be saturated by matchings in the $E, [m]$-bigraph G. that is the incidence bigraph of the family.

By definition, then, the rank of a set $X \subseteq E$ is the maximum size of a matching in the subgraph $G[X \cup [m]]$, which we denote by G_X. By the König–Egerváry Theorem, $\alpha'(G_X)$ equals the minimum size of a vertex cover in G_X. For $S \subseteq X$, the smallest vertex cover Q such that $S = X - Q$ is $(X - S) \cup N(S)$. Hence the minimum size of a vertex cover of G_X is $|X| - \max_{S \subseteq X}(|S| - |N(S)|)$. The quantity $|S| - |N(S)|$ is the *deficiency* of S, denoted $\mathrm{def}(S)$, and the fact that $\alpha'(G_X) = |X| - \max_{S \subseteq X} \mathrm{def}(S)$ is due to Ore (Exercise 3.1.32).

Now consider subsets $X, Y \subseteq E$. For the submodularity inequality, we must bound $r(X \cup Y) + r(X \cap Y)$ by $r(X) + r(Y)$. For this we begin by studying the neighborhoods of the union and intersection of two sets $S \subseteq X$ and $T \subseteq Y$. The key to the inequality is that for $S, T \subseteq E$, we have $N(S \cap T) \subseteq N(S) \cap N(T)$ (equality need not hold!). Also $N(S \cup T) = N(S) \cup N(T)$. Thus

$$|N(S \cup T)| + |N(S \cap T)| \le |N(S) \cup N(T)| + |N(S) \cap N(T)| = |N(S)| + |N(T)|$$

Since $|S \cup T| + |S \cap T|) = |S| + |T|$, this yields $\mathrm{def}(S \cup T) + \mathrm{def}(S \cap T) \ge \mathrm{def}(S) + \mathrm{def}(T)$. Furthermore, the deficiency of a set S is the same in each G_X such that $X \supseteq S$. Therefore, if we let S and T be subsets of X and Y with maximum deficiency in G_X and G_Y, we obtain

$$\begin{aligned} r(X) + r(Y) &= |X| - \mathrm{def}(S) + |Y| - \mathrm{def}(T) \ge |X| + |Y| - \mathrm{def}(S \cup T) - \mathrm{def}(S \cap T) \\ &\ge |X \cup Y| - \max_{U \subseteq (X \cup Y)} \mathrm{def}(U) + |X \cap Y| - \max_{V \subseteq (X \cap Y)} \mathrm{def}(V), \end{aligned}$$

using in the last step that $S \cup T$ and $S \cap T$ are particular subsets of $X \cup Y$ and $X \cap Y$, respectively. Thus the submodularity inequality holds.

8.2.12. *For a digraph D with distinguished source s and sink t, and $r(X)$ defined for $X \subseteq V(D) - \{s, t\}$ to be the number of edges from $s \cup X$ to $\overline{X} \cup t$,* *the function r is submodular.* When we view D as a network by giving each edge capacity 1, the statement of submodularity for r is precisely the statement of part (a) of Exercise 4.3.12.

8.2.13. *For an element x in a hereditary system, the following properties are equivalent and characterize loops.* The definition of a loop (an element comprising a circuit of size 1) is statement C.

A) $r(x) = 0$. D) x belongs to no base.
B) $x \in \sigma(\varnothing)$. E) Every set containing x is dependent.
C) x is a circuit. F) x belongs to the span of every $X \subseteq E$.

F $\Rightarrow$ B. If $x \in \sigma(X)$ for all $X \subseteq E$, then $x \in \sigma(\varnothing)$.

B $\Rightarrow$ C. If $x \in \sigma(\varnothing)$, then x completes a circuit with $\varnothing$; hence $\{x\}$ is a circuit.

C $\Rightarrow$ A. The rank of a circuit C is $|C| - 1$.

A $\Rightarrow$ D. Every subset of every base is independent. If x belongs to a base, then $r(\{x\}) = 1$.

D $\Rightarrow$ E. If x belongs to an independent set, then it can be augmented to a maximal independent set (a base) containing x.

E $\Rightarrow$ F. Let Y be a maximal independent subset of X. If every set containing x is dependent, then $Y \cup x$ contains a circuit C, which must contain x since Y is independent. Hence x completes a circuit with a subset of X, so $x \in \sigma(X)$.

8.2.14. *The following characterizations of parallel elements in a hereditary system are equivalent, assuming that $x \neq y$ and neither is a loop.* Property B is the definition of parallel elements, given that neither is a loop.

A) $r(\{x, y\}) = 1$.
B) $\{x, y\} \in \mathbf{C}$.
C) $x \in \sigma(y)$, $y \in \sigma(x)$, $r(x) = r(y) = 1$.

B $\Leftrightarrow$ A. The rank of a circuit C is $|C| - 1$, so B $\Rightarrow$ A. Conversely, if $r(\{x, y\}) < 2$ with x and y being non-loops, then $\{x, y\}$ is a minimal dependent set.

B $\Leftrightarrow$ C. Since neither is a loop, $r(x) = r(y) = 1$, and each element by itself forms an independent set. Now $\{x, y\} \in \mathbf{C}$ is equivalent to $x \in \sigma(\{y\})$ and $y \in \sigma(\{x\})$, by the definition of the span function.

If x and y are parallel elements in a matroid and $x \in \sigma(X)$, then $y \in \sigma(X)$. From $x \in \sigma(X)$, we have $x \in X$ or $Y \cup \{x\} \in \mathbf{C}$, where $Y \subseteq X$. If $x \in X$, then $y \in \sigma(X)$, since y completes a circuit with x. If $Y \cup \{x\} \in \mathbf{C}$ and $\{x, y\} \in \mathbf{C}$, then the weak elimination property guarantees a circuit in $Y \cup \{y\}$, and hence $y \in \sigma(X)$.

8.2.15. *If $r(X) = r(X \cap Y)$ in a matroid, then $r(X \cup Y) = r(Y)$.*

Proof 1 (submodularity). Submodularity yields $r(X \cup Y) + r(X \cap Y) \le r(X) + r(Y)$. Cancelling $r(X) = r(X \cap Y)$ leaves $r(X \cup Y) \le r(Y)$, but $r(X \cup Y) \ge r(Y)$ always, so equality holds.

Proof 2 (span function and absorption). The hypothesis implies $X \subseteq \sigma(X \cap Y)$, which in turn is contained in $\sigma(Y)$ since σ is order-preserving. Now $X \subseteq \sigma(Y)$ and the absorption property yield $r(X \cup Y) = r(Y)$.

8.2.16. *If M is a hereditary system that satisfies the base exchange property (B), then the greedy algorithm generates a maximum-weighted base whenever the elements have nonnegative weights.* This is actually more direct using the dual version of the base exchange property (Lemma 8.2.33): if $B_1, B_2 \in \mathbf{B}$ and $e \in B_1 - B_2$, then there exists $f \in B_2 - B_1$ such that $B_2 + e - f$ is a base. This follows from the induced circuit property in Lemma 8.2.33, and the induced circuit property follows directly from the base exchange property in Exercise 8.2.17.

Since the weights are nonnegative, the greedy algorithm generates a base. Let B be a base generated by the greedy algorithm. Among the bases of maximum weight, let B^* be one having largest intersection with B. If $B^* \neq B$, then there exists an element $e \in B - B^*$, since the bases form an antichain. Let e be a heaviest element of $B - B^*$. By the dual base exchange property, there exists $f \in B^* - B$ such that $B^* + e - f$ is a base. Since B^* is optimal, $w(f) \ge w(e)$. Since the greedy algorithm chose e after choosing the heavier elements of B, even though f was also avaiable, $w(e) \ge w(f)$. Hence $w(e) = w(f)$, and $B^* + e - f$ is an optimal base having larger intersection with B than B^* does. Hence in fact $B = B^*$.

8.2.17. *Exercises in axiomatics.*

a) In a hereditary system, the submodularity property implies the weak absorption property. Applying submodularity to $X + e$ and $X + f$ yields $r(X + e + f) + r(X) \le r(X + e) + r(X + f)$. If $r(X + e) = r(X + f) = r(X)$, then monotonicity of r implies $r(X + e + f) = r(X)$.

b) In a hereditary system, the strong absorption property implies the submodularity property. We use induction on $k = |X \triangle Y|$. If $r((X \cap Y) + e) = r(X \cap Y)$ for all $e \in X \triangle Y$, then $r(X \cup Y) = r(X \cap Y)$, by strong absorption. Monotonicity of r then implies $r(X \cap Y) + r(X \cup Y) \le r(X) + r(Y)$. This case includes the basis step $k = 0$.

Hence when $k > 0$ we may select $e \in X - Y$ (by symmetry) such that $r((X \cap Y) + e) = r(X \cap Y) + 1$. Let $Y' = Y + e$. By the induction hypothesis, $r(X \cap Y') + r(X \cup Y') \le r(X) + r(Y')$. The left side equals $r(X \cap Y) + 1 + r(X \cup Y)$ and the right side is bounded by $r(X) + r(Y) + 1$, so subtracting 1 from both sides yields the desired inequality.

c) The base exchange property (B) implies the induced circuit property (J). **Proof 1** (contradiction). For $I \in \mathbf{I}$, if $I + e$ contains distinct circuits C_1, C_2, then each consists of e plus a subset of I. Since $C_1 \neq C_2$, we may choose $a \in C_1 - C_2$. Both $C_1 - a$ and $(C_1 \cup C_2) - e$ are independent; augment them to bases B_1 and B_2, respectively.

Since $C_1 - e \subseteq B_2$, every element of $B_1 - B_2$ except e is outside $C_1 - a$. Using (B), delete such elements from B_1, replacing them with elements of $B_2 - B_1$. This transforms B_1 to a base B such that the only element of $B - B_2$ is e, and still $C_1 - a \subseteq B$ and $a \notin B$.

Since (B) implies that bases have the same size, also $|B_2 - B| = 1$. Since $a \in B_2 - B$, the rest of B_2, including $C_2 - e$, is in B. However, $e \in B$, so $C_2 \in B$, contradicting that B is a base.

Proof 2 (extremality). Since every independent set lies in a base, it suffices to prove for $B \in \mathbf{B}$ that $B + e$ contains exactly one circuit. Let A be a minimal subset of B containing an element of each circuit in $B + e$. Thus $(B - A) + e \in \mathbf{I}$, but $(B - A) + e + a \notin \mathbf{I}$ for all $a \in A$.

Let B' be a base containing $(B - A) + e$; note that $B - B' = A$. If $B' - B$ has an element b other than e, then (B) yields an element $a \in B - B'$ such that $B' - b + a \in \mathbf{B}$, but this contradicts the dependence of $(B - A) + e + a$. Hence $B' - B = \{e\}$, and therefore $|A| = 1$. Since every minimal transversal of the circuits in $B + e$ has one element, there is only one such circuit.

d) The uniqueness of induced circuits (J) implies the weak elimination property (C). Suppose that $C_1, C_2 \in \mathbf{C}$ and $e \in (C_1 cap C_2)$. If $(C_1 \cup C_2) - e$ is independent, then adding e creates a unique circuit, which contradicts the distinctness of C_1 and C_2.

e) In a hereditary system, uniqueness of induced circuits (J) implies the augmentation property (I). Choose $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$. We obtain the augmentation by induction on $|I_1 - I_2| = k$. If $I_1 \subseteq I_2$, any element of $I_2 - I_1$ works; this is the basis step $k = 0$.

For $k > 0$, select $e \in I_1 - I_2$. If $I_2 + e \in \mathbf{I}$, then the induction hypothesis allows us to augment I_1 from $I_2 + e$. Hence we may assume that $I_2 + e$ contains a unique circuit C. Choose $f \in C \cap I_2$, and let $I' = I_2 + e - f$; we have $I' \in \mathbf{I}$. Now $\left|I'\right| = |I_2|$ and $\left|I_1 - I'\right| = k - 1$, so the induction hypothesis guarantees an augmentation of I_1 from I'. Any such element is also in I_2.

8.2.18. *A hereditary system is a matroid if and only if it satisfies the following: If $I_1, I_2 \in \mathbf{I}$ with $|I_2| > |I_1|$ and $|I_1 - I_2| = 1$, then $I_1 + e \in \mathbf{I}$ for some $e \in I_2 - I_1$.* This is a weaker form of the augmentation property, so it suffices to show that this implies the augmentation property. The stated property provides the basis for induction on $k = |I_1 - I_2|$. If $k > 1$, select $x \in I_1 - I_2$, and let $I = I_1 - x$. The induction hypothesis yields $e_1 \in I_2 - I$ such that $I + e_1 \in \mathbf{I}$. Also $|I + e_1| = |I_1| < |I_2|$ and $|I + e_1 - I_2| = k - 1$, so again the induction hypothesis yields $e_2 \in I_2$ such that $I' = I \cup \{e_1, e_2\} \in \mathbf{I}$. Since $\left|I'\right| = |I_1| + 1$ and $I_1 - I' = \{x\}$, the original hypothesis ($k = 1$) yields $e \in \{e_1, e_2\} \subseteq I_2$ such that $I_1 + e \in \mathbf{I}$.

8.2.19. *If $\mathbf{I}$ is the family of independent sets of a matroid on E, and $\mathbf{I}'$ is obtained from $\mathbf{I}$ by deleting the sets that intersect a fixed subset A of E, then $\mathbf{I}'$ is also the family of independent sets of a matroid on E.* If $I \in \mathbf{I}'$, then $I \cap A = \varnothing$, and also $J \cap A = \varnothing$ for $J \subseteq I$. Also $J \in \mathbf{I}$, since $I \in \mathbf{I}$, so J remains in $\mathbf{I}'$. Hence $\mathbf{I}'$ is an ideal. Also $\varnothing \in \mathbf{I}'$, since $\varnothing \in \mathbf{I}$ and $\varnothing \cap A = \varnothing$.

Consider $I_1, I_2 \in \mathbf{I}'$ with $|I_2| > |I_1|$. In fact also $I_1, I_2 \in \mathbf{I}$, since $\mathbf{I}'$ is a subset of $\mathbf{I}$. The augmentation property in $\mathbf{I}$ yields $e \in I_2$ such that $I_1 + e \in \mathbf{I}$. In fact, also $I_1 + e \in \mathbf{I}'$, since I_1 and I_2 are both disjoint from A, so $I_1 + e \cap A = \varnothing$.

8.2.20. Given a matroid on E and $e \notin B \in \mathbf{B}$, let $C(e, B)$ denote the unique circuit in $B \cup e$.

a) For $e \notin B$, the set $B - f + e$ is a base if and only if f belongs to $C(e, B)$. If $f \in C(e, B)$, then $B - f + e$ contains no circuit, because $C(e, B)$ is the only circuit in $B + e$. Hence $B - f + e$ is an independent set of size $r(E)$. By the uniformity property, $B - f + e$ is a base.

If $f \notin C(e, B)$, then $B - f + e$ contains the circuit $C(e, B)$ and hence is not a base.

b) If $e \in C \in \mathbf{C}$, then $C = C(e, B)$ for some base B. The set $C - e$ is independent and hence can be augmented to a base B. This base cannot contain e. Adding e must complete a unique circuit. It completes C, so it completes no other, and hence $C = C(e, B)$.

8.2.21. *If B_1 and B_2 are bases of a matroid such that $|B_1 \triangle B_2| = 2$, then there is a unique circuit C such that $B_1 \triangle B_2 \subseteq C \subseteq B_1 \cup B_2$.* Let e_1 be the element of $B_1 - B_2$, and let e_2 be the element of $B_2 - B_1$. Since $B_1 \cup B_2 = B_2 + e_1$, the union contains a unique circuit, C. Since B_2 is independent, $e_1 \in C$. Furthermore, $e_2 \in C$, since $B_1 \cup B_2 - \{e_2\} = B_1$, which is independent.

8.2.22. *If B_1 and B_2 are bases of a matroid and $X_1 \subseteq B_1$, then there exists $X_2 \subseteq B_2$ such that $(B_1 - X_1) \cup X_2$ and $(B_2 - X_2) \cup X_1$ are both bases of M.* This is easy using Exercise 8.2.24; otherwise that argument must be generalized. We use induction on $|X_1|$; when X_1 is empty the claim is trivial. Otherwise, choose $e \in X_1$ and let $X_1' = X_1 - \{e\}$. By Exercise 8.2.24, there exists $f \in B_2$ such that $B_1 - e + f$ and $B_2 + e - f$ are both bases. Let $B_1' = B_1 - e + f$ and $B_2' = B_2 + e - f$. By the induction hypothesis, there exists X_2' such that $(B_1' - X_1') \cup X_2'$ and $(B_2' - X_2') \cup X_1'$ are both bases. Now let $X_2 = X_2' \cup f$. This set has the desired property, since $(B_1 - X_1) \cup X_2 = (B_1' - X_1') \cup X_2'$ and $(B_2 - X_2) \cup X_1 = (B_2' - X_2') \cup X_1'$.

8.2.23. Consider distinct bases B_1 and B_2 of a matroid M.

a) The B_1, B_2-bigraph G with $e \in B_1$ adjacent to $f \in B_2$ when $B_2 + e - f \in \mathbf{B}$ has a perfect matching. It suffices to verify Hall's condition. Since $|B_1| = |B_2|$, we may verify Hall's Condition for either partite set.

Proof 1: For $S \subseteq B_2$, suppose that $|N(S)| < |S|$. This yields $|B_1 - N(S)| > |B_2 - S|$. Both $B_1 - N(S)$ and $B_2 - S$ are independent, so the augmentation property yields $e \in B_1 - N(S)$ such that $B_2 - S + e \in \mathbf{I}$. Hence S must contain a member of the circuit formed by added e to B_2. This contradicts $e \notin N(S)$, and hence $|N(S)| \geq |S|$.

Proof 2: We seek a transversal of $\{I(e): e \in B_1\}$, where $I(e) \cup e$ is the unique circuit in $B_2 + e$ if $e \in B_1 - B_2$, and $I(e) = \{e\}$ if $e \in B_1 \cap B_2$. For $X \subseteq B_1$, let $Y = \bigcup_{e \in X} I(e)$. Since $e \in \sigma(I(e))$, we have $X \subseteq \sigma(Y)$. Since $X, Y \in \mathbf{I}$, the incorporation property yields $|Y| = r(Y) = r(\sigma(Y)) \geq r(X) = |X|$. Hence Hall's Condition holds.

(Comment: we can similarly establish a bijection $\pi\colon B_1 \to B_2$ such that $B_1 - e + \pi(e) \in \mathbf{B}$ for all $e \in B_1$.)

b) There is a bijection $\pi\colon B_1 \to B_2$ such that for each $e \in B_1$, the set $B_2 - \pi(e) + e$ is a base of M. Such a bijection is given by the perfect matching obtained in part (a). Elements of $B_1 \cap B_2$ yield isolated edges in G.

8.2.24. *For any $e \in B_1$, there exists $f \in B_2$ such that $B_1 - e + f \in \mathbf{B}$ and $B_2 - f + e \in \mathbf{B}$.* If $e \in B_1 \cap B_2$, then let $f = e$. Hence we may assume $e \in B_1 - B_2$.

Proof 1 (transitivity of dependence). Let $I(e) + e$ be the unique circuit in $B_2 + e$, so $I(e) = \{f \in B_2\colon B_2 - f + e \in \mathbf{B}\}$. If $B_1 - e + f \notin \mathbf{B}$ for all $f \in I(e)$, then $I(e) \subseteq \sigma(B_1 - e)$. Since $I(e) + e$ is a circuit, this implies $e \in \sigma(I(e)) \subseteq \sigma(B_1 - e)$, which is impossible since B_1 is independent.

Proof 2 (cocircuits). In $B_2 + e$ there is a unique circuit C containing e. Since $\overline{B}_1$ is a cobase, $\overline{B}_1 + e$ contains a unique cocircuit C^* containing e. Since $|C \cap C^*| = 1$ is forbidden, there exists another element $f \in C \cap C^*$. Hence $B_2 - f + e$ is independent, has size $|B_2|$, and therefore is a base. Similarly $\overline{B}_1 - f + e$ is independent in the dual, has size $|\overline{B}_1|$, and is a cobase. Therefore $B_1 - e + f$ is a base and f is the desired element.

b) There may be no bijection $\pi\colon B_1 \to B_2$ such that e and $f = \pi(e)$ satisfy part (a) for all $e \in B_1$. Consider the cycle matroid $M(K_4)$. Let B_1 and B_2 be the edge sets of two complementary 4-vertex paths. If e is a pendant edge of B_1, then e can only be matched with the central edge of B_2, since one pendant edge of B_2 completes a triangle with $B_1 - e$, and the other is not in the triangle of $B_2 + e$. This argument applies for both pendant edges, but they cannot both be paired with the one central edge of B_2.

8.2.25. *Every matroid has a fundamental set of circuits* (a collection of $|E| - r(E)$ circuits such that C_i contains $e_{r(E)+i}$ but no higher-indexed element). If the elements $e_1, \ldots, e_r$ form a base B, then addition of any other $e \in E - B$ creates a unique circuit in $B + e$. The set of these generated by the elements of $E - B$ form a fundamental set of circuits.

8.2.26. *If $C_1, \ldots, C_k$ are distinct circuits in a matroid, with none contained in the union of the others, and X is a set with $|X| < k$, then $\bigcup_{i=1}^{k} C_i - X$ contains a circuit.* We use induction on k. For $k = 1$ the statement is trivial, and for $k = 2$ it is the statement of the weak elimination property (if $x \notin C_1 \cap C_2$, then C_1 or C_2 itself is the desired circuit). For $k > 2$, choose $x \in X$, and let $X' = X - \{x\}$. By the induction hypothesis, $\bigcup_{i=1}^{k-1} C_i - X'$ contains a circuit C'. The case $k = 2$ yields a circuit in $(C' \cup C_k) - x$; this circuit has the desired properties.

8.2.27. $(+)$ For a hereditary system, prove that the weak elimination property implies the strong elimination property, by induction on $|C_1 \cup C_2|$.

8.2.28. *Min-max formula for maximum weighted independent set.* Given weight $w(e) \in \mathbb{N} \cup \{0\}$ for each element e, we prove $\max_{I \in \mathbf{I}} \sum_{e \in I} w(e) = \min \sum_i r(X_i)$, where the minimum is taken over all chains $X_1 \subseteq X_2 \subseteq \cdots$ of sets in E such that each element $e \in E$ appears in at least $w(e)$ sets in the chain (sets may repeat).

Max $\leq$ min. This inequality holds for every I and every acceptable chain $\{X_i\}$. Independence of I implies $r(X_i) \geq |I \cap X_i|$. Now the appearance of each $e \in I$ in at least $w(e)$ sets of $\{X_i\}$ yields $\sum_i |I \cap X_i| \geq \sum_{e \in I} w(e)$.

To establish equality, let I be a maximum weighted independent set, and define a chain by $X_i = \{e \in E\colon w(e) \geq W + 1 - i\}$, where W is the maximum weight and $1 \leq i \leq W$ (repetition occurs if the weights are not consecutive integers). For this chain, each element e appears in the $w(e)$ largest sets; hence $\sum |I \cap X_i| = \sum_{e \in I} w(e)$.

Thus it suffices to prove that $|I \cap X_i| = r(X_i)$. This holds by induction on i, with trivial basis for $X_0 = \varnothing$. Having selected $I \cap X_i$ of maximum weight and maximum size from the set X_i of elements with weight at least $W + 1 - i$, the greedy algorithm next considers elements of weight $W - i$, adding as many to I as fail to produce a circuit. Hence $|I \cap X_{i+1}| - |I \cap X_i| = r(X_{i+1}) - |I \cap X_i|$, as desired.

8.2.29. *If r and σ are the rank function and span function of a matroid, then $r(X) = \min\{|Y|: Y \subseteq X$ and $\sigma(Y) = \sigma(X)\}$.* Let M be a matroid. Given $Y \subseteq X$ with $\sigma(Y) = \sigma(X)$, two applications of the incorporation property yield $r(X) = r(\sigma(X)) = r(\sigma(Y)) = r(Y) \leq |Y|$. On the other hand, if Y is a maximum independent subset of X, then $\sigma(Y) \subseteq \sigma(X)$, since σ is order-preserving. Now the choice of Y implies $X \subseteq \sigma(Y)$, and transitivity of dependence implies $\sigma(X) \subseteq \sigma(Y)$.

8.2.30. *A matroid of rank r has at least 2^r closed sets.* A base B in such a matroid has size r. For each $X \subseteq B$, the span $\sigma(X)$ is closed. These closed sets are all distinct, because their intersections with B are distinct, since gaining an element of B in $\sigma(X)$ would increase the rank.

8.2.31. *A matroid is simple if and only if no element appears in every hyperplane and every set of two elements intersects some hyperplane exactly once.* If e is a loop, then e is spanned every set. Hence e belongs to every closed set, including all hyperplanes. If e is not a loop, then augment $\{e\}$ to a base B, and let $H = \sigma(B - e)$; now H is a hyperplane avoiding e.

If e and f are parallel, then $\{e, f\}$ is a circuit, so a closed set contains e if and only if it contains f. If $\{e, f\}$ is independent, augment is to a base B, and let $H = \sigma(B - e)$; now H is a hyperplane containing f but not e.

8.2.32. (•) Prove that in a matroid, a set is a hypobase if and only if it is a hyperplane.

8.2.33. *A family of sets is the family of hyperplanes of some matroid if it is an antichain and, for distinct members H_1 and H_2 both avoiding an element e, there is another member H containing $(H_1 \cap H_2) + e$.* A hyperplane is a maximal set containing no base, and hence its complement is a minimal set contained in no cobase, which by definition is a cocircuit. Hence the hyperplanes are the complements of the cocircuits. A family of sets is the set of cocircuits of a matroid M if and only if it is the set of circuits of a matroid, namely M^*. Hence the characterization of families of cocircuits is the same as the characterization of families of circuits. In particular, a family is the set of cocircuits of some matroid if and only if it is an antichain and, for distinct members C_1 and C_2 with a common element e, there is another member contained in $(C_1 \cup C_2) - e$.

Translating this by complementation, let $H_1 = \overline{C_1}$ and $H_2 = \overline{C_2}$, the condition on members H_1 and H_2 of the family, if they both omit e (so that $e \in C_1 \cap C_2$, is the existence of another member H such that $\overline{H} \subseteq (C_1 \cap C_2) - e$. Thus

$$H \supseteq \overline{(\overline{H}_1 \cup \overline{H}_2) - e} = (H_1 \cap H_2) + e.$$

8.2.34. *The closed sets of a matroid are the complements of the unions of its cocircuits.* The closed sets of a matroid are the intersections of its hyperplanes. The hyperplanes are the complements of the cocircuits. Since $\overline{A \cup B} = \overline{A} \cap \overline{B}$, the desired statement holds.

8.2.35. *Closed sets and hyperplanes.*

a) If X and Y are closed sets in a matroid M, with $Y \subseteq X$ and $r(Y) = r(X) - 1$, then there exists a hyperplane H in M such that $Y = X \cap H$. Let Z be a maximal independent subset of Y, and let $e \in X$ be an element that augments Z to a maximal independent subset of X. Augment $Z + e$ to a base B. Let $H = \sigma(B - e)$; we claim H is the desired hyperplane. Since $Z \subseteq B - e$ and $Y = \sigma(Z)$, we have $Y \subseteq H$. Because $B \subseteq H \cup X$, we have $r(H \cup X) = r(M)$. Applying submodularity to H and X yields $r(H \cap X) \le r(X) - 1$. Since $Y \subseteq H \cap X$, we have $r(H \cap X) = r(X) - 1$. Since Y is closed and has rank $r(X) - 1$, we have $H \cap X = Y$.

b) If X is a closed set in a matroid M, then there exist $r(M) - r(X)$ distinct hyperplanes in M whose intersection is X. By induction on $r(M) - k$. If $r(M) - k = 1$, then X is a hyperplane. If $r(M) - k > 1$, than take $e \notin X$ and $Z = \sigma(X + e)$. Now $r(Z) = k + 1$, and by the induction hypothesis there are $r(M) - k - 1$ distinct hyperplanes whose intersection is Z. Since Z is closed, part (a) guarantees an additional hyperplane H such that $X = Z \cap H$.

8.2.36. *Properties of closed sets in a matroid.*

a) The intersection of two closed sets is closed. Let X and Y be closed sets, so $\sigma(X) = X$ and $\sigma(Y) = Y$. Since σ is order-preserving, $\sigma(X \cap Y) \subseteq \sigma(X) = X$ and $\sigma(X \cap Y) \subseteq \sigma(Y) = Y$. Hence $\sigma(X \cap Y) \subseteq X \cap Y$. Equality holds, because σ is expansive. Hence $\sigma(X \cap Y) = X \cap Y$, and $X \cap Y$ is closed.

b) The span of a set is the intersection of all closed sets containing it. Consider $\sigma(X)$, and let X be the intersection of all the closed sets containing X. A closed set has the form $\sigma(Y)$. If $X \subseteq \sigma(Y)$, then $\sigma(X) \subseteq \sigma^2(Y) = \sigma(Y)$, by the order-preserving and idempotence properties of σ. Hence $\sigma(X)$ is contained in all the closed sets containing X, so $\sigma(X) \subseteq Z$. On the other hand, since $\sigma(X)$ itself is a closed set containing X, also $Z \subseteq \sigma(X)$.

c) The union of two closed sets need not be a closed set. Let M be the cycle matroid of a 4-cycle. Any two consecutive edges on the 4-cycle form a closed set in the matroid. The union of two consecutive such sets is not closed, because it spans the remaining edge of the cycle.

8.2.37. *For a matroid M, $M.X$ has no loops if and only if $\overline{X}$ is closed.* An element is a loop in $M.X$ if and only if it completes a circuit with a subset of $\overline{X}$. There is no such element if and only if $\sigma(\overline{X}) = \overline{X}$.

8.2.38. *Bases and cocircuits in matroids.*

a) When e belongs to a base B in a matroid M, there is exactly one cocircuit of M disjoint from $B - e$, and it contains e. The complement of B is a base in M^*. Adding the element e to it creates a unique circuit in M^*. This is the unique cocircuit of M disjoint from $B - e$, and it contains e.

b) If C is a circuit of a matroid M and x, y are distinct elements of C, then there is a cocircuit $C^* \in \mathbf{C}^*$ with $C^* \cap C = \{x, y\}$. Augment the independent set $C - x$ to a base B; this base B contains y but not x. By the first statement, M has a unique cocircuit C^* disjoint from $B - y$, and it contains y. Since a cocircuit cannot intersect a circuit in exactly one element, and x is the only element of C not contained in B, $C^* \cap C = \{x, y, \}$.

c) Part (b) is trivial for cycle matroids. If e and f are edges in a cycle C, then $V(C)$ splits into sets A and B that are the vertex sets of the paths on C connecting e and f. For every minimal edge cut B that separates A and

B, the intersection of the cocircuit B with the circuit C in the cycle matroid is $\{e, f\}$.

8.2.39. *The dual of a simple matroid need not be simple.* The independent sets of a matroid M^* are the complements of the spanning sets of M. Hence an element e is a loop in M^* if and only if its complement is not spanning in M, which means that e belongs to every base in M. Let M be the hereditary system on E in which the only circuit is $E-e$. Since M has only one circuit, M vacuously satisfies weak elimination and is a matroid. If $|E| \ge 4$, then M is simple. The bases of M are the sets of size $|E|-1$ containing e. Since $E-e$ does not span, e is a loop in M^*.

A set of elements in a matroid can be both a circuit and a cocircuit. Suppose C is a both circuit and a cocircuit in M. Hence C is a minimal dependent set in M^*, which means $\overline{C}$ is a hyperplane (maximal nonspanning set) in M. Given $e \in C$, we thus have $C-e$ independent and $\overline{C}+e$ spanning. Since bases of M have the same size, we have $\left|\overline{C}+e\right| \ge |C-e|$, which implies $|C| \le |E|/2+1$. This suggests considering $U_k(2k)$. Indeed, the dual of $U_k(2k)$ is $U_k(2k)$ itself, in which the circuits and cocircuits are all sets of size $k+1$.

8.2.40. *Proof of Euler's Formula by matroids.* The cycle matroid of a connected n-vertex graph G has rank $n-1$. The dual matroid, also defined on the edge set, has rank $e-n+1$. When the graph is planar, the dual matroid is the cycle matroid of the dual graph G^*. Since G^* has f vertices (one for each face of G), the rank of that matroid is $f-1$. Hence $f-1 = e-n+1$, or $n-e+f = 2$, as desired.

8.2.41. *Restrictions and contractions of matroids commute.* If e is to be deleted and f is to be contracted away, then we can do these in either order without affecting the resulting matroid. More precisely, for a matroid M on E and $Y \subseteq X \subseteq E$, we use rank functions to prove that $(M|X).Y = (M.\overline{X-Y})|Y$ and $(M.X)|Y = (M|\overline{X-Y}).Y$.

For the first equation, we have two matroids defined on Y. For $Z \subseteq Y$, we have $r_{(M|X).Y}(Z) = r_{M|X}(Z \cup (X-Y)) - r_{M|X}(X-Y)$. Also $r_{M.\overline{X-Y}|Y}(Z) = r_{M.\overline{X-Y}}(Z) = r_M(Z \cup (X-Y)) - r_M(X-Y)$. Since $r_{M|X} = r_M$ on subsets of X, the rank function is the same for the two matroids on Y.

For the second equation, we use the first and duality: For the matroid on the left, $[(M.X)|Y]^* = (M.X)^*.Y = (M^*|X).Y$. For the matroid on the right, similarly $[(M|\overline{X-Y}).Y]^* = (M^*.\overline{X-Y})|Y$. By applying the first equation to M^*, the duals of the two matroids in the second equation are the same, and hence they are the same matroid.

8.2.42. *Rank function for matroid contraction.* We apply the formula for the rank function of the dual and the equality $(M.F)^* = M^*|F$. We compute

$$\begin{aligned} r_{M.F}(X) &= r_{(M^*|F)^*}(X) = |X| - r_{M^*|F}(F) + r_{M^*|F}(F-X) \\ &= |X| - r_{M^*}(F) + r_{M^*}(F-X) \\ &= |X| - [|F| - r_M(E) + r_M(\overline{F})] + |F-X| - r_M(E) + r_M(\overline{F-X}) \\ &= r_M(X \cup \overline{F}) - r_M(\overline{F}) \end{aligned}$$

($\bullet$) Also derive the formula directly by proving that X is independent in $M.F$ if and only if adding X to $\overline{F}$ increases the rank by $|X|$.

8.2.43. *The cycle matroid of a graph G is the column matroid over $\mathbb{Z}_2$ of the vertex-edge incidence matrix of G.* A set of edges is dependent in the cycle matroid if and only if it contains a cycle. A set of columns of the incidence matrix (which correspond to edges) is dependent in the column matroid if and only it contains a subset of columns summing to an even number in each row. If a set of edges contains a cycle C, then the corresponding columns have two 1's in the rows for the vertices of C and no 1's in other rows; hence this set of columns is dependent. If a set of columns is dependent, then the subset with even sum correspondence to a nonempty even subgraph of the graph. Every nonempty even subgraph is a union of cycles; hence the corresponding edges of the graph form a dependent set.

8.2.44. *a) The matrix $\binom{1\,0\,1\,1}{0\,1\,1\,2}$ represents $\mathbf{U}_{2,4}$ over $\mathbb{Z}_3$.* Since the matrix has two rows, there are no three independent columns. Since no column is a multiple of another, all pairs of columns are independent.

b) $\mathbf{U}_{2,4}$ has no representation over $\mathbb{Z}_2$. Suppose that $\mathbf{U}_{2,4} = M(A)$ for some binary matrix A, and let x_1, x_2, x_3, x_4 denote the four column vectors. Since the columns corresponding to circuits must sum to 0, we have $x_1 + x_2 + x_3 = \overline{0}$ and $x_1 + x_2 + x_4 = \overline{0}$, modulo 2. This yields $x_3 + x_4 = \overline{0}$, which contradicts the independence of $\{3, 4\}$.

8.2.45. *The three operations below preserve the cycle matroid of G.*

a) Decompose G into its blocks $B_1, \ldots, B_k$, and reassemble them to form another graph G' with blocks $B_1, \ldots, B_k$.

b) In a block B of G that has a two-vertex cut $\{x, y\}$, interchange the neighbors of x and y in one of the components of $B - \{x, y\}$.

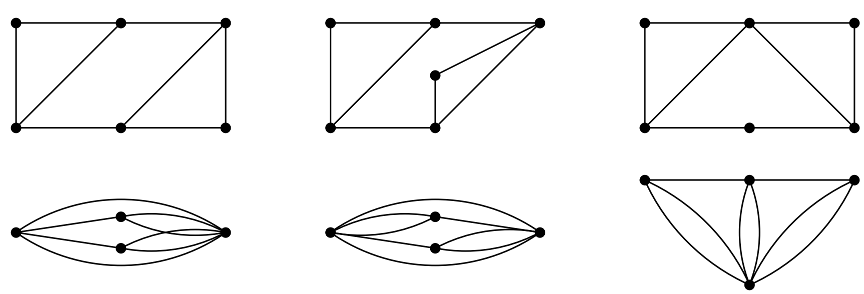
c) Add or delete isolated vertices.

Operation (a) can be described as a succession of "splitting" and "splicing" operations, where a cut-vertex is split into two vertices belonging to separate components or vertices from distinct components are merged. The blocks remain the same. This does not change the cycle matroid, because edge sets of cycles like in a single block, and a matroid is determined by its elements and its circuits. Similarly, adding or deleting isolated vertices does not affect the circuits or the set of elements.

The operation in (b) is a "vertex twist". A vertex twist at $\{x, y\}$ switches the identity of x and y in a component of $G - \{x, y\}$. The sets of edges forming x, y-paths do not change, and every cycle through x and y consists of two internally disjoint x, y-paths. Other cycles do not change. Hence the edge sets forming cycles do not change.

(Comment: Whitney's 2-Isomorphism Theorem [1933b] states that G and H have the same cycle matroid if and only if some sequence of these operations turns G into H. Thus every 3-connected planar graph has only one dual graph, meaning essentially only one planar embedding.)

8.2.46. *An abstract dual that is not a geometric dual.* Among the plane graphs below, H_i is the geometric dual of G_i. Graphs G_1 and G_2 are isomorphic. They are the only distinguishable ways to embed G_1 and G_2 in the plane, so H_1 and H_2 are the only geometric duals of G_1. However, G_3 is obtained from G_1 by an instance of operation (b) of Exercise 8.2.45, so G_1 and G_3 have the same cycle matroid; call it M. Any graph whose cycle matroid is dual to M is an abstract dual of G_1, by Corollary 8.2.37. This includes every geometric dual of G_3. Hence H_3 is an abstract dual of G_1 that is not a geometric dual of G_1.



8.2.47. *For every matroid M, the base exchange graph $\beta(M)$ is Hamiltonian ($\beta(M)$ has a vertex for each base, adjacent when their symmetric difference has size 2).* The proof is by induction on $|E|$, proving the stronger statement that there is a Hamiltonian path connecting the endpoints of any edge. If there is only one base, then the graph is trivial. If there are two bases, then they induce an edge, by the base exchange property.

Suppose that there are more than two bases, which requires $|E| \geq 3$. If M has an element e in no circuit, then every base of M contains e, and $\beta(M)$ is isomorphic to $\beta(M \cdot e)$. Similarly, if e is a loop (itself a circuit), then no base of M contains e, and $\beta(M)$ is isomorphic to $\beta(M - e)$. Hence we may assume that M has no loops and that every element of E belongs to a circuit (no co-loops).

If M has exactly one circuit C, then addition of any element to a base

generates C, so every base lacks exactly one element of E, and that element always belongs to C. Thus $\beta(M)$ in this case is a complete graph and has the desired cycle. Hence we may assume that M has more than one circuit.

Let (B_1, B_2) be an arbitrary edge of $\beta(M)$, with $B_1 - B_2 = e$ and $B_2 - B_1 = f$. The subgraph of $\beta(M)$ induced by the bases containing e is isomorphic to $\beta(M \cdot e)$, and the subgraph induced by the bases not containing e is isomorphic to $\beta(M - e)$. The induction hypothesis will yield Hamiltonian paths in these subgraphs starting at B_1 and B_2. We will connect the opposite ends of these paths to obtain a Hamiltonian path from B_1 to B_2 in $\beta(M)$.

Because $B_1 + f$ contains a circuit C but B_1 and $B_1 + f - e$ do not, we conclude that e and f both belong to C and that $C - \{e, f\}$ belongs to both B_1 and B_2. If every element of M that is not in B_1 or B_2 is parallel to e or f, then again $\beta(M)$ is a clique and has the desired cycle. Otherwise, we may select an element h that is not in B_1 or B_2 and is not parallel to e or f.

The set $B_1 + h$ contains a unique cycle C'; because h is not parallel to e we can select an edge $g \neq e, h$ from C'. Note that g cannot be parallel to e or to f, the former because $g, e \in B_1$ and latter because $C - f + g \subseteq B_1$. Let $B_3 = B_1 + h - g$, so (B_1, B_3) is an edge of $\beta(M)$. Now B_1, B_2, B_3 agree outside $\{e, f, g, h\}$, and they intersect $\{e, f, g, h\}$ in $\{e, g\}$, $\{f, g\}$, and $\{e, h\}$, respectively. We claim that the set $B_4 = B_2 + h - g = B_3 + f - e$ that intersects $\{e, f, g, h\}$ in $\{f, h\}$ is also a base of M, so that (B_2, B_4) and (B_3, B_4) are edges of $\beta(M)$.

To show this, keep in mind that $C - f \subseteq B_1$ and $C' - h \subseteq B_1$. Suppose first that $f \notin C'$. In this case $C' - h \subseteq B_2$, so $C' \subseteq B_2 + h$. Since adding h to B_2 introduces a unique circuit, this circuit is C', which contains g, and B_4 is independent.

To eliminate the possibility that $f \in C'$, we use strong elimination. In the situation at hand, we have $f \in C \cap C'$ and $h \in C' - C$, and the strong elimination property guarantees a circuit C'' in $C \cup C' - f$ that contains h. However, $C \cup C' - f \subseteq B_1 + h$. Thus C'' must be the unique circuit C' obtained by adding h to B_1, contradicting the assumption that $f \in C'$.

Hence we can apply the induction hypothesis to $\beta(M \cdot e)$ and $\beta(M - e)$ to obtain paths from B_1 to B_3 and from B_2 to B_4 through all the bases containing e and omitting e, respectively. Adding the edge (B_3, B_4) completes a Hamiltonian path in $\beta(M)$ between B_1 and B_2.

For graphic matroids, the Hamiltonian circuit is a cyclic listing of the maximal forests by changing one edge at a time. For uniform matroids, the result is perhaps more interesting; it guarantees a cyclic listing of the k-sets of an n-set by changing one element at each step. (This can also be done by omitting the non-k-sets from the standard "Gray code" listing of all subsets as produced by Exercise 7.2.17.)

8.2.48. (•) Use weak duality of linear programming to prove the weak duality property for matroid intersection: $|I| \le r_1(X) + r_2(\overline{X})$ for any $I \in \mathbf{I}_1 \cap \mathbf{I}_2$ and $X \subseteq E$. (Hint: Consider the discussion of dual pairs of linear programs in Remark 8.1.7.)

8.2.49. *Common independent and spanning sets in two matroids M_1 and M_2 on E.*

a) The minimum size of a set in E that is spanning in both M_1 and M_2 is $\max_{X \subseteq E}(r_1(E) - r_1(X) + r_2(E) - r_2(\overline{X}))$. A common spanning set contains a base of each matroid. Thus a smallest common spanning set is a smallest union of bases of the two matroids. We minimize $|B_1 \cup B_2|$ by maximizing $|B_1 \cap B_2|$, which is the size of I, a largest common independent set. The minimum size of a common spanning set is thus $|B_1| + |B_2| - |I|$. By the Matroid Intersection Theorem, the size is as claimed.

b) For a U, V-bigraph G without isolated vertices, $\alpha(G) = \beta'(G)$ (König's Other Theorem).

Since G has no isolated vertices, $r_1(E) = |U|$ and $r_2(E) = |V|$. For $X \subseteq E$, let A_1 be the subset of U not touched by X; we have $|A_1| = r_1(E) - r_1(X)$. Similarly, $|A_2| = r_2(E) - r_2(\overline{X})$, where A_2 is the subset of V not touched by $\overline{X}$. If $a_1 \in A_1$ and $a_2 \in A_2$, then every edge of $X \cup \overline{X}$ misses one or the other, so $A_1 \cup A_2$ is a stable set. This holds for all X, so

$$\alpha(G) \ge \max_{X \subseteq E}\{r_1(E) - r_1(X) + r_2(E) - r_2(X)\} = \beta'(G).$$

c) The maximum size of a common independent set plus the minimum size of a common spanning set equals $r_1(E)+r_2(E)$, and thus $\alpha'(G)+\beta'(G) = n(G)$ for a U, V-bigraph G without isolated vertices (Gallai's Theorem). Continuing the argument in part (a), adding $|I|$ to the formula yields $|B_1|+|B_2|$, which equals $r_1(E) + r_2(E)$.

Now let M_1 and M_2 be the partition matroids on $E(G)$ induced by U and V. Since G has no isolated vertices, a set $S \subseteq E$ is spanning in both matroids if and only if it covers all the vertices; hence $\beta'(G) = \min |S|$. As has been remarked repeatedly, a set I is a common independent set if and only if it is a matching. Thus $\alpha'(G) = \max |I|$. We obtain $\alpha'(G) + \beta'(G) = \max |I| + \min |S| = r_1(E) + r_2(E) = |U| + |V| = n(G)$.

Equality holds by letting X be the set of edges with endpoints in $A \cap T$, where A is a maximum stable set. (Alternatively, every edge cover requires at least $\alpha(G)$ edges to cover the vertices in a maximum stable set.)

8.2.50. *In every acyclic orientation of G, the vertices can be covered with at most $\alpha(G)$ pairwise-disjoint paths.* Given an acyclic digraph D, let M_1 and M_2 be the head partition matroid and the tail partition matroid of D. Every common independent set I is the edge set of a family of disjoint paths in D, since D is acyclic. The number of paths is $n(D) - |I|$. By the Matroid Intersection Theorem, the maximum such $|I|$ equals $\min_{X \subseteq E} r_1(X) + r_2(\overline{X})$. For each $X \subseteq E$, the vertices not covered by the head of an edge in X or the tail of an edge in $\overline{X}$ form a stable set of vertices in D. Hence for each $X \subseteq E$ there is a stable set in D of size at least $n - r_1(X) - r_2(\overline{X})$. Hence the maximum size of a stable set in D is at least the minimum number of paths needed to partition the vertices of D. Thus $V(D)$ can be covered using at most $\alpha(D)$ disjoint paths. (Comment: This is the special case of the Gallai–Milgram Theorem for acyclic digraphs.)

8.2.51. *For the transversal matroid M on E induced by subsets $A_1, \dots, A_m$ of E, the rank function r on subsets of E is defined by $r(X) = \min_{Y \subseteq X}\{|X| - (|Y| - |N(Y)|)\}$, where $N(Y)$ indicates the neighborhood in the incidence bigraph with partite sets E and $\{A_1, \dots, A_m\}$.* Let $T = \{A_1, \dots, A_m\}$. Ore's Theorem (Exercise 3.1.32) for an X, T-bigraph H states that $\alpha'(H) = |X| - \max_{Y \subseteq X}(|Y| - |N(Y)|)\}$. In this setting, $r(X) = \alpha'(G[X \cup T])$, so we merely set $H = G[X \cup T]$.

8.2.52. *Hall's Theorem from the rank function.* Let G be a bipartite graph with partite sets E, $[m]$ and with no isolated vertices. For $X \subseteq E$, let $r(X) = \min_{J \subseteq [m]}\{|N(J) \cap X| - |J| + m\}$. The following are equivalent for X.

A) Hall's Condition holds for X ($|N(S)| \ge |S|$ for all $S \subseteq X$).

B) $r(X) \ge |X|$.

C) X is saturated by some matching in G.

C $\Rightarrow$ A. This is the trivial part of Hall's Theorem; the members of each set $S \subseteq X$ have distinct neighbors in the matching.

A $\Rightarrow$ B. For $J \subseteq [m]$, we show that $|N(J) \cap X| - |J| + m \ge |X|$. Let $S = X - N(J)$, so $N(S) \subseteq [m] - J$. We are given $|N(S)| \ge |S|$, so $m - |J| \ge |S| = |X| - |N(J) \cap X|$. Moving $|N(J) \cap X|$ yields the desired inequality.

B $\Rightarrow$ C. By restricting our attention to $G[X \cup [m]]$, we may assume that X is the entire partite set E. Hence we are given $|E| - |N(J)| \le m - |J|$ for all $J \subseteq [m]$. (Comment: Showing that this condition is equivalent to Hall's Condition yields this expression for the rank function of a transversal matroid from Hall's Theorem. The proof of Hall's Theorem from this condition is not very different from the usual proofs of Hall's Theorem.)

Let M be a maximum matching in G. If M does not saturate all of E, let R be the set of vertices reachable from unsaturated vertices of E by paths that alternate between edges not in M and edges in M. Let $S = R \cap E$, and let $T = R \cap [m]$. Alternating paths reach T along edges not in M and continue to S under edges of M. If any vertex of T is unsaturated, then we have an augmenting path and M is not a maximum matching. Hence every vertex of T has a mate in S under M. This yields $|S| > |T|$, since S includes at least one unsaturated vertex.

Now let $J = [m] - T$. Since the edges of M incident to S come from T, the definition of T yields $N(S) \subseteq T$. Hence $N(J) \subseteq E - S$. We obtain $|E| - |N(J)| \ge |S| > |T| = m - |J|$, which contradicts the given inequality.

8.2.53. (!) Let G be an $E,[m]$-bigraph without isolated vertices. For $X \subseteq E$ and $J \subseteq [m]$, let $g(X, J) = |N(J) \cap X| - |J|$, and let $r(X) = \min\{g(X, J) + m\colon J \subseteq [m]\}$. Say that J is X-*optimal* if $r(X) = g(X, J) + m$.

a) Prove that $r(\emptyset) = 0$ and that $r(X) \le r(X + e) \le r(X) + 1$.

b) Prove that r satisfies the weak absorption property.

8.2.54. *Restrictions and unions of transversal matroids are transversal matroids.* If M is the transversal matroid on E induced by the bipartite graph G with partite sets E, T, then $M|X$ is the transversal matroid induced on X by the induced subgraph $G' = G[X \cup T]$. By definition, $Y \in \mathbf{I}(M|X)$ if $Y \subseteq X$ and $Y \in \mathbf{I}(M)$. No matching of Y in G uses vertices of $E - X$, so Y has a matching in G if and only if it has a matching in $G[X \cup F]$.

The union of matroids M_1, M_2 on E has X independent in $M_1 \cup M_2$ if $X = X_1 \cup X_2$, where $X_i \in \mathbf{I}(M_i)$. If M_1, M_2 are transversal matroids with set systems $\{A_i\}, \{B_j\}$, then the union of partial transversals of $\{A_i\}$ and $\{B_j\}$ is a partial transversal of the set system $\{A_i\} \cup \{B_j\}$. In the graph context, this is equivalent to identifying corresponding vertices of E in two graphs with partite sets E, F_1 and E, F_2.

Contractions and duals of transversal matroids need not be transversal matroids. We first construct a non-transversal matroid. Define a matroid M of rank 2 on six elements $E = \{a, b, c, d, e, f\}$ by letting the bases of M be all 15 pairs except $\{\{a, b\}, \{c, d\}, \{e, f\}\}$. To verify that M is a matroid, it is easy to check that the remaining twelve pairs satisfy the base axioms, or use the fact that the dual discussed below is a matroid.

Suppose that M is a transversal matroid. Singletons are independent, so a set system $\{A_i\}$ realizing M has each element in some set. A dependent pair appears in only one set. Since each element appears in some dependent pair, each element therefore appears in only one A_i. Hence the elements of any 3-element circuit appear in only two sets. Consider the circuit $\{a, c, e\}$. By symmetry, we may assume a, c appear in the same set. However, a, c appearing in only one set contradicts the independence of $\{a, c\}$.

The dual of this matroid is a transversal matroid of rank 4. The set system realizing it is $A_1 = \{a, b, c, d, e, f\}$, $A_2 = \{a, b\}$, $A_3 = \{c, d\}$, $A_4 = \{e, f\}$. Any set of 4 elements containing one element each from A_2, A_3, A_4 is a transverval. In other words, the only 4-sets of E that are not bases are $\{a, b, c, d\}, \{c, d, e, f\}, \{a, b, e, f\}$. Hence this transversal matroid is M^*.

To define a transversal matroid N whose contraction is M, add an element g. Let $A_1 = \{g, a, b\}$, $A_2 = \{g, c, d\}$, $A_3 = \{g, e, f\}$. Then the tranversal matroid N induced on $E \cup \{g\}$ by A_1, A_2, A_3 has rank 3, and $N \cdot E$

has rank 2. The bases of N are the pairs that with g form a transversal of $\{A_i\}$; these are all pairs except $\{a, b\}, \{c, d\}, \{e, f\}$. Hence $N \cdot E = M$. Note that $N|E$ is a transversal matroid of rank 3.

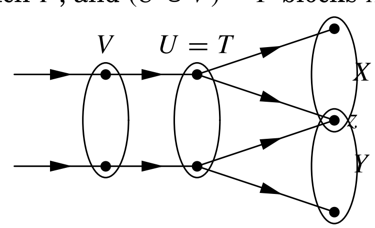
8.2.55. *Gammoids.* Give a digraph D and sets $F, E \subseteq V(D)$, the *gammoid* on E induced by (D, F) is the hereditary system given by $\mathbf{I} = \{X \subseteq E\colon$ there exist $|X|$ pairwise disjoint paths from F to $X\}$; equivalently, $r(X)$ is the maximum number of pairwise disjoint F, X-paths.

a) Every transversal matroid is a gammoid. A transversal matroid M arises from an E, Y-bigraph G by letting the independent sets be the subsets of E that are saturated by matchings. Let D be the orientation of G directing all edges from Y to E, and let $F = Y$. Now M is the gammoid on E induced by (D, F).

b) Every gammoid is a matroid.

Proof 1 (submodularity of the rank function). Say that S *blocks* X if S intersects all F, X-paths. By Menger's Theorem, $r(X)$ is the minimum size of a set blocking X. Let U, V be minimum sets blocking X, Y, respectively. We will obtain a set $T \subseteq U \triangle V$ such that $(U \cap V) \cup T$ blocks $X \cap Y$ and $(U \cup V) - T$ blocks $X \cup Y$. The sizes of such sets sum to $|U| + |V|$, which will yield $r(X \cap Y) + r(X \cup Y) \le r(X) + r(Y)$.

If $U \cap V$ does not block $X \cap Y$, then for some $z \in X \cap Y$ there is an F, z-path using $U \triangle V$. Let T be the set of vertices that are the last vertex of $U \triangle V$ on some such path. Then $(U \cap V) \cup T$ blocks $X \cap Y$; we need only show that $(U \cup V) - T$ blocks $X \cup Y$. If there is a path P from F to $X \cup Y$ that is not blocked by $(U \cup V) - T$, then all vertices of P in $U \cup V$ belong to T. Let P' be the portion of P up to its first vertex v in T. By the definition of T, for some $z \in X \cap Y$ there is a v, z-path Q that avoids $U \triangle V$ after v. Following P' by Q yields an walk and hence a path from F to $z \in X \cap Y$ that has only one vertex of $U \cup V$, which is a vertex $v \in T$. This path avoids V if $v \in U$ and avoids U if $v \in V$. This is impossible, because U and V each block $X \cap Y$. Hence there is no such P, and $(U \cup V) - T$ blocks $X \cup Y$.

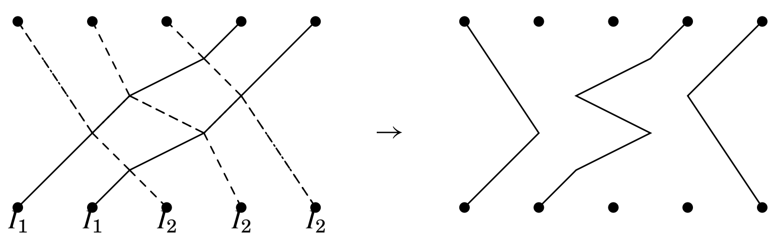


Proof 2 (augmentation property). Let $I_1, I_2 \in \mathbf{I}$ be independent sets with $|I_2| - 1 = |I_1| = k$. We have disjoint F, I_1-paths $P_1, \ldots, P_k$ and disjoint F, I_2-paths $Q_1, \ldots, Q_{k+1}$. Let $U = \{P_i\} \cup \{Q_j\}$. Partition the edges of each path Q_j or P_i into segments that are maximal subpaths for which no internal vertex belongs to another path in U. From these segments we form

$k+1$ disjoint paths with sources in F and sinks I_1+e, for some $e \in I_2$. Since each of $\{P_i\}$ and $\{Q_j\}$ is a set of disjoint paths, each intersection point v is shared by one P_i and one Q_j. Except for sources and sinks, one P-segment and one Q-segment enters v, and one segment of each type leaves v. If v is a source or sink of one path, then the entering or departing segment of that type is missing, and we include v as a trivial segment, declared to meet the other path at its sink or source, respectively. Thus every intersection point has one entering and one departing segment of each type.

Define a bipartite graph H with the segments as vertices. A P-segment and a Q-segment are adjacent in H if they have the same source or have the same sink. A segment from u to v meets at most 1 segment of the other type at each of u, v. Hence every vertex of H has degree at most two, and H consists of alternating paths and alternating (i.e. even) cycles.

Counting the nontrivial segments by endpoints counts each segment twice. Since there is an extra Q-source and Q-sink, the number of Q-segments in H exceeds the number of P-segments by one. Hence there must be some path R that starts and ends at Q-segments. To obtain the augmentation of I_1, we turn each Q-segment along R into a segment of a P-path and delete each P-segment of R, which we view as switching the ownership of the segments along R. This preserves the disjointness of the P-paths and the disjointness of the Q-paths. However, since R starts and ends at Q-segments, we have changed the number of P-paths from k to $k+1$. The sources are still in F, and the sinks are I_1+e for some $e \in I_2$.



8.2.56. *A matroid is a strict gammoid if and only if it is the dual of a transversal matroid,* where a *strict gammoid* is a gammoid on E induced by (D, F) with the additional property that $E = V(D)$.

Given a transversal matroid M of rank n on E, let $G \subseteq K_{E,T}$ be a bipartite graph realizing it. Let $B = \{b_1, \dots, b_n\} \subseteq E$ be a base of M, and let $L = \{b_1t_1, \dots, b_nt_n\}$ be a matching of B. Define a directed graph D with vertex set E and edges $e \to b_j$ if and only if $e \leftrightarrow t_j$ in G. Let $F = E - B$. Any path P starting in F starts with a vertex of $E-B$, but thereafter stays in B. Traversal of the edge $e \to b_j$ in D can be interpreted as traversal of the path $e \leftrightarrow t_j \leftrightarrow b_j$ in G. Thus the path P of length k starting in F corresponds to a path Q of length $2k$ that starts outside B and alternates

between edges not in L and edges in L. The symmetric difference $L \Delta P$ is a matching in G in which the source of P is now matched and the sink is not. A set of disjoint paths $\mathbf{P}$ corresponds to paths Q that use disjoint edges in L, so again $L\Delta \cup \mathbf{P}$ is a matching in G in which the sources are matched and the sinks are not.

Let N be the strict gammoid on D with source set F. Since $E = V(D)$, the bases of N are the sink sets of all collections of $|F|$ disjoint paths $\mathbf{P}$ with sources F. As discussed above, for each such collection there is a matching of size n that leaves the sink vertices of $\mathbf{P}$ unmatched. Hence the complement of any base of N is a base of M. Conversely, if L' is a matching in G for a base B' of M, then $L' \Delta L$ is a collection of disjoint alternating paths and cycles, including paths from $B' - B$ to $B - B'$ that alternate between edges of L' and L. These paths collapse to a set of paths in D, half as long, from $F = \overline{B}$ to $\overline{B}'$. Hence also the complement of any base in M is a base in N, and $N = M^*$.

Conversely, given any strict gammoid N on D with vertices E and source set F, we can reverse this construction. First note that $r(N) = |F|$. This means that no edge between vertices of F can be used in paths corresponding to a base, so if we discard or ignore all edges between vertices of F we get the same gammoid (similarly, in the previous construction we ignored edges of G not involving $\{t_1, \dots, t_n\}$). Letting $B = E - F = \{b_1, \dots, b_n\}$, we define a bipartite graph $G \subseteq K_{E,T}$, where $T = \{t_1, \dots, t_n\}$, with edges b_it_i, and also edges et_j if $e \to b_j$ in D. Let M be the transversal matroid induced on E by G. The correspondence between paths in D originating in F and alternating paths is G with respect to the matching $L = \{b_it_i\}$ is the same as above, and once again $N = M^*$.

8.2.57. *If M_1 and M_2 are matroids with spanning sets $\mathbf{S}_1$ and $\mathbf{S}_1$, then the hereditary system $M_1 \wedge M_2$ whose spanning sets are $\{X_1 \cap X_2 \colon X_1 \in \mathbf{S}_1, X_2 \in \mathbf{S}_2\}$ is $(M_1^* \cup M_2^*)^*$.* We have X as the intersection of spanning sets X_1, X_2 in M_1 and M_2 if and only if $\overline{X}$ is the union of independent sets $\overline{X}_1, \overline{X}_2$ in M_1^* and M_2^*. Hence $X \in \mathbf{S}_{M_1 \wedge M_2}$ if and only if $\overline{X} \in \mathbf{I}_{M_1^* \cup M_2^*}$, which implies $M_1 \wedge M_2 = (M_1^* \cup M_2^*)^*$.

8.2.58. *Generalized transversal matroids.* Let M be a matroid on E.

a) For $A_1, \dots, A_m \subseteq E$, the hereditary system M' defined by $\mathbf{I}_{M'} = \{X \subseteq [m] \colon \{A_i \colon i \in X\}$ has a transversal belonging to $\mathbf{I}_M\}$ is a matroid with rank function $r'(X) = \min_{Y \subseteq X}\{|X-Y| + r(A(Y))\}$, where $A(Y) = \bigcup_{i \in Y} A_i$. We show first that r' is the rank function of M'; that is, the family of sets indexed by X has a transversal in $\mathbf{I}_M$ if and only if $|X-Y| + r(A(Y)) \ge |X|$ for all $Y \subseteq X$, or equivalently $r(A(Y)) \ge |Y|$; call this condition $(*)$.

If there is such a transversal T, then its subsets are also independent in M. The subset of T representing the sets indexed by Y has size Y, and hence $r(A(Y)) \ge |Y|$.

For the converse, (∗) is given. We use induction on $\sum_{i=1}^m |A_i|$ to prove that X has an independent transversal. From (∗), $|A_i| \ge 1$ for all i. If equality always holds, then the sets are distinct elements and their union is an independent transversal in M. This completes the basis step.

Hence we may assume that $|A_1| \ge 2$. By the induction hypothesis, it suffices to find $e \in A_1$ such that replacing A_1 with $A_1 - e$ yields a system that also satisfies (*). If not, then for all $x_i \in A_1$ there exists $Y_i \subseteq [m] - \{1\}$ such that $r((A_1 - x_i) \cup A(Y_i)) < |Y_i| + 1$. Taking distinct elements $x_1, x_2 \in A_1$ and applying the submodularity of r and (∗) yields

$$\begin{aligned}|Y_1| + |Y_2| &\ge r(A_1 \cup A(Y_1) \cup A(Y_2)) + r((A_1 - \{x_1, x_2\}) \cup A(Y_1 \cap Y_2)) \\ &\ge r(A(\{1\} \cup Y_1 \cup Y_2) + r(A(Y_1 \cap Y_2)) \ge 1 + |Y_1 \cap Y_2| + |Y_1 \cap Y_2|\end{aligned}$$

The contradiction completes the proof that r' is the rank function of the hereditary system M'.

It thus suffices to prove that r' is submodular. For $U \subseteq X$ and $V \subseteq Y$, note (as used in the proof of the Matroid Union Theorem) that $|X - U| + |Y - V| = |(X \cap Y) - (U \cap V)| + |(X \cup Y) - (U \cup V)|$. Also, $A(U) \cup A(V) = A(U \cup V)$, while $A(U) \cap A(V) \supseteq A(U \cap V)$. Using first the submodularity of r, these observations yield

$$\begin{aligned}r'(X) + r'(Y) &= \min_{U \subseteq X}[|X - U| + r(A(U))) + \min_{V \subseteq Y}(|Y - V| + r(A(V))] \\ &\ge \min_{U,V}[|X - U| + |Y - V| + r(A(U) \cup A(V)) + r(A(U) \cap A(V))] \\ &\ge \min_{U,V}[|(X \cup Y) - (U \cup V)| + r(A(U \cup V)) + |(X \cap Y) - (U \cap V)| + r(A(U \cap V)] \\ &\ge \min_{S \subseteq X \cup Y}[|(X \cup Y) - S| + r(A(S))] + \min_{T \subseteq X \cap Y}[|(X \cap Y) - T| + r(A(T)) \ge r'(X \cup Y) + r'(X \cap Y)\end{aligned}$$

b) If f is a function from E to a finite set F, and M' is the hereditary system on F defined by $\mathbf{I}_{M'} = \{f(X) \colon X \in \mathbf{I}_M\}$, then M' is a matroid with rank function $r'(X) = \min_{Y \subseteq X}\{|X - Y| + r(f^{-1}(Y))\}$ when f is surjective. Let $A_i = f^{-1}(i)$ for $i \in F$. Now X is independent in M' if and only if the sets indexed by F have a transversal in $\mathbf{I}_M$, since f is a function. By part (a), M' is a matroid with the specified rank function.

8.2.59. (•) Apply matroid sum and Exercise 8.2.58 to prove the Matroid Union Theorem.

8.2.60. *Matroid Intersection from Matroid Union.*

The maximum size of a common independent set in matroids M_1 and M_2 on E is $r_{M_1 \cup M_2^*}(E) - r_{M_2^*}(E)$. Let $a = r_{M_1 \cup M_2^*}(E)$, $b = r_{M_1 \cap M_2}$, and $c = r_{M_2^*}$; we seek $b = a - c$. If $Z \in \mathbf{I}_1 \cap \mathbf{I}_2$ with $|Z| = b$, then $\overline{Z}$ contains a base A of M_2^*, which means that $Z \cup A$ is independent in $M_1 \cup M_2^*$; thus $a \ge b + c$. On the other hand, if $X \in \mathbf{I}_{M_1 \cup M_2^*}$ with $|X| = a$, then we may assume $X = B_1 \cup B_2$, where $B_1 \in \mathbf{B}_1$ and $B_2 \in \mathbf{B}_2^*$. Since $|B_1 \cup B_2| = |B_2| + |B_1 - B_2|$, we have $a \le c + b$, because $B_1 \in \mathbf{I}_1$ and $B_1 - B_2 \subseteq \overline{B}_2 \in \mathbf{B}_2$.

b) Matroid intersection. Using the rank formula in the Matroid Union Theorem, $a - c = \min_{X \subseteq E}\{|\overline{X}| + r_1(X) + r_2^*(X)\} - r_2^*(E)\}$. Since $r_2^*(X) = |X| - (r_2(E) - r_2(\overline{X}))$ and $r_2^*(E) = |E| - r_2(E)$, the formula for $a - c$ simplifies to $\min_{X \subseteq E}\{r_1(X) + r_2(\overline{X})$, which by part (a) equals $\max\{|I| \colon I \in \mathbf{I}_1 \cap \mathbf{I}_2\}$.

8.2.61. *If G is an n-vertex weighted graph, and $E_1, \ldots, E_{n-1}$ is a partition of $E(G)$ into $n - 1$ sets, then there a polynomial-time algorithm to find a spanning tree having exactly one edge in each subset E_i, if one exists.* Such a spanning tree is a set of edges that is independent in both the cycle matroid of G and the partition matroid on $E(G)$ induced by the specified partition. The Matroid Intersection Algorithm finds a common independent set of maximum size, and the size will be $n - 1$ if such a spanning tree exists. However, this may not be a common independent set of maximum weight.

8.2.62. *Every $2k$-edge-connected graph G has k pairwise edge-disjoint spanning trees avoiding any specified set of at most k edges.* By Corollary 8.2.59, a necessary and sufficient condition for having k pairwise edge-disjoint spanning trees is that for every vertex partition, the number of edges of G with endpoints in different blocks of the partition is at least $k(p-1)$, where p is the number of blocks.

If G is $2k$-edge-connected and S is a block in the partition, then at least $2k$ edges lie in the edge cut $[S, \overline{S}]$. Summing over all p parts counts each crossing edge twice. Hence at least kp edges crossing between parts. If at most k edges are deleted, then there are still at least $k(r - 1)$ crossing edges, which is enough.

The result is sharp, because K_{2k+1} is $2k$-edge-connected but does not have $k + 1$ pairwise edge-disjoint spanning trees. Such spanning trees would require $(k + 1)2k$ edges, and K_{2k+1} has $(2k + 1)k$ edges, which is less.

8.2.63. (•) Given matroids $M_1, \ldots, M_k$ on E, the *Matroid Partition Problem* is the problem of deciding whether an input set $X \subseteq E$ partitions into sets $I_1, \ldots, I_k$ with $I_i \in \mathbf{I}_i$.

a) Use the Matroid Union Theorem to show that X is partitionable if and only if $|X - Y| + \sum r_i(Y) \ge |X|$ for all $Y \subseteq X$, and that all maximal partitionable sets are maximum partitionable sets.

b) Let M' be the union of k copies of a matroid M on E, and let X be a maximum partitionable set. Prove that there are disjoint sets $F_1, \ldots, F_k \subseteq X$ such that $\{F_i\} \subseteq \mathbf{I}$ and $\overline{X} \subseteq \sigma(F_1) = \cdots = \sigma(F_k)$.

8.3. RAMSEY THEORY

8.3.1. *Two concentric discs, each with 20 radial sections half red and half blue, can be aligned so that at least 10 sections on the inner disc match color with the corresponding sections on the outer disc.* Over all positions, each section provides 10 agreements, for 200 agreements in total. Since the agreements occur during 20 positions, there must be some position where the number of agreements is at least 200/20, which equals 10.

8.3.2. *Every set of $n+1$ numbers in $[2n]$ contains a pair of relatively prime numbers.* Any two consecutive numbers are relatively prime, since an integer greater than 1 cannot both. Hence it suffices to partition $[2n]$ into the n pairs of the form $(2i-1, 2i)$. Since there are only n such pairs, the pigeonhole principle guarantees that a set of $n+1$ numbers in $[2n]$ must use two from some pair, and these are relatively prime.

The result is best possible, because in the set of n even numbers in $[2n]$, every pair has a common factor. Note that since each pair of even numbers is not relatively prime, a solution to the problem by partitioning $[2n]$ into n classes and applying the pigeonhole principle must put the n even numbers into n different classes.

Every set of $n+1$ numbers in $[2n]$ contains two numbers such that one divides the other. This is best possible in that the n largest numbers in $[2n]$ do not contain such a pair. To apply the Pigeonhole Principle, we partition $[2n]$ into n classes such that for every two numbers in the same class, one divides the other.

Every natural number has a unique representation as an odd number times a power of two. For fixed k, the set $\{(2k-1)2^{j-1}\colon j \in \mathbb{N}\}$ has the desired property; the smaller of any two divides the larger. Since there are only n odd numbers less than $2n$, we have n such classes. The kth class is $\{m \in [2n]\colon m = (2k-1)2^{j-1} \text{ and } j \in \mathbb{N}\}$.

8.3.3. *a) Every set of n integers has a nonempty subset summing to a multiple of n.* Let a_i be the sum of the first i integers in the set. If n divides any a_i, we are finished. So, the a_i fall into $n-1$ congruence classes mod n. Hence there must be two of them in the same class. If these are a_j and a_k, then $a_k - a_j$, which is the sum of the $j+1$th through kth numbers, is divisible by n. So, in fact we have found a consecutive subset summing to a multiple of n. The example showing this is best possible is a set of $n-1$ copies of 1. Since n divides a sum of 1s only if the number of 1s is a multiple of n, the condition fails for this example.

b) At least one of $\{x, \ldots, (n-1)x\}$ differs by at most $1/n$ from an integer. Consider the fractional parts of these numbers and the n intervals of the form $[(i-1)/n, i/n)$. If some fractional part falls in the first or last interval, we are done. Otherwise, we have $n-1$ objects in $n-2$ classes, and some pair jx and kx fall in the same interval. Now $(k-j)x$ is within $1/n$ of an integer.

8.3.4. *Private club needing 990 keys.*

990 keys permit every set of 90 members to be housed. Suppose 90 members receive one key apiece, each to a different room, and the remaining 10 members receive keys to all 90 rooms. Each set of 90 members that might arrive consists of k members of the first type and $90-k$ members of the second type. When the k members of the first type go to the rooms for which they have keys, there are $90-k$ rooms remaining, and the $90-k$ members of the second type that are present have keys to those rooms.

No scheme with fewer keys works. If the number of keys is less than 990, then by the pigeonhole principle (every set of numbers has one that is at most the average) there is a room for which there are fewer than 11 keys. Since the number of keys to each room is an integer, there are at most 10 keys to this room. Hence there is a set of 90 of the 100 members that has no one with a key to this room. When this set of 90 members arrives, they have keys to at most 89 rooms among them and cannot all be housed.

8.3.5. *The center of a tree T is a vertex or an edge.* For each vertex v, let v' be a vertex farthest from v in T, and mark the edge incident to v that leaves v on the unique path to v' in T. This makes $n(T)$ marks, so some edge uw has been marked twice.

The graph $T - uw$ consists of two components. If y is a vertex of the component of $T - uw$ containing u, then $d(y, u') > d(u, u')$, since the unique y, u'-path passes through u'. Similarly, if x is a vertex of the other component of $T - uw$, then $d(x, w') > d(w, w')$.

Hence the only candidates for the minimum eccentricity are the adjacent vertices u and w, and the center is a vertex or an edge.

8.3.6. *Every set of $2^m + 1$ integer lattice points in $\mathbb{R}^m$ contains two points whose centroid (mean vector) is also an integer lattice point.* Define 2^m classes by parity; each class is an m-tuple from {odd, even}. With $2^m + 1$ integer points and 2^m classes, there must be two in the same class. When two integer points having the same parity in each coordinate are averaged, the result is an integer point.

8.3.7. *Every red/blue-coloring of $\mathbb{R}^m$ has n integer lattice points with the same color whose centroid also has that color.* From any $2n-1$ lattice points whose coordinates are multiples of n, choose n points $a_1, \ldots, a_n$ with the same color, say red; their centroid $\frac{1}{n}\sum a_i$ is also an integer point. Let C denote the centroid. If C is blue, then let $b_j = (n+1)a_j - \sum a_i$ for $1 \le j \le n$.

Now a_j is the centroid of the set obtain by replacing a_j by b_j. If any b_j is red, then we have the desired set in red. Otherwise, $\{b_1, \ldots, b_n\}$ is a blue set with blue centroid, since $\frac{1}{n}\sum b_j = C$.

8.3.8. *If S is a multiset of $n+1$ positive integers with sum k, and $k \le 2n+1$, then S has a subset with sum i for each $i \in [k]$.* The result is sharp, since $n+1$ copies of 2 have sum $2n+2$ but no subset with sum 1.

Proof 1 (counting argument). Let $r = \max S$, and suppose that S has m copies of the number 1. If $m >= r-1$, then we can add 1s successively, increasing the subset sum by one each time until reaching $r-1$. The next set is the number r alone. Then 1s can be introduced again until the next largest number in S can be substituted for them, and so on. To obtain $m >= r-1$, observe that the bound on the sum yields $1\cdot r + (n-m)\cdot 2 + m\cdot 1 \le k \le 2n+1$, which simplifies to $m >= r-1$.

Proof 2 (induction on n). For $n = 0$, the only example is $\{1\}$, which works. For $n \ge 1$, if $\max S = 1$ then all sums can be achieved, so we may assume that $\max S = a > 1$. Since a exceeds 1, the sum of the remaining n elements is at most $2n-1$, so we can apply the induction hypothesis to obtain subsets of $S - \{a\}$ summing to all integers from 1 to $k-a$. For $k-a+1 \le i \le k$, adding the element a to a subset summing to $i-a$ will construct a subset of S summing to i, if $i - a >= 0$. This requires $a \le (1+k)/2$. The needed inequality holds, since $2a > k+1$ and $a+n \le k$ would imply $a \ge n+2$ and then $k \ge 2n+2$.

8.3.9. *For even n, Theorem 8.3.4 is sharp, in that there is an ordering of $E(K_n)$ so that the maximum length of an increasing trail is $n-1$.* When n is even, K_n has a 1-factorization. Define the linear ordering on $E(K_n)$ so that for each 1-factor in a specified factorization, the edges of that 1-factor occur consecutive. This ensures that each 1-factor contributes at most one edge to an increasing trail, and hence the maximum length of an increasing trial in this ordering is $n-1$.

8.3.10. *Every set of nine points in the plane with no three collinear contains the vertex set of a convex 5-gon, and this is sharp.* The four points $\{(\pm1, \pm2)\}$ and the four points $\{(\pm2, \pm3)\}$ together form no convex 5-gon. For nine points, our case analysis forcing a convex pentagon is due to L.H. Mak and D.B. West.

If at least 5 points lie on the convex hull, then we are finished, so we consider the two cases of four points and three points on the hull. The observation that simplifies the analysis is the following LEMMA: The segments ("spokes") from an interior point to the vertices of the hull partition the region into triangles. If some segment between two other interior points crosses two spokes, then these two points and the three endpoints of those spokes form a pentagon.

If the hull is a quadrilateral Q, we may assume that some interior point P is a convex combination of interior points XYZ, else the interior points form a pentagon. Since the triangle XYZ separates P from Q, one of its three edges must cross at least two of the four spokes from P to vertices of Q, yielding a pentagon by the lemma.

If the hull is a triangle T, we may again assume that the hull of the interior points has at most four points, so that at least two of the interior points R, S are convex combinations of the others. Let ABC be the vertices of T, with the line RS cutting AB and AC, oriented with BC horizontal, A above it, and R to the left of S. The segments from R to ABC partition the interior into three triangles. If there is no pentagon, then the lemma implies that any three interior points enclosing R have a point in each of these triangles; in particular, there is a point inside ARB. Similarly, there is a point in ASC. If the point X in ARB is below the line RS, then $BXRSC$ is a pentagon. If the point Y in ASC is below RS, then $BRSYC$ is a pentagon. If both are above, then $AXRSY$ is a pentagon.

8.3.11. *Every nondegenerate set of $R(m, m; 3)$ points in $\mathbb{R}^2$ has m points forming a convex m-gon.*

Proof 1 (stronger result). We can tilt the point set slightly, if necessary, to assume that the horizontal coordinates of the points are distinct. Color each triple of points by whether the line determined by the leftmost and rightmost points is above or below the middle point. With $R(m, m; 3)$ points, there are m points such that all triples have the same color. If all triples have the left-right line above the middle point, then the piecewise-linear function determined by the horizontally consecutive pairs of points is convex, and these points form a convex m-gon in which the line between the leftmost and rightmost is above all the other points. If all triples have the other color, then the piecewise-linear function is concave, and again the points determine a convex m-gon.

Proof 2 (using the full point set). For each triple $T \subseteq S$, color T by the parity of the number of points in $S - T$ that are in the interior of the triangle formed by T. Let Q be a set of four points in S whose triples have the same color. If Q is not convex, then the triangular region R formed by the triple T on the convex hull of Q contains the other three triples. If the three inside triangles are odd, then T is even. If the three inside triangles are even, then T is odd. Hence every homogeneous 4-set is convex. Hence taking at least $R(m, m; 3)$ points yields m points whose 4-sets are all convex, and these m points form a convex m-gon.

8.3.12. *Monotone tournaments: If N is sufficiently large, then every simple digraph with vertex set $[N]$ has an independent set of order m or a monotone tournament of order m or a complete loopless digraph of order m.* Let

$N = R(m, m, m, m)$, the Ramsey number for 4-coloring 2-sets to obtain a homogeneous set of size m. Given a simple digraph D on $[N]$, form a coloring of $E(K_N)$ by letting ij with $i < j$ have color 1 if i and j are nonadjacent in D, color 2 if $i \to j$ in D, color 3 if $j \to i$ in D, and color 4 if D has edges in both directions on this pair. By Ramsey's Theorem, there is a homogeneous set of size m, and this is the vertex set of the desired induced subgraph.

8.3.13. *Given $k > 0$, there exists an integer s_k such that every k-coloring of the integers $1, \ldots, s_k$ yields monochromatic (but not necessarily distinct) x, y, z solving $x + y = z$.* Let $r_k = R_k(3; 2)$. We show that $s_k < r_k$ by showing that every k-coloring f of $[r_k - 1]$ has a monochromatic solution to $x + y = z$. From f, we define a k-coloring f' of $E(K_{r_k})$. Let $V(K_{r_k}) = [r_k]$. Let the color of edge ab in f' be $f(|a - b|)$.

By Ramsey's Theorem, f' yields a monochromatic triangle with vertices a, b, c. We may assume that $a < b < c$. Let $x = b - a$, $y = c - b$, $z = c - a$. Since the triangle is monochromatic, $f(x) = f(y) = f(z)$. By construction, they satisfy $x + y = z$.

b) $s_k \ge 3s_{k-1} - 1$, and hence $s_k \ge (3^k + 1)/2$. Let f be a k-coloring of $[n]$ with no monochromatic $x + y = z$. Define a $(k+1)$-coloring f' of $[3n + 1]$ by

$$f'(i) = \begin{cases} f(i) & \text{if } i \le n, \\ f(i - 2n - 1) & \text{if } i \ge 2n + 2, \\ k + 1 & \text{if } n + 1 \le i \le 2n + 1. \end{cases}$$

Under f' there is no monochromatic $x + y = z$. The resulting recurrence $s_k \ge 3s_{k-1} - 1$ yields $s_k \ge (3^k + 1)/2$.

8.3.14. *Application of the lexicographic product (composition) to Ramsey numbers.* The graph $G[H]$ has a copy of H for each vertex of G, with all edges present between copies that correspond to edges of G, and no edges present between copies that correspond to non-adjacent vertices in G.

a) $\alpha(G[H]) \le \alpha(G)\alpha(H)$. Let S be a largest independent set in $G[H]$. No two vertices in S can have first coordinates u and u' with $u \leftrightarrow u'$ in G, since all such vertices in the product are adjacent. So, the vertices u that appear as first coordinates of vertices in S form an independent set in G; hence there are at most $\alpha(G)$ of them.

The vertices of S using a fixed vertex u of G as first coordinate must have as their second coordinates a set of vertices that are independent in H, since these vertices inherit adjacencies from H. Therefore, each $u \in V(G)$ that appears among the first coordinates of vertices in an independent set in $G[H]$ is used at most $\alpha(H)$ times. Since at most $\alpha(G)$ such vertices appear, each at most $\alpha(H)$ times, $\alpha(G[H]) \le \alpha(G)\alpha(H)$. (Actually, equality holds, because $S \times T$ is independent in $G[H]$ whenever S is independent in G and T is independent in H.)

b) The complement of $G[H]$ is $\overline{G}[\overline{H}]$. Nonadjacency in $G[H]$ requires $u \not\leftrightarrow u'$ when $u \ne u'$, and it requires $v \not\leftrightarrow v'$ when $u = u'$. Thus is simply the definition of adjacency in $\overline{G}[\overline{H}]$.

c) $R(pq+1, pq+1) - 1 \ge [R(p+1, p+1) - 1] \cdot [R(q+1, q+1) - 1]$. Let G be a graph on $R(p+1, p+1) - 1$ vertices with no clique or independent set of size $p + 1$, and let H be a graph on $R(q + 1, q + 1) - 1$ vertices with no clique or independent set of size $q + 1$. By part (a), $\alpha(G[H]) \le \alpha(G)\alpha(H) \le pq$ and $\alpha(\overline{G}[\overline{H}]) \le \alpha(\overline{G})\alpha(\overline{H}) = \omega(G)\omega(H) \le pq$. By part (b), $\omega(G[H]) = \alpha(\overline{G[H]}) = \alpha(\overline{G}[\overline{H}]) \le pq$. Thus $G[H]$ has no clique or independent set of size $pq + 1$, which yields the desired bound.

d) $R(2^n + 1, 2^n + 1) \ge 5^n + 1$ for $n \ge 0$, We use induction on n. If $n = 0$, then $R(2, 2) = 2$, as desired. For the induction step, let $k = 2^{n-1}$ and $l = 2$ and apply (c). This yields

$$\begin{aligned} R(2^n + 1, 2^n + 1) = R(kl + 1, kl + 1) &\ge [R(2^{n-1} + 1, 2^{n-1} + 1)][R(3, 3) - 1] + 1 \\ &\ge 5^{n-1} \cdot 5 + 1 = 5^n + 1. \end{aligned}$$

To compare this with the nonconstructive lower bound $R(p, p) \le cp2^{p/2}$, let $p = 2^n + 1$. Since $5^n + 1 = 5^{\lg(p-1)} + 1 = 1 + (p - 1)^{\lg 5}$. Since $\lg 5 < 2.5$. This construction gives only a low-degree polynomial lower bound, while the nonconstructive lower bound is exponential.

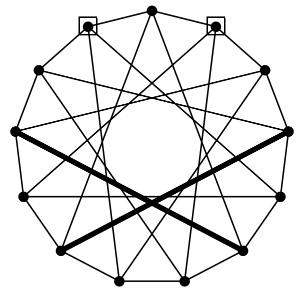
8.3.15. *$R(p, 2) = R(2, p) = p$, and hence $R(p, q) \le \binom{p+q-2}{p-1}$.* In a 2-coloring of $E(K_p)$, all edges have one color or there is an edge with the other color. In $E(K_{p-1})$, making all the edges the first color yields neither a p-set whose edges have the first color nor a single edge of the other color.

For the general upper bound, we use induction on $p + q$. When $\min\{p, q\} = 2$, we have $R(p, 2) = p = \binom{p+2-2}{p-1} = \binom{p}{1}$. Otherwise, we have $R(p, q) \le R(p, q - 1) + R(p - 1, q) \le \binom{p+q-3}{p-1} + \binom{p+q-3}{p-2} = \binom{p+q-2}{p-1}$.

8.3.16. $R(3, 5) = 14$. For the upper bound, $R(3, 5) \le R(3, 4) + R(2, 5) = 9 + 5 = 14$. To show that 13 vertices do not force a red triangle or blue K_5, it suffices to show that the graph G below is triangle-free and has no independent set of size 5.

The graph is vertex-transitive, and the neighborhood of a vertex is independent, so the graph is triangle-free.

Let S be a set of five vertices in G. By the pigeonhole principle, there are two vertices in S separated by distance at most two along the outside cycle. If S is independent, it thus has a pair with distance exactly 2, as marked below. Deleting two such vertices and their neighbors leaves the graph $2K_2$ in bold below. Thus we cannot add three more vertices to obtain an independent 5-set.



8.3.17. *Ramsey numbers for $r = 2$ and multiple colors.*

a) $R(p) \le \sum_{i=1}^{k} R(q_i) - k + 2$, *where* $p = (p_1, \ldots, p_k)$ *and* q_i *is obtained from* p *by subtracting 1 from* p_i *but leaving the other coordinates unchanged.* We show that $R(p) \le 2 + \sum(R(q_i) - 1)$. Consider a fixed vertex x in a coloring of the edges of a complete graph on $2 + \sum(R(q_i) - 1)$ vertices. Partition the other vertices into color classes by the color of the edges joining them to x. With k classes and thresholds $R(q_1), \ldots, R(q_k)$, by the pigeonhole principle at least one of the thresholds must be met. If this occurs for color j, then the definition of q_j implies that the original coloring, restricted to the edges on the vertices of color j, has a $(p_j - 1)$-clique in color j or a p_i-clique in some other color i. In the former case, x can be added to obtain a p_j-clique in color j, so either way the coloring has a monochromatic complete subgraph of the desired size in some color.

b) $R(p_1 + 1, \ldots, p_k + 1) \le \frac{(p_1+\cdots+p_k)!}{p_1!\cdots p_k!}$. We use induction on $\sum p_i$. If all $p_i = 0$, then $R(1, \ldots, 1) = 1 = (\sum 0)!/\prod 0!$. For $\sum p_i > 0$, let $r_i = R(p_1 + 1, \ldots, p_{i-1} + 1, p_i, p_{i+1} + 1, \ldots, p_k + 1)$. The induction hypothesis gives $r_i \le s_i$, where $s_i = p_i \left(\sum p_j - 1\right)!/\prod p_j!$. Since $k \ge 2$, we now have

$$R(p_1{+}1, \ldots, p_k{+}1) \le 2{-}k{+}\sum r_i \le \sum s_i = \frac{\sum p_i(\sum p_j - 1)!}{\prod p_j!} = \frac{(p_1 + \cdots + p_k)!}{p_1! \cdots p_k!}.$$

8.3.18. *a)* $r_k \le k(r_{k-1} - 1) + 2$, *where* $r_k = R_k(3; 2)$ (the minimum n such that k-coloring $E(K_n)$ forces a monochromatic triangle). Consider a k-coloring with no monochromatic triangle, and let x be some vertex. There are at most $r_{k-1} - 1$ neighbors of x along edges of the ith color, for each i. Otherwise, avoiding a monochromatic triangle (in any other color) within those vertices would force having at least one edge of color i, and its endpoints would form a triangle in color i with x. Thus $1 + k(r_{k-1} - 1)$ is an upper bound on the number of vertices for which it is possible to avoid a monochromatic triangle.

b) $r_k \le \lfloor k!e \rfloor + 1$. Proof by induction. Note that $r_2 = 6$, which satisfies the formula (any irrational number at least 2.5 could be used in place of e). By induction, $r_k \le k(r_{k-1} - 1) + 2 \le k \lfloor (k-1)!e \rfloor + 2$. Since e is irrational, $k!e$ cannot be an integer, so the 2 can be reduced to 1 when k is brought inside the $\lfloor\ \rfloor$.

8.3.19. ($\bullet$) Prove that $R_k(p; r + 1) \le r + k^M$, where $M = \binom{R_k(p;r)}{r}$.

8.3.20. *Off-diagonal Ramsey numbers.*

a) Let $h(n, p) = \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}}$. *For fixed n, If* $h(n, p) < 1$ *for some* $p \in (0, 1)$, *then* $R(k, l) > n$. *Furthermore,* $R(k, l) > n - h(n, p)$ *for all* $n \in \mathbb{N}$ *and* $p \in (0, 1)$. Produce a coloring of $E(K_n)$ at random, by letting each edge be red with probability p, independently. For any k-set, the probability that it induces a red complete graph in the resulting coloring is $p^{\binom{k}{2}}$. Since there are $\binom{n}{k}$ choices of k-sets, the linearity of expectation (see Section 8.5) implies that the expected number of red k-cliques is $\binom{n}{k} p^{\binom{k}{2}}$. Similarly, the expected number of blue l-cliques is $\binom{n}{l}(1 - p)^{\binom{l}{2}}$. Letting X be the random variable that counts the monochromatic cliques of threshold size, we have $\mathrm{E}(X) = h(n, p)$.

If $h(n, p) < 1$, then $\mathrm{E}(X) < 1$. This means that in some outcome of the experiment there are no such cliques. That is, there exists a 2-coloring of $E(K_n)$ proving that $R(k, l) > n$. It suffices to have any value of p in $(0, 1)$ with this property.

Similarly, always $R(k, l) > n - h(n, p)$. Since $\mathrm{E}(X) = h(n, p)$, in some outcome of the experiment $X \le h(n, p)$. Deleting one vertex from each bad clique in the resulting coloring yields with at least $n - h(n, p)$ vertices, showing that $R(k, l) > n - h(n, p)$.

b) $R(3, k) > k^{3/2 - o(1)}$. To obtain a lower bound from $R(3, k) > n - \binom{n}{3} p^3 - \binom{n}{k}(1 - p)^{\binom{k}{2}}$, we choose p and n in terms of k so that the subtracted terms are less than $n/2$ (constant factors won't matter). Using upper bounds on these terms, we have $R(3, k) > n(1 - \frac{1}{6}n^2p^3 - \frac{1}{k!}(ne^{-pk/2})^{k-1}$. The first term suggests setting $p = cn^{-2/3}$ (it suffices to make c a constant as small as 1). Since k may be large, we also want $e^{pk/2} > n$ (again we can make $e^{pk/2} = c'n$, with c' a constant as large as 1).

Letting $c = c' = 1$ and taking the natural logarithm, we want to choose n so that $k = 2n^{2/3} \ln n$. We want $n^{2/3}$ to cancel $2 \ln n$ and leave k, so we set $n = (\frac{k}{3 \ln k})^{3/2}$ to obtain $2n^{2/3} \ln n = k(1 - c'' \frac{\ln \ln k}{\ln k}$.

This choice of n (and p) yields $R(3, k) > n(1 - \frac{1}{6} - \frac{1}{k!})$. Since $(\ln k)^{-1} = -\ln \ln k/ \ln k$, we can write this as

$$R(3, k) > \frac{2n}{3} = \frac{2}{9\sqrt{3}} k^{(3/2)(1 - \ln\ln k/\ln k)} = k^{3/2 - o(1)}.$$

We could optimize the argument by letting c depend on n and c' depend on k, but this won't improve the exponent.

The first part of (a) yields no useful lower bound on $R(3,k)$. To obtain p such that $\binom{n}{3}p^3 < 1$, we need $p < c/n$. We also need $e^{p(k-1)/2} > n$, which leads to $k > 1 + (2/c)n\ln n$. Unfortunately, this works only when n is smaller than k, and we already know trivially that $R(3,k) > k$.

c) Use part (a) to obtain a lower bound for $R_k(q)$. We have k colors with thresholds all equal to q. We give each edge the ith color with probability $1/k$, for each i, independently, and let X be the number of monochromatic q-cliques. Thus $\mathrm{E}(X) = k\binom{n}{k}k^{-\binom{q}{2}}$. By deleting one vertex of each such clique in an outcome with at most the expected number, we obtain $R_k(q) > n - \mathrm{E}(X)$.

Thus $n - \mathrm{E}(X)$ is a lower bound on $R_k(q)$, for each n. Since $\binom{n}{k} < \left(\frac{ne}{k}\right)^k$, we also have the simpler $n - k\left(\frac{ne}{k}\right)^k k^{-\binom{q}{2}}$ as a lower bound. We seek n to maximize this bound. Differentiating suggests choosing n to satisfy $1 = k\frac{e}{k}\left(\frac{ne}{k}\right)^{k-1}k^{-\binom{q}{2}}$, or $n \approx (k/e)k^{(\binom{q}{2}-1)/(k-1)}$.

At this value of n, $\mathrm{E}(X)$ is near 1, and our lower bound is just a bit less than $(k/e)k^{(\binom{q}{2}-1)/(k-1)}$.

8.3.21. *$R(K_{1,m},K_{1,n})$ is $m+n$ unless m and n are both even, in which case it is $m+n-1$.* The value of $R(K_{1,m},K_{1,n})$ is the least N such that, for every N-vertex graph G, either $\Delta(G) \ge m$ or $\Delta(\overline{G}) \ge n$. By the pigeonhole principle, this property occurs when $N = m+n$, because at every vertex there must be m neighbors or n nonneighbors. This is the least such N if and only if there exists an $(m-1)$-regular graph with $m+n-1$ vertices. If m or $m+n$ is odd, then such a graph exists. If m and n are both even, then we are seeking a regular graph of odd order and degree, which does not exist.

8.3.22. *If T is a tree with m vertices, and $m-1$ divides $n-1$, then $R(T,K_{1,n}) = m+n-1$.* With $m+n-1$ vertices, if every vertex in the blue graph has degree at most $n-1$, then every vertex in the red graph has degree at least $m-1$. This implies that the red graph contains every tree with m vertices, by Proposition 2.1.8(easily proved by deleting a leaf and using induction).

For the lower bound, since $m-1$ divides $n-1$, a set of $m+n-2$ vertices splits into sets of size $m-1$. The components of the red graph are K_{m-1}, so there is no red tree with m vertices. Each vertex has $m-2$ red neighbors, so it has $n-1$ blue neighbors, and there is no blue star with n edges.

8.3.23. *$(m-1)(n-1)+1$ is the minimum value of p such that every 2-coloring of $E(K_p)$ in which the red graph is transitively orientable contains a red m-clique or a blue n-clique.* More than $(m-1)(n-1)$ vertices are needed, since $(n-1)K_{m-1}$ is transitively orientable.

Proof 1 (perfect graphs). The red graph is a comparability graph and hence is perfect. If it has no m-clique, then it has a proper $(m-1)$-coloring. If it has more than $(m-1)(n-1)$ vertices, then by the pigeonhole principle some color class has (at least) n vertices. This class yields an n-clique in the blue graph.

Proof 2 (induction on m). Immediate for $m=1$. For $m>1$, consider such a coloring with $(m-1)(n-1)+1$ vertices. Let F be a transitive orientation of the red graph, with sources S. The sources induce a blue clique, so $|S| \le n-1$ if the claim does not hold. However, $F - S$ is then a transitive orientation of a graph G' with more than $(m-2)(n-1)$ vertics. By the induction hypothesis, G has a blue n-clique or a red $m-1$-clique Q. A transitive orientation of such a clique Q has a unique source u. Since u is not a source of F, there is an edge from some $v \in S$ to u, and then transitivity guarantees that v can be added to Q to obtain a red m-clique.

8.3.24. *If T is a tree with m vertices, then $R(T,K_{n_1},\dots,K_{n_k}) = (m-1)(R(n_1,\dots,n_k)-1)+1$.* Let the colors be $0,\dots,k$ corresponding to T and $K_{n_1},\dots,K_{n_k}$, respectively.

For the lower bound, begin with a k-coloring of K_q points that has no copy of K_{n_i} in color i for any i, for $1 \le i \le k$. Replace each vertex by a complete graph of order $m-1$ whose edges all get color 0. Each edge in the original graph expands into a copy of $K_{m-1,m-1}$. Give all edges in this subgraph the same color that its original edge had. This coloring has no m-tree in color 0, because the components of the graph in color 0 have only $m-1$ vertices. It meets no clique quota in any other color, because such a monochromatic clique can be collapsed to a monochromatic clique of that size in the original coloring. This holds because the vertices must come from copies of distinct points in the original point set, since copies of the same point are joined by edges of color 0.

For the upper bound, let f be a $(k+1)$-coloring of $E(K_{(m-1)q+1})$, where $q = R(n_1,\dots,n_k)-1$. Define f' on the same edges by letting an edge be red if it gets color 0 in f and blue if it gets a nonzero color in f. By Chvatál's Theorem for $R(T,Kn)$ (Theorem 8.3.14), this coloring has a red T, in which case we are done, or a blue $(q+1)$-clique. In the latter case, we return to f, restricted to this set of $q+1$ vertices. The definition of the Ramsey number says that f has a monochromatic copy of K_{n_i} in color i on these vertices, for some $i \in [k]$.

8.3.25. $R(C_4,C_4) = 6$.

Claim 1: A 2-colored complete graph with at least 6 vertices containing a monochromatic 5-cycle or 6-cycle also contains a monochromatic 4-cycle. Given a red 5-cycle, avoiding a red 4-cycle makes every chord blue, so the coloring on these 5 points consists of 5-cycles in each color. For a vertex outside these five, its edges to these have at least three in one color, which we may assume by symmetry is red. These three points include two that are nonadjacent on the red 5-cycle. The 2-edge path between them on that cycle, together with the edges to the extra vertex, yield a red 4-cycle.

Given a red 6-cycle, the chords joining opposite vertices on the cycle must be blue, or we already have a red 4-cycle. The remaining chords cannot all be blue, since this yields a blue 4-cycle. consisting entirely of chords. However, if one of them is red, then we have a red 5-cycle and can apply the argument above.

Claim 2: $R(P_4, C_4) = 5$. With four vertices, the color classes may be a triangle and a claw. With five vertices, if there is no monochromatic triangle, then every vertex has two incident vertices of each color. Being 2-regular, each color class is a disjoint union of cycles, which with five vertices can only be C_5. However, C_5 contains P_4.

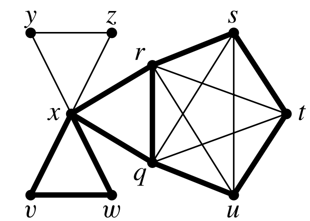
Hence we may assume a monochromatic triangle $[x, y, z]$. If it is red, then any additional red edge incident to the triangle yields a red P_4; if there is no such red edge, then the blue edges from $[x, y, z]$ to the remaining vertices u and v contain a blue 4-cycle. If $[x, y, z]$ is blue, then two blue edges from one of $\{u, v\}$ to $\{x, y, z\}$ would complete a blue 4-cycle. By symmetry, we may thus choose x with no blue edges to $\{u, v\}$, so the path $\langle u, x, v\rangle$ is red. Now vy or vz is red to complete a red P_4.

Claim 3: $R(C_4, C_4) = 6$. Since the 5-cycle and its complement contain no monochromatic C_4, $R(C_4, C_4) \ge 6$. For the upper bound, consider a 2-coloring of $E(K_6)$. Since $R(P_4, C_4) = 5$ by Claim 2, we may assume that the coloring has a red P_4, with vertices u, v, w, x in order. If both of the remaining vertices, y and z, have both edges to $\{u, x\}$ blue, then $[x, y, u, z]$ is a blue C_4. So, we may assume that one of the edges is red and extend the path to a red $P5$, such as u, v, w, x, y. The chords uy, vy, ux must all be blue, else we have a red 4-cycle or 5-cycle, which suffices, by Claim 1. Now consider z. The edges zv and zx cannot be both red or both blue, else $[z, v, w, x]$ is a red 4-cycle or $[z, w, y, u, x]$ is a blue 5-cycle. Hence we may assume by symmetry that zv is blue and zx is red. Now we cannot color zu.

8.3.26. $R(2K_3, 2K_3) = 10$. The lower bound is provided by the construction in Theorem 8.3.15: the red graph is $K_{1,3}+K_5$. For the upper bound, consider a 2-coloring of $E(K_{10})$. Any six vertices contain a monochromatic triangle, and the seven vertices outside that triangle yield another monochromatic triangle. We are finished if they have the same color; if not, then the edges joining them are used to collapse the configuration to a "bow tie" as in the proof of Theorem 8.3.15. Thus we have vertices $\{v, w, x, y, z\}$ such that $[x, v, w]$ is a bold triangle and $[x, y, z]$ is a solid triangle.

To avoid a monochromatic triangle on the remaining five vertices, we must have a bold 5-cycle $[q, r, s, t, u]$ and a solid 5-cycle $[q, s, u, r, t]$, as

shown below.



Vertex x must have three neighbors of the same color in $\{q, r, s, t, u\}$, of which two must be adjacent on the cycle in that set having that color. By symmetry, we may thus assume that $[x, q, r]$ is a bold triangle. If any edge e with endpoints in $\{v, w\}$ and $\{q, r\}$ is solid, then the edges joining the endpoint of e in $\{v, w\}$ to the solid neighbors in $\{s, t, u\}$ of the other endpoint of e (for example, vt and vu when $e = vr$) must be bold to avoid disjoint solid triangles, but this makes disjoint bold triangles ($[x, q, r]$ and $[v, t, u]$ in the example $e = vr$. Hence the edges joining $\{5, 6\}$ to $\{1, 2\}$ are all bold.

Now consider the edges vs and vu. If both are solid, then $[x, y, z]$ and $[v, s, u]$ are disjoint solid triangles. Hence one is bold; we still have symmetry and may assume that it is vu. Now we have disjoint solid triangles $[v, q, u]$ and $[x, w, r]$. Hence a monochromatic $2K_2$ is forced.

8.3.27. $R(mK_2, mK_2) = 3m - 1$. For the lower bound, let the red graph be $K_{2m-1} + \overline{K}_{m-1}$. Every red edge has both endpoints in the $(2m-1)$-clique, so there cannot be m independent red edges. The complementary blue graph is the join of the complete graph on $m-1$ vertices with an independent set on $2m-1$ vertices $(\overline{K}_{2m-1} \vee K_{m-1}$. Every edge has at least one endpoint in the $(m-1)$-clique, so again there cannot be m independent edges.

For the upper bound, we use induction. Note that $R(K_2, K_2) = 2$. For $m > 1$, consider an arbitrary 2-coloring of the edges of K_{3m-1}. There must be incident edges of differing colors, else the entire clique gets one color and has enough points to contain mK_2. Remove the three points hit by these two incident edges of different color, and apply the induction hypothesis. To the resulting monochromatic $(m-1)K_2$, add the edge of the appropriate color from the deleted three vertices.

8.3.28. *If G_i is a graph of order p_i, for $1 \le i \le k$, then $R(m_1G_1, \ldots, m_kG_k) \le \sum(m_i - 1)p_i + R(G_1, \ldots, G_k)$.* Each m_iG_i is the disjoint union of m_i copies of G_i. Given a 2-coloring with the specified number of vertices, we iteratively extract disjoint monochromatic copies of these graphs in the specified colors. As long as $R(G_1, \ldots, G_k)$ vertices remain that have not been touched by the extracted graphs, we can find another monochromatic G_i in color i for some i. If ever we obtain m_i copies of G_i, then we are done. Otherwise, we have obtained at most $m_i - 1$ copies of each G_i, so we have eliminated at most $\sum(m_i - 1)p_i$ vertices from consideration. In this case at least $R(G_1, \dots, G_k)$ vertices remain, and we can continue. Hence the process terminates only by finding $m_i G_i$ in color i, for some i.

8.3.29. *Graphs with n vertices having no clique or independent set with size as large as $2^{c\sqrt{\log n \log\log n}}$ yield a lower bound for $R(p,p)$ in terms of p that grows faster than every polynomial in p but slower than every exponential in p.* The existence of such a graph implies that $R(p,p) > n$, where $p = 2^{c\sqrt{\log n \log\log n}}$. To find the behavior of the lower bound, we need to solve this equation for n in terms of p, but we do not need the complete solution to answer the question.

Taking logs and squaring both sides yields $c'(\log p)^2 = \log n \log\log n$, where $c' = 1/(c\log 2)^2$. To study the form of the function, we express n in terms of p and a parameter t.

First suppose that $n \le p^t$. In this case $c'(\log p)^2 \le (t\log p)(\log t + \log\log p)$. If t is bounded, then this inequality is false. Hence n cannot be bounded by any polynomial function of p.

Now suppose that $n \ge t^p$. In this case $c'(\log p)^2 \ge (p\log t)(\log p + \log\log t)$. Again, this is impossible when t is a constant. Hence n cannot grow faster than any exponential function of p.

8.3.30. *If G is an n-vertex graph such that $\alpha'(\overline{G}) = k$, then $R(P_3, G) = \max\{n, 2n-2k-1\}$.* We seek the minimum r such that red/blue-colorings of $E(K_r)$ yield a red P_3 or a blue G.

If $r < n$, then we color all of $E(K_r)$ blue yields $R(P_3, G) > r$, so n is a lower bound. If $r = 2(n-k-1)$, then we color $E(K_r)$ with a perfect matching in red and the rest in blue. The red matching avoids P_3, and every set of n vertices contains at least $k+1$ pairs from the red matching (if it has s vertices whose mates are omitted and t matched pairs, then $s + 2t = n$ and $s + t \le n-k-1$, so $n + t \le n-k-1$). There is no blue G on such a set of vertices, since $\overline{G}$ has no matching of size $k+1$.

For the upper bound, consider a 2-coloring of $E(K_r)$ with $r = \max\{n, 2n-2k-1\}$. If there is no red P_3, then the red graph is restricted to a matching. Thus all edges are blue except for 1) at most $n-k-1$ pairwise disjoint edges and an isolated vertex if $k < n/2$, or 2) at most $n-k$ pairwise disjoint edges if $k = n/2$. In either case, we choose $n-k$ vertices that span no red edges, and then we augment these with any k other vertices. The coloring induced by these n vertices has at most k red edges, and the red edges are pairwise disjoint. Since $\alpha'(\overline{G}) = k$, the graph G can be mapped into an n-vertex complete graph to avoid any matching of size k, and hence the blue graph on these n vertices contains G.

8.3.31. *If r and s are natural numbers with $r + s \not\equiv 0 \pmod 4$, then every 2-coloring of $E(K_{r,s})$ has a monochromatic connected graph with at least $\lceil r/2\rceil + \lceil s/2\rceil$ vertices.* We prove a slightly stronger result: If the edges of $K_{r,s}$ are 2-colored, then there is a monochromatic connected subgraph with at least half the vertices from each side, with one side exceeding half unless each color forms $2K_{r/2,s/2}$.

To prove this, first delete a vertex or two (if necessary) to leave an odd number of vertices on each side. Now let X and Y be the partite sets, and consider an arbitrary edge-coloring. Give each vertex of X the color occuring on a majority of its incident edges. Let blue be the color thus assigned to a majority of the vertices in X. Any two blue vertices in X have a common neighbor in Y along blue edges. Hence the blue vertices in X and their incident blue edges form a connected subgraph; it has more than half of X (by pigeonhole choice of blue) and more than half of Y (by the blue neighbors of each vertex of X).

With the deleted vertex of each original partite set of even size replaced, we have obtained an connected monochromatic subgraph with at least $(r+s)/2$ vertices. Equality requires that all $r/2$ vertices of this subgraph in X have the same $s/2$ neighbors in Y.in Y. Equality forbids additional incident blue edges, so the red edges incident to these vertices form $2K_{r/2,s/2}$. To avoid having a spanning connected red subgraph, all edges not incident to our original blue subgraph must be red, forming another red copy of $K_{r/2,s/2}$.

Every 3-coloring of $E(K_{r+s})$ contains a monochromatic connected subgraph with more than $(r+s)/2$ vertices, except maybe when $r+s \equiv 0 \pmod 4$. Given a 3-coloring of $E(K_n)$, let G be a maximal monochromatic connected subgraph in color 0; let r be its order, with $n = r+s$. On edges joining $V(G)$ and $V(K_n) - V(G)$, only colors 1 and 2 are used. The preceding argument guarantees a monochromatic connected subgraph with more than half the vertices unless r and s are even and the subgraphs between $V(G)$ and $V(K_n) - V(G)$ are $2K_{r/2,s/2}$ in colors 1 and 2.

In the exceptional case, these subgraphs in color 1 (red) and color 2 (blue) partition $V(G)$ into A_0 and A_1 and $V(K_n) - V(G)$ into A_2 and A_3 so that all edges joining A_0 to A_2 or joining A_1 to A_3 are red and all edges joining A_0 to A_3 or joining A_1 to A_2 are blue. If any edge joining A_0 to A_1 or A_2 to A_3 does not have color 0, then we have a monochromatic n-vertex connected subgraph. Otherwise, we have monochromatic connected subgraphs in color 0 with r and s vertices. Hence we have the desired configuration unless $r = s = n/2$, which now implies $n \equiv 0 \pmod 4$.

Furthermore, we have shown that when $n \equiv 0 \pmod 4$ the claim fails only for the following coloring: given the three pairings of four sets A_0, A_1, A_2, A_3 of size $n/4$, assign color i to all edges between groups paired in the ith pairing. Any coloring within each graph can be used; all the monochromatic connected subgraphs have exactly $n/2$ vertices.

8.3.32. *Forcing 4-cycles.*

a) If $\sum_{v\in V(G)}\binom{d(v)}{2} > \binom{n(G)}{2}$, then G contains a 4-cycle. The sum $\sum_{v\in V(G)}\binom{d(v)}{2}$ counts the triples u, v, w such that v is a common neighbor of u and w. If G has no 4-cycle, then every pair of vertices has at most one common neighbor.

b) If $e(G) > \frac{n(G)}{4}(1+\sqrt{4n(G)-3})$, then G contains a 4-cycle. Since $\binom{x}{2}$ is a convex function of x, the minimum of $\sum_{v\in V(G)}\binom{d(v)}{2}$ for fixed $\sum d(v)$ occurs numerically when the values for $d(v)$ are equal (even though this may not be realized by a graph). Since $\sum d(v) = 2e(G)$, we conclude that $\sum_{v\in V(G)}\binom{d(v)}{2} \ge n(G)\binom{2e(G)/n(G)}{2}$. If $e(G)(\frac{2e(G)-n(G)}{n(G)} > \binom{n(G)}{2}$, then the condition of part (a) holds. This inequality reduces to the stated condition.

c) $R_k(C_4) \le k^2+k+2$. If $n > k^2+k+2$, then $\binom{n}{2} > k\frac{n}{4}(1+\sqrt{4n-3})$. Hence some color class is as large as $\frac{n}{4}(1+\sqrt{4n-3})$, and the result of part (b) applies.

8.3.33. *$R(C_m, K_{1,n}) = \max\{m, 2n+1\}$, except possibly if m is even and does not exceed $2n$.* For the lower bound, first consider $m \ge 2n+1$. Form a red clique on $m-1$ vertices; it has no red C_m and no blue edge, hence no vertex with blue degree n. If $m < 2n+1$ and m is odd, form two disjoint blue cliques on n vertices, and let all the edges between them be red. There is no vertex with blue degree n, and all the red cycles have even length. The case with $m < 2n+1$ and m even is unsettled, although the argument for the upper bound is still valid.

For the upper bound, first consider $m \ge 2n+1$ and any 2-coloring of $E(K_m)$. If no vertex has blue degree at least n, then the red degree of every vertex is at least $m-n$, which exceeds $m/2$ since $n < m/2$. We now invoke Bondy's Theorem, stating that if $x \not\leftrightarrow y$ implies $d(x)+d(y) \ge n(G)$, then G is a complete bipartite graph with equal-sized partite sets or G has a cycle of each length from 3 to $n(G)$. The red graph satisfies this hypothesis, so in either case it is Hamiltonian, which yields a red C_m.

If $m < 2n+1$, consider a 2-coloring of $E(K_{2n+1})$. Again having no vertex of blue degree at least n implies that the minimum red degree is at least half the number of vertices. Since $2n+1$ is odd, Bondy's Theorem now yields red cycles of all lengths, including length m.

8.3.34. *Every 2-coloring of $E(K_n)$ contains a monochromatic Hamiltonian cycle or a Hamiltonian cycle consisting of two monochromatic paths.* This is immediate for $n = 3$; we proceed by induction. If $n > 3$, consider the coloring on $E(K_n - v)$. If this has a monochromatic cycle, then we can replace an arbitrary edge of the cycle by the edges from its endpoints to v.

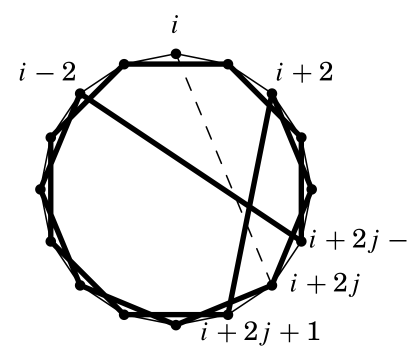
If it has two monochromatic paths whose union is a cycle, then let x, y, z be three consecutive vertices on the cycle with xy red and yz blue. We may assume that yv is red. Now the cycle obtained by replacing yz with $\langle y, v, z\rangle$ has the desired property.

8.3.35. *Ramsey numbers for cycles.*

a) A 2-coloring of $E(K_n)$ that contains a monochromatic C_{2k+1} for some $k \ge 3$ also contains a monochromatic C_{2k}. Let C be a red $(2k+1)$-cycle, with vertices $v_0, \dots, v_{2k}$ in order. If there is no monochromatic $2k$-cycle, then each x_ix_{i+2} is blue, which yields a blue $2k+1$-cycle C' and implies that each x_ix_{i+4} is red, where indices are mod $2k$. For each i, consider the cycle obtained from C by replacing the path $\langle x_i, x_{i+1}, \dots, x_{i+5}\rangle$ with $\langle x_i, x_{i+3}, x_{i+2}, x_{i+1}, x_{i+5}\rangle$. Skipping x_{i+4}, it has length $2k$, and all edges except x_ix_{i+3} are red; hence x_ix_{i+3} is blue. Now we can replace the path $\langle x_i, x_{i+2}, x_{i+4}, x_{i+6}\rangle$ on C' with $\langle x_i, x_{i+3}, x_{i+6}$ from C to obtain a blue $2k$-cycle. Note that this requires $k \ge 3$.

b) A 2-coloring of $E(K_n)$ that contains a monochromatic C_{2k} for some $k \ge 3$ also contains a monochromatic C_{2k-1} or $2K_k$. Let C be a red $2k$-cycle, with vertices $v_0, \dots, v_{2k-1}$ in order. We prove that if there is no monochromatic $(2k-1)$-cycle, then all edges of the form x_ix_{i+2j} are blue. This suffices, since it implies that the odd-indexed vertices and even-indexed vertices along C both induce blue copies of K_k.

If there is no monochromatic $(2k-1)$-cycle, then each x_ix_{i+2} is blue, so we may assume that $2 \le j \le k-2$. Replacing $x_{i+2j}x_{i+2j+1}$ and $\langle x_ix_{i+1}x_{i+2}\rangle$ on C with x_ix_{i+2j} and $x_{i+2}x_{i+2j+1}$ yields a cycle of length $2k-1$ in which every edge except the two new edges is red. If x_ix_{i+2j} is red, then $x_{i+2}x_{i+2j+1}$ must therefore be blue to avoid a red $(2k-1)$-cycle. (In the figure below, bold means blue and solid means red.) Similarly, replacing $x_{i+2j}x_{i_2j-1}$ and $\langle x_ix_{i-1}x_{i-2}\rangle$ on C with x_ix_{i+2j} and $x_{i-2}x_{i+2j-1}$ forces $x_{i-2}x_{i+2j-1}$ to be blue. Now replacing $\{x_{i-2}x_i, x_ix_{i+2}, x_{i+2j-1}x_{i+2j+1}\}$ with $\{x_{i-2}x_{i+2j-1}, x_{i+2}x_{i+2j+1}\}$ in the set of edges of the form $\{x_rx_{r+2}\}$ yields a blue $(2k-1)$-cycle avoiding x_i. Hence if there is no monochromatic $(2k-1)$-cycle, then x_ix_{i+2j} is blue.



c) If $m \geq 5$, then $R(C_m, C_m) \leq 2m - 1$. (Note that $R(C_3, C_3) = R(C_4, C_4) = 6$; Exercise 8.3.25.) Consider a 2-coloring of $E(K_{2m-1})$. One color has at least half the edges; we may assume it is red. Erdős–Gallai [1959] (Theorem 8.4.35) proved that $e(G) > \frac{1}{2}(m-1)(n(G)-1)$ forces a cycle of length at least m in G. Since $\frac{1}{2}\binom{2m-1}{2} > \frac{1}{2}(m-1)[2m-2]$, we conclude that the coloring has a red cycle of length at least m. By parts (a) and (b), there is also a red m-cycle or two disjoint blue complete graphs of equal order exceeding $m/2$; we may assume the latter.

Let Q_1 and Q_2 be disjoint sets inducing blue complete graphs, each of order exceeding $m/2$, chosen to maximize $|Q_1 \cup Q_2|$. If two nonincident blue edges join Q_1 and Q_2, then we can take $P_{\lceil m/2 \rceil}$ from Q_1 and $P_{\lfloor m/2 \rfloor}$ from Q_2 to form a blue m-cycle with these edges. Hence all blue edges joining Q_1 and Q_2 are incident to a single vertex x, which we may assume is in Q_1. If m is even, then we can now take $m/2$ vertices from each Q_i, avoiding x, and form a red m-cycle using the edges between them.

We may therefore assume that m is odd. Let $T = \overline{Q_1 \cup Q_2}$ and $S = Q_1 - \{x\}$. If there is no blue m-cycle within Q_1 or Q_2, then $|Q_1 \cup Q_2| < 2m - 1$, and $T \neq \varnothing$. For $v \in T$, if all edges from v to Q_1 or to Q_2 are blue, then we contradict the maximality of $|Q_1 \cup Q_2|$. Hence there are red edges from v to both Q_1 and Q_2.

Since all the edges joining Q_1 and Q_2 are red except those incident to x, we can complete a red m-cycle through v and alternating between Q_1 and Q_2 unless vx is the only red edge from v to Q_1 and all edges from x to Q_2 are blue (except possibly one incident to the only neighbor of v in Q_2 along a red edge). This is true for all $v \in T$, so every edge from x to T is red and all of $[S, T]$ is blue. Also let R be the subset of Q_2 whose edges to x are blue; R is all of Q_2 except possibly one vertex.

Since $\lceil m/2 \rceil \geq 3$, there are at least two blue edges from x to R. Hence there is a blue cycle spanning $Q_2 \cup \{x\}$, and we may assume that $|Q_2 \cup \{x\}| \leq m - 1$. Therefore $|S \cup T| \geq (2m-1) - (m-1) = m$.

We now have a blue m-cycle in the graph induced by $S \cup T$ if $|S| \geq \lceil m/2 \rceil$, so we may assume that $|S| = (m-1)/2$ and $|T| \geq (m+1)/2 \geq 3$. If an edge within T is blue, then we complete a blue cycle by using it and otherwise alternating between S and T. Hence we may assume that all edges induced by T are red.

If there is a blue edge in $[T, R]$, then we can form a blue cycle by following it with any portion of R, then x, then any portion of S. The length is any value from 4 to at least $|Q_1 \cup Q_2|$, which exceeds m. Hence we may assume that all of $[T, R]$ is red. Now we can form a red m-cycle by using a path alternating between S and R using $(m-3)/2$ vertices of S, $(m-1)/2$ vertices of R, and two vertices of T.

8.3.36. *The Ramsey multiplicity of K_3 is 2*, where the *Ramsey multiplicity* of G is the minimum number of monochromatic copies of G in a 2-coloring of $E(K_{R(G,G)})$. To color $E(K_6)$ with only two monochromatic triangles, let the red graph be $K_{3,3}$, which is triangle-free. The complementary graph is $2K_3$, with two triangles.

Now we show that every coloring has at least two monochromatic triangles. Since $R(3,3) = 6$, there is at least one monochromatic triangle T, say in red. If we delete one vertex of T, then there remains a monochromatic triangle on the remaining five vertices unless the color classes on that subgraph are complementary 5-cycles. Let C be the red 5-cycle, and let z be the deleted vertex. To form T, we have edges from z to consecutive vertices on C, which we call x and y. Let u, x, y, v be the consecutive vertices on C including the edge xy. A red edge from z to u or v completes another red triangle, but if uz and vz are both blue they complete a blue triangle with uv.

8.3.37. *Each point in a triangular region has a unique expression as a convex combination of the vertices of the triangle.* We observe first that each point on a segment has a unique expression as a convex combination of the endpoints. Now, given a point x inside the triangle with corners u, v, w, let y be the point at which the ray from u through x reaches the opposite side. Now $y = \lambda v + (1-\lambda)w)$, for a unique λ, and $x = \mu y + (1-\mu)u$, for a unique μ. Hence $x = (1-\mu)u + (\lambda\mu)v + \mu(1-\lambda)w$. The coefficients are uniquely determined in terms of λ and μ, and these constants also are uniquely determined by x and the corners.

8.3.38. *Sperner's Lemma in higher dimensions. In a proper labeling of a simplicial subdivision of a k-dimensional simplex, there is a cell receiving all $k+1$ labels*, where "proper labeling" is a labeling such that label i does not appear at any vertex on the ith outer face.

We prove the stronger result, by induction on k, that there are an odd number of completely labeled cells. When $k = 1$, we have a 0, 1-labeling of a path segment with 0 and 1 on the ends, and there must be an odd number of switches between 0 and 1 along the path.

For $k > 1$, define a graph G with a vertex for each cell plus one vertex v for the outside region. Two vertices of G are adjacent if the corresponding regions share a $(k-1)$-dimensional face with corners having labels $0, \ldots, k-1$. If the vertex for a cell is nonisolated, then the cell has all these k labels among its $k+1$ corners. If it repeats one of the labels, then it has two incident edges in G. Otherwise, it is a completely labeled cell and has degree 1.

Hence the only cells with odd degree are the completely labeled cells. To prove that there are an odd number of them, it suffices to prove that the vertex v also has odd degree. A cell having a $(k-1)$ dimensional face on the ith outside face cannot have label i on it. Therefore, having an edge to v happens only through the $(k+1)$th face, where label $k+1$ is forbidden.

This face is a simplicial subdivision of a $(k-1)$-dimensional simplex. The labeling is proper on this face, as it inherits the needed properties from the full labeling (think of the edges of a triangle in the 2-dimensional case). By the induction hypothesis, this lower-dimensional labeling has an odd number of completely labeled cells. Hence the full-dimensional labeling has an odd number of cells with edges to v.

8.3.39. *The badwidths of P_n, K_n, and C_n are 1, $n-1$, and 2, respectively.* A nontrivial graph has bandwidth 1 if and only if its vertices can be ordered so that no nonconsecutive vertices are adjacent, which means that its components are paths. In ordering, the vertices of K_n, the first and last vertices are adjacent. Since C_n is not a path, its bandwidth is at least 2. To achieve this, number the vertices around the cycle $\ldots, 5, 3, 1, 2, 4, \ldots$ in order, reaching to $n-1$ in one direction and to n in the other direction.

8.3.40. *The bandwidth of $K_{n_1,\ldots,n_k}$ is $n-1-\lfloor n'/2 \rfloor$, where $n=\sum_{i=1}^k n_i$ and* $n' = \max_i n_i$. Consider an optimal numbering. If the vertices given labels 1 and n come from different partite sets, then the bandwidth is $n-1$. If they come from the same partite set, with c vertices of this partite set at the beginning and c' at the end of the labeling, then the bandwidth is at least $\max n-c, n-c'$. To minimize this lower bound, we split the largest partite set between the front and back. The lower bound becomes $n-1-\lfloor n'/2 \rfloor$. Also, splitting the largest partite set in this way achieves equality for any ordering of the remaining vertices in the remaining middle positions.

8.3.41. *The bandwidth of a tree with k leaves is at most $\lceil k/2 \rceil$.* Let $m-\lceil k/2 \rceil$. We use the fact that every tree with k leaves is the union of m pairwise intersecting paths (Exercise 2.1.40). We repeat the proof: Let T be a tree with k leaves. By pairing leaves arbitrarily, we form a set of m paths that together cover the leaves. Among all such sets of paths, choose one with maximum total length; we claim it has the desired properties. If some pair of paths is disjoint, say an x,y-path P and a u,v-path Q, consider the path R in T from $V(P)$ to $V(Q)$. Replace P and Q with the x,u-path and the y,v-path in T. The new paths still cover the leaves, and the total length has increased by twice the length of R. If some edge e of T is omitted by the longest covering set of paths, then consider the two components of $T-e$. Each contains a leaf of T, so each contains at least one path in the set. Again making the switch increases the total length.

To prove the upper bound, we provide an injective integer embedding in which the difference along every edge is at most m; the set of labels need not be consecutive. Let $P_0, \ldots, P_{m-1}$ be a set of pairwise-intersecting paths with union T, and let $T_j = \bigcup_{i=0}^j P_i$. Because the paths are pairwise intersecting, each T_j is connected. Because T_{j-1} is connected and T_j contains no cycle, a traversal of P_j cannot leave T_{j-1} and then return to it.

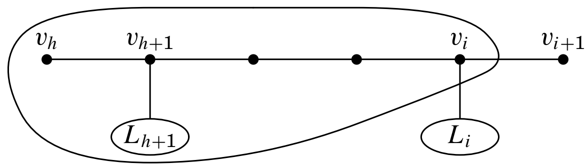
First assign successive multiples of m to the vertices along P_0. For $j>0$, suppose that the vertices of T_{j-1} have received labels congruent to $1, \ldots, j-1$ modulo m so that edges have dilation at most m. We use labels congruent to j on vertices of $V(P_j) - V(T_{j-1})$. Let u, v be the vertices of $V(P_j) \cap V(T_{j-1})$ that are closest to the two ends of P_j (these may be equal). By symmetry, we may assume that $f(u) \le f(v)$. Let a be the largest integer less than $f(u)$ congruent to $j \pmod m$, and let b be the smallest integer greater than $f(v)$ congruent to $j \pmod m$. From the neighbor of u [or v] out to the corresponding leaf of P_j, assign the label $a-(i-1)m$ [or $b+(i-1)m$] to the ith vertex of $V(P_j) - V(T_{j-1})$ encountered. (If k is odd, then for one value of j, one of these subpaths is empty). The new labels are in the new congruence class, and the newly-included edges have difference at most m.

8.3.42. *If G is a caterpillar with $\left\lceil \frac{n(H)-1}{\operatorname{diam} H} \right\rceil \le m$ for all $H \subseteq G$, then $B(G) \le m$.* (Note that the least such m is a lower bound, so equality with hold.)

Let P be the spine of G, having vertices $\langle v_0, \ldots, v_p \rangle$ in order, where v_0 and v_p are leaves. Assign the number im to v_i for $0 \le i \le p$. It suffices to show that this allows us to assign numbers to the remaining vertices so that all leaf neighbors of v_i receive numbers between $(i-1)m$ and $(i+1)m$. We can then compress the numbering to eliminate gaps without increasing any edge difference.

Let $L_i = N(v_i) - V(P)$, and let $l_i = |L_i|$. For $1 \le i \le k-1$, iteratively label L_i as follows. Use $\min\{l_i, c\}$ labels between $(i-1)m$ and im, where c is the number of labels between $(i-1)m$ and im not already assigned to L_{i-1}. If $l_i > c$, give the remaining vertices $l_i - c$ labels starting with $im+1$.

We show that this works by proving that $l_i - c \le m-1$ at step i. For $j<i$, the algorithm has assigned a label above jm to some vertex of L_j only if it has assigned all labels between $(j-1)m$ and jm. At step i, let h be the least index such that all labels between hm and $(i-1)m$ have been assigned. We have $h \le i-1$; equality is possible. Since the interval between $(h-1)m$ and hm is not "full", no label above hm is assigned to L_h. Let H be the subgraph of G induced by $\{v_{h+1}, \ldots, v_i\}$ and their neighbors; note that $\operatorname{diam} H = i+1-h$. Between hm and im, $V(H)$ has received $n(H)-1-l_i$ labels for the vertices circled in the figure below. Hence $c = (i-h)m+1-[n(H)-1-l_i]$. By the local density computation, $n(H)-1 \le (i+1-h)m$. Thus $l_i - c = n(H)-1-(i-h)m-1 \le m-1$, as desired.



8.3.43. *Bandwidth of grids.*

a) Local density bound for $P_m \square P_n$. We consider only the bound that comes from the subgraph $P_n \square P_n$ and omit some details of that.

It suffices to consider induced subgraphs; adding edges cannot increase diameter. For a subgraph H with diameter $2k$, let u and v be vertices of H at distance $2k$ (the odd case is similar). Let w be a vertex halfway along a shortest u,v-path. Any vertex at distance more than k from w will have distance more than $2k$ from u or v, by the nature of the grid (details omitted). Hence we get the best lower bound by including all the vertices in $P_n \square P_n$ that are within distance k of w. For $k \le (n-1)/2$, the number of vertices in this subgraph is $\sum_{i=1}^{k+1}(2i-1) + \sum_{i=1}^{k}(2i-1)$. This equals $2k^2+2k+1$. Subtracting 1 and dividing by $2k$ yields $B(P_n \square P_n) \ge k+1$.

When k is larger than $(n-1)/2$, the full set we have described does not fit inside the n-by-n grid. We must subtract $4\sum_{i=1}^{k-(n-1)/2}(2i-1)$ vertices after putting w in the center of the grid. The largest subgraph now has $2k(2n-1-k)-(n-1)^2+1$ vertices. After subtracting 1 and dividing by the diameter $2k$, we have a lower bound of $2n-1-[k+(n-1)^2/2k]$. This is maximized by setting k to be about $(n-1)/\sqrt{2}$, where the resulting lower bound is $(n-1)[2-\sqrt{2}]$. This is about $.59n$, which is still short of the desired lower bound of n.

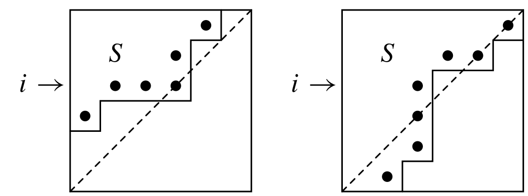
b) Sliding the elements of a vertex subset of $P_n \square P_n$ to the extreme left within their rows does not increase the size of the boundary. Choose $S \subseteq V(P_n \square P_n)$ with a_i vertices in the ith row, for each i. Let T consist of the first a_i in the ith row, for each i. We show that $|\partial T| \le |\partial S|$.

If $a_j = n$, then each set has the same number of boundary elements in row j. Furthermore, no vertex outside row j becomes a boundary element due to an edge to row j. Therefore, we may assume that $a_i < n$ for all i, but by convention we define $a_0 = a_{n+1} = n$. For $1 \le i \le n$, the number of boundary elements in row i of T are the rightmost elements, exactly 1 of them if $a_i = \min\{a_{i-1}, a_i, a_{i+1}\}$, and otherwise the maximum of $a_i - a_{i-1}$ and $a_i - a_{i+1}$. Since $a_i < n$, also S in row i has at least one boundary element due to row i, at least $a_i - a_{i-1}$ boundary elements due to row $i-1$, and at least $a_i - a_{i+1}$ boundary elements due to row $i+1$, since these are lower bounds on the sizes of the set differences. Hence S has at least as many boundary elements in each row as T has.

c) $|\partial S|$ is minimized over k-sets in $V(P_n \square P_n)$ by some S such that $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$, and hence Harper's lower bound for $B(P_n \square P_n)$ is n. By part (b), sliding vertices to the left within their rows does not increase the boundary, and it produces a set whose column populations are in nonincreasing order. By symmetry, sliding vertices to the top within their columns also does not increase the boundary. This leaves the column populations unchanged and produces a set whose row populations also are in nondecreasing order.

To show that the boundary bound is at least n, it suffices to prove that in $P_n \square P_n$ there is some k such that every k-set of vertices has boundary at least n. We choose k such that $\binom{n}{2} < k < \binom{n+1}{2}$. View $V(P_n \square P_n)$ as positions in a matrix. We may restrict our attention to a set S in the upper left as discussed above. Let a_i be the number of vertices of S in the ith row.

If $a_1 = n$ and $a_n = 0$, then S has a boundary element in each column. If $a_1 < n$ and $a_n > 0$, then S has a boundary element in each row. We illustrate the other cases for $n = 6$; the diagonal corresponds to $a_i = n+1-i$.



Case 1: *$a_1 < n$ and $a_n = 0$.* If $a_i \le n-i$ for all i, then $|S| \le \sum(n-i) = \binom{n}{2} < k$. Hence $a_i > n-i$ for some i, and ∂S has distinct elements in rows $1,\ldots,i$ and columns $1,\ldots,a_i-1$. We obtain $|\partial S| \ge n$.

Case 2: *$a_1 = n$ and $a_n > 0$.* If $a_i \ge n+1-i$ for all i, then $|S| \ge \sum(n+1-i) = \binom{n+1}{2} > k$. Hence $a_i \le n-i$ for some i. Now ∂S has distinct elements in rows $i,\ldots,n$ and columns $a_i+1,\ldots,n$, and $|\partial S| \ge n+1$.

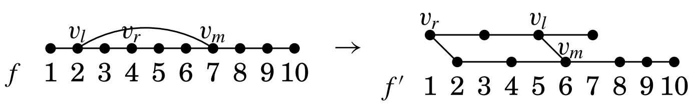
d) $B(P_m \square P_n) = \min\{m,n\}$. Numbering the vertices in successive ranks along the short direction yields maximum difference $\min\{m,n\}$. Since $P_n \square P_n$ is a subgraph of $P_m \square P_n$, it suffices to prove that $B(P_n \square P_n) \ge n$, which is the result of part (c).

Comment: A much shorter proof. For $B(P_n \square P_n)$, consider an optimal numbering f, minimizing the maximum dilation of edges. It suffices to show that some initial segment of f has boundary of size at least n.

Let S be the maximal initial segment of f that does not contain a full row or column. Adding the next element of f completes a row or column, say row r. We claim that S has a boundary element in each column. It has one boundary element in row r. In every column not containing that element it has an element in row r but does not contain all of that column, so it has a boundary element in the column.

8.3.44. *Change in bandwidth under edge addition.* Let G be a simple graph with order n and bandwidth b.

a) If $e \in \overline{G}$, then $B(G+e) \le 2b$. Let f be an optimal numbering of G, let $v_i = f^{-1}(i)$, and let $v_l v_m$ be the added edge e. We define a new numbering f' to prove that $B(G+e) \le 2b$. Let $r = \lfloor (l+m)/2 \rfloor$, and set $f'(v_r) = 1$ and $f'(v_{r+1}) = 2$. Number outward from v_r, setting $f'(v_i) = f'(v_{i+1}) + 2$ if $i < r$ and $f'(v_i) = f'(v_{i-1}) + 2$ if $i > r$ until the vertices on one side of v_r are exhausted. The remaining vertices ($v_{2r}, \ldots, v_n$ if $2r \le n$, or $v_{2r-n}, \ldots, v_1$ if $2r \ge n+1$) receive the remaining high labels in order. Edges between vertices on the same side of v_r may be stretched by a factor of 2; no other edges stretch as much. Since we began midway between v_l and v_m, we also have $|f'(v_l) - f'(v_m)| = 1$.



b) If $n \ge 6b$, then $B(G+e)$ can be as large as $2b$. Let G be a maximal n-vertex graph with bandwidth b; that is, G is the graph P_n^b obtained by adding edges to P_n joining any two vertices whose distance in P_n is at most b. Now let e be the edge joining the two vertices that are b positions from the ends of the ordering. Let S be the set of vertices consisting of the first $2b+1$ vertices and last $2b+1$ vertices in the ordering. Note that the subgraph of G induced by S has diameter 3.

Let f' be an optimal numbering of $G+e$. If the vertices labeled 1 and n by f' are both in S, then the path of length at most 3 joining these vertices has some edge whose endpoints differ under f' by at least $(n-1)/3$, which is at least $2b$.

Otherwise, the vertex x assigned number 1 or number n by f' belongs to $G - S$. Now all neighbors of x have numbers lying to one side of $f'(x)$. Since each vertex of $G - S$ has $2b$ neighbors in G, this again forces an edge difference of at least $2b$ under f'.

8.4. MORE EXTREMAL PROBLEMS

8.4.1. *The intersection number of an n-vertex graph G is at most $n^2/4$, using sets of size at most 3.* By Proposition 8.4.2, we need only show $E(G)$ can be covered with $\lfloor n^2/4 \rfloor$ complete subgraphs. The graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ shows that this is best possible. We show by induction that $\lfloor n^2/4 \rfloor$ complete subgraphs suffice to cover $E(G)$; this holds by inspection for $n = 1, 2$ (or $n = 3$). We

use only edges and triangles; this is equivalent to providing an intersection representation by finite sets such that each element appears in at most three sets.

For larger n, select $xy \in E(G)$. The difference between $\lfloor n^2/4 \rfloor$ and $\lfloor (n-2)^2/4 \rfloor$ is $n-1$. By the induction hypothesis, it suffices to show the edges incident to $\{x, y\}$ can be covered by $n-1$ edges and triangles. If $v \notin \{x, y\}$ is incident to exactly one of $\{x, y\}$, we use that edge; if both, we use the triangle $\{v, x, y\}$. If xy is in no triangle, then we add xy itself as a complete subgraph. (Note: the proof can also be phrased in terms of building a discrete intersection representation, without using the equivalence to clique covering.)

8.4.2. *Equivalent conditions for intersection number $\theta'(G)$ when G has no isolated vertices:*
A) $\theta'(G) = \alpha(G)$,
B) $\theta'(G \vee G) = (\theta'(G))^2$,
C) $\theta'(G) = \theta(G)$,
D) Every clique in any minimum clique cover of $E(G)$ contains a simplicial vertex of G.

Let Θ, Θ' denote minimum clique covers of $V(G)$ and $E(G)$, respectively. When discussing $H = G \vee G$, let G_1, G_2 denote the two copies of G in H, and let $\Theta(G_i), \Theta'(G_i)$ denote the copy of Θ, Θ' in G_i. Since G has no isolated vertices, Θ' covers both $E(G)$ and $V(G)$.

A $\Rightarrow$ B. The inequality $\theta'(H) \le (\theta'(G))^2$ always holds, because the join of complete subgraphs in G_1 and G_2 is a complete subgraph in H. Since Θ' covers both $E(G)$ and $V(G)$, $\{Q \vee Q' \colon Q \in \Theta'(G_1), Q' \in \Theta'(G_2)\}$ covers $E(H)$. For equality when $\theta'(G) = \alpha(G)$, consider the two copies in H of a maximum stable set in G. This set induces a complete bipartite subgraph with $\alpha(G)$ vertices in each partite set, so $(\alpha(G))^2$ complete subgraphs of H are needed to cover these edges, and $(\alpha(G))^2 = (\theta'(G))^2$.

B$\Rightarrow$C. Since G has no isolated vertices, $\theta'(G) \ge \theta(G)$. We form a cover of $E(H)$ using fewer than $(\theta'(G))^2$ complete subgraphs if $\theta'(G) > \theta(G)$. To cover $E(G_1) \cup E(G_2)$, for each $Q \in \Theta'$ we take its occurrences in $\Theta'(G_1)$ and $\Theta'(G_2)$ and form their join; this contributes $\theta'(G)$ complete subgraphs. To cover the edges joining $V(G_1)$ and $V(G_2)$, we take $\{Q \vee Q' \colon Q \in \Theta(G_1), Q' \in \Theta(G_2)\}$; this contributes $(\theta(G))^2$ complete subgraphs. Now $\theta'(H) \le \theta'(G) + (\theta(G))^2 < (\theta'(G))^2$, which contradicts the hypothesis.

C$\Rightarrow$D. Let $r = \theta'(G) = \theta(G)$, and let Θ be any set of r complete subgraphs covering $E(G)$ and hence $V(G)$. Every element of Θ has a vertex appearing in no other complete subgraph, else we could omit it and obtain a smaller covering of $V(G)$. A vertex appearing in only one member of a clique cover of $E(G)$ must be simplicial.

D⇒A. Since $\theta'(G) \ge \alpha(G)$ for any graph without isolated vertices, it suffices to obtain a stable set consisting of one vertex from each member of in a minimum clique cover Θ of $E(G)$. We may assume that every member of Θ is a maximal clique. Since a simplicial vertex belongs to only one maximal clique, this implies that each clique of Θ contains a vertex belonging only to that clique. The $\theta'(G)$ vertices from distinct cliques thus selected must be independent, because no edge among them is covered by Θ.

8.4.3. *If $b(G)$ is the minimum number of bipartite graphs needed to partition the edges of G, and $a(G)$ is the minimum number of classes needed to partition $E(G)$ such that every cycle of G contains a non-zero even number of edges in some class, then $b(G) = a(G) = \lceil \lg \chi(G) \rceil$.* We prove $\lg \chi(G) \le b(G) \le a(G) \le \lceil \lg \chi(G) \rceil$. Since $\lg \chi(G)$ and $\lceil \lg \chi(G) \rceil$ differ by less than 1, the integers in this string of inequalities must be the same.

Let $E_1 \cup \cdots \cup E_{b(G)}$ be a minimum partition of $E(G)$ into bipartite subgraphs; we may assume these are spanning subgraphs. We can define a proper $2^{b(G)}$-coloring f by giving each $v \in V(G)$ a binary $b(G)$-sequence $f(v)$ in which $f_i(v)$ indicates which partite set in E_i contains v. Since each edge belongs to some E_i, the endpoints of each edge receive different labels. This proves $\chi(G) \le 2^{b(G)}$, i.e. $\lceil \lg \chi(G) \rceil \le b(G)$.

Let $E_1 \cup \cdots \cup E_{a(G)}$ be a minimum partition having the cycle intersection property defined above. If E_i contains an odd cycle, then this cycle in G does not contain a non-zero even number of edges of any color. Hence each E_i is bipartite, and $b(G) \le a(G)$.

Let f be an optimal vertex coloring of G. Encode the colors in f of G by distinct binary sequences of length $k = \lceil \lg \chi(G) \rceil$. Partition $E(G)$ into $E_1 \cup \cdots \cup E_k$ by using the coordinates of this encoding: put $uv \in E_i$ if i is the first coordinate for which $f_i(u) \ne f_i(v)$. Given any cycle C in G, let j be the lowest-indexed color used on $E(C)$. While traversing C, coordinate j changes a non-zero even number of times, but since every other color on $E(C)$ is higher, when traversing edges of C coordinate j can change only along edges that actually belong to E_j. Hence this partition has the cycle intersection property, and $a(G) \le k$.

8.4.4. (•) Determine all the n-vertex graphs that have product dimension $n-1$. (Lovász–Nešetřil–Pultr [1980])

8.4.5. pdim $G \le 2$ *if and only if G is the complement of the line graph of a bipartite graph.* Given a 2-dimensional encoding of G, define a bipartite graph H with vertices $X \cup Y$ and edges $x_i y_j$ such that (i, j) is one of the vectors in the encoding. Then the vertex for (i, j) is adjacent to the vertex for (k, l) in G if and only if $i \ne k$ and $j \ne l$, which happens if and only if $x_i y_j$ and $x_k y_l$ are not incident, which happens if and only if the vertices for $x_i y_j$ and $x_k y_l$ are adjacent in the complement of the line graph of H. Conversely,

if G is the complement of the line graph of a bipartite graph H whose bipartition is $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$, then the vertices of G correspond to the edges of H, and we obtain a two-dimensional encoding by assigning (i, j) to the vertex of G corresponding to $x_i y_j \in E(H)$. This is an encoding because the vertices of G corresponding to $x_i y_j$ and $x_k y_l$ are adjacent if and only if those edges of H are non-incident, which happens if and only if $i \ne j$ and $k \ne l$.

8.4.6. *For $r \ge 2$,* $\text{pdim}\,(K_r + mK_1) = \begin{cases} r & \text{if } 1 \le m \le (r-1)! \\ r+1 & \text{if } m > (r-1)! \end{cases}$.

Allowing $r+1$ coordinates, we can represent this graph using $(i, \ldots, i)$ for the ith vertex of the clique and $(1, \ldots, r, j)$ for the jth vertex of the stable set, with the extra coordinate included to ensure that the vertices of the stable set get distinct encodings. Since $K_r + K_1$ is an induced subgraph, the answer thus is always r or $r+1$.

In every coordinate, the vertices of the clique must have distinct values. By permuting the labels used within a coordinate, we may assume that the code for the ith vertex of the clique is $(i, \ldots, i)$. If pdim $G = r$, then each vertex of the stable set must be encoded by a permutation of $[r]$ in order to establish all non-adjacencies to clique vertices. These permutations must be distinct, and each pair of them must agree in some coordinate to avoid edges in the stable set. Hence no pair of the permutations can be cyclic permutations of each other.

This partitions the $r!$ permutations into $(r-1)!$ classes of size r, from each of which we can take at most 1. Therefore, $\text{pdim}\,(K_r + mK_1) = r$ requires $m \le (r-1)!$. When $m \le (r-1)!$, r coordinates do suffice; give each vertex of the stable set value 1 in coordinate r to prevent edges, and use the $(r-1)!$ distinct permutations of $[r-1]$ in coordinates $1, \ldots, r-1$.

8.4.7. *The product dimension of the three-dimensional cube Q_3 is 2.* Since Q_3 is not a complete graph, we need at least two coordinates, and we can encode it with two coordinates by using the binary triples. Each triple x is a vertex that is adjacent to every vertex of opposite parity except the complement of x. We use coordinate 1 to destroy edges to vertices of the same parity and coordinate 2 to destroy edges between complements. In coordinate 1, assign 0 to each sequence of even weight and 1 to each sequence of odd weight. In coordinate 2, assign 0 to 000 and 111, and assign i to each sequence in which the ith coordinate has a value that appears only once in the sequence. The resulting vectors are $\{(i, j)\colon 0 \le i \le 1,\ 0 \le j \le 3\}$.

8.4.8. *The product dimension of the Petersen graph is 3 or 4.* The Petersen graph is $\overline{L(K_5)}$. It is not the complement of the line graph of a bipartite graph, so by Exercise 8.4.5 its product dimension is at least 3.

The encoding in the table below shows that the product dimension is at most 4. The vertices are named by the 2-element subsets of [5], adjacent when they are disjoint. Hence the codes of two vertices should agree in some coordinate if and only if their names have a common element. In the ith coordinate, value 0 is assigned to the four doubletons that contain element i. The remaining three doubletons that contain element 5 have value 1 in the ith coordinate. The three doubletons not containing 5 or i have value 2.

If two doubletons share an element, then their codes agree in coordinate i (with value 0) if their shared element i is not 5 (and in another coordinate with value 2). If the shared element is 5, then the union omits two elements from [4], and the codes agree in those coordinates (with value 1). If two doubletons are disjoint, then they agree in no coordinate, because in the ith coordinate, value 0 goes only to doubletons sharing element i, value 1 goes only to doubletons sharing element 5, and value 2 goes only to doubletons chosen from the set of three elements outside $\{i, 5\}$, which pairwise intersect.

12: 0022	23: 2002	34:2200
13: 0202	24: 2020	34:1101
14: 0220	25: 1011	34:1110
15: 0111		

8.4.9. *Maximum product of* $\text{pdim}\, G$ *and* $\text{pdim}\, \overline{G}$ *when* G *is an* n*-vertex graph.* Let $f(n)$ be the desired value. From $\max\{\text{pdim}\, G\} = n-1 = \max\{\text{pdim}\, \overline{G}\}$, we have $f(n) \le (n-1)^2$. Form G by identifying one leaf of $K_{1,\lceil n/2\rceil}$ with one vertex of $K_{\lfloor n/2\rfloor}$. Since G contains $K_{\lfloor n/2\rfloor} + K_1$ and $\overline{G}$ contains $K_{\lceil n/2\rceil} + K_1$ as induced subgraphs, we have $\text{pdim}\, G \ge \lfloor n/2\rfloor$ and $\text{pdim}\, \overline{G} \ge \lceil n/2\rceil$, yielding $f(n) \ge (n^2-1)/4$.

8.4.10. *If* $n \ge 4$, *then* $\text{pdim}\, P_n = \lceil \lg(n-1)\rceil$. *If* $n \ge 3$, *then* $\text{pdim}\, C_{2n} = 1 + \lceil \lg(n-1)\rceil$ *and* $1 + \lceil \lg n\rceil \le \text{pdim}\, C_{2n+1} \le 2 + \lceil \lg n\rceil$. Given a path induced by $x_1, \dots, x_m$ in G, set $u_i = x_i$ and $v_i = x_{i+1}$ for $1 \le i \le m-1$. This yields $u_i \leftrightarrow v_i$ for all i and $u_i \not\leftrightarrow v_j$ for $i < j$. By the LNP lower bound, this yields $\text{pdim}\, G \ge \lceil \lg m - 1\rceil$. For paths, we obtain $\text{pdim}\, P_n \ge \lceil \lg(n-1)\rceil$. Since C_m contains P_{m-1} as an induced subgraph, we obtain $\text{pdim}\, C_m \ge \lceil \lg(m-2)\rceil$. Thus $\text{pdim}\, C_{2n} \ge \lceil \lg(2n-2)\rceil = 1 + \lceil \lg(n-1)\rceil$ and $\text{pdim}\, C_{2n+1} \ge \lceil \lg(2n-1)\rceil = \lceil \lg(2n)\rceil = 1 + \lceil \lg n\rceil$.

We complete the proof for paths by embedding P_{2^k+1} in the weak product of k triangles, beginning with $k=2$. Let $x_k(i)$ be the encoding of the ith vertex on the path, for $1 \le i \le 2^k$. When $k=2$, we set $x_2(0) = 00$, $x_2(1) = 11$, $x_2(2) = 02$, $x_2(3) = 10$, $x_2(4) = 01$. For $k > 2$, we obtain $x_k(i)$ from the previous codes, by appending a suitable value in the new coordinate. Here i runs from 0 to 2^k.

index i	parity of i	in first $k-1$ coords	in kth coord
$i < 2^{k-1}$	even	$x_{k-1}(i)$	0
$i < 2^{k-1}$	odd	$x_{k-1}(i)$	1
2^{k-1}	even	$x_{k-1}(2^{k-1})$	2
$i > 2^{k-1}$	odd	$x_{k-1}(2^k - i)$	0
$i > 2^{k-1}$	even	$x_{k-1}(2^k - i)$	1

Codes for consecutive vertices come from codes for consecutive vertices at the previous stage, with distinct values in the new coordinate, so the desired edges exist. The only codes that are distinct throughout the first $k-1$ coordinates are those coming from consecutive vertices at the previous stage. If the distance from the old vertex 2^{k-1} is even, we append a 0 in the first half of the path, a 1 in the second half. If the distance is odd, we append a 1 in the first half, a 0 in the second half. Two vertices whose codes disagree in the first $k-1$ coordinates but are in opposite halves of the path arise from vertices at the previous stages whose distances from the last vertex have opposite parity. Thus their codes agree in the kth coordinate, and the undesired edge is destroyed.

To obtain the encodings for cycles, we need some observations about the above encoding for paths. Since each code is obtained by extending the code of a previous vertex whose index has the same parity, the first coordinate of a code is 1 if and only if the index is odd. For the same reason plus attention to when 2's are introduced, a code contains a 2 at some coordinate after the first if and only if the index of its vertex is even and is not the first or last vertex.

To encode C_{2j+2} in k dimensions, where $1 \le j < 2^{k-1}$, we use $x_k(0), \dots, x_k(j-1)$ and $x_k(2^k - j + 1), \dots x_k(2^k)$; these codes induce a disjoint union of two paths. Between $x_k(j-1)$ and $x_k(2^k - j + 1)$ we put $x_{k-1}(j)$ with a 2 appended. Between $x_k(0)$ and $x_k(2^k)$ we put $122\cdots 2$. By the observations above, this encodes C_{2j+2} in k dimensions.

To encode C_{2j+3} in $k+1$ dimensions, where $1 \le j < 2^{k-1}$, we use $u_k(i)$ for $0 \le i \le 2j$, alternately appending 0 and 2, and then complete the cycle using $u_k(2j+1)$ with 1 appended, followed by the code of all 2's.

(Comment: LNP improved the lower bound for odd cycles to agree with the upper bound when the length is one more than an even power of 2. On the other hand, Křivka [1978] showed that the lower bound is the correct answer for asymptotically at least 1/3 of all odd cycles.

8.4.11. *If* $k > 1$, *then* C_{2k+1} *has no isometric embedding in a cartesian product of complete graphs.* In such a cartesian product, the vertices correspond to integer vectors, and the distance between them is the number of coordinates where the vectors differ. Suppose that G is isometrically embeddable. If P is a shortest x, y-path in G, then the distances from x and y change by one with each step along the path, and each coordinate in the encoding therefore changes at most once along the path.

In any isometric embedding of C_{2k+1}, therefore, the edges along any path of length k change distinct coordinates. This implies that along any path of length $k+1$, the last edge changes the same coordinate as the first; repeating any other coordinate violates the previous statement, and changing another new coordinate creates a difference in $k+1$ coordinates for the encoding of two vertices at distance k (along the other part of the cycle). Since $k+1$ is relatively prime to $2k+1$, this implies that all edges change the same coordinate. Hence the only clique product in which C_{2k+1} embeds isometrically has one factor, which implies $k=1$.

8.4.12. $\text{qdim}\,(C_5) = 4$. By Winkler's result (Theorem 8.4.18), it suffices to prove that $\text{qdim}\,(C_5) > 3$. Let f be a 3-dimensional encoding, if one exists. Since C_5 is not bipartite, some code $f(v)$ has a star in some position. No code has more than one star, since each vertex has nonneighbors. By symmetry of 0s and 1s in a given coordinate of the encoding, we may assume that $f(v) = {*}11$. Let u, v, w, x, y be the vertices in cyclic order. Since the nonneighbors of v are adjacent, their codes must be 000 and 100; by symmetry, let $f(x) = 000$ and $f(y) = 100$. Since w is farther from y than from x, we have $f_1(w) = 0$; similarly, $f_1(u) = 1$. To obtain the correct distances from x and y, each of $f(u)$ and $f(w)$ has exactly one 1 after the first position. Since $d(u, v) = d(v, w) = 1$, the remaining entry in $f(u)$ and $f(w)$ is 0. Now the distance between $f(u)$ and $f(w)$ is 1 or 3, which contradicts $d(u, w) = 2$.

8.4.13. *The squashed-cube dimension of $K_{3,3}$ is 5.* By Theorem 8.4.18, it suffices to assume a 4-dimensional encoding f and obtain a contradiction. Let the partite sets be $A = \{a, b, c\}$ and $B = \{x, y, z\}$. If f encodes some vertex without stars, we may assume $f(a) = 0000$. Now each code for B has exactly one 1, and distance two among them forces these 1's to be in distinct coordinates with matching 0's. Hence $\{f(x), f(y), f(z)\} = \{100?, 010?, 001?\}$, where $? \in \{0, *\}$. Since $d(a, b) = 2$, $f(b)$ has exactly two 1's. Placing them in the first three coordinate violates $d(b, w) = 1$ for some $w \in B$. Hence $f_4(b) = 1$; by symmetry $f(b) = 1??1$, where $? \in \{0, *\}$. Now $d(b, y) = d(b, z) = 1$ forces $f(b) = 1{*}{*}1$. The same argument shows that $f(c)$ also has no zeros, which violates $d(b, c) = 2$.

Hence we may assume that every vertex code has a star. None can have more than two stars, since all have eccentricity 2. Suppose $f(a) = 00{*}{*}$. Now distance two among $\{a, b, c\}$ requires $f(b), f(c) = 1101, 1110$. Since $B \subseteq N(a)$, each code for B has exactly one 1 in the first two coordinates. By the pigeonhole principle, we may assume $f_1(x) = f_1(y) = 1$ and $f_2(x) \neq 1 \neq f_2(y)$. Now $d(x, y) = 2$ requires $\{f_3(x), f_3(y)\} = \{f_4(x), f_4(y)\} = \{0, 1\}$. To ensure distance 1 to b and c, we now must have $\{f(x), f(y)\} = \{1{*}00, 1{*}11\}$. If $f_1(z) = 1$, this argument would force $f(z)$ to end both in 00 and in 11. If $f_1(z) = *$, then $d(z, x) = d(z, y) = 2$ forces the same result. If $f_1(z) = 0$, then $d(z, b) = d(z, c) = 1$ implies $f(z) = 01{*}{*}$, which violates $d(z, x) = 2$.

Hence we may assume that every vertex code has exactly one star, with $f(a) = 000{*}$. Now each of $f(b), f(c)$ has exactly two 1's in the first three coordinates. Hence they have a common 1, which we may put in the first coordinate. Now $d(b, c) = 2$ forces their stars into the same coordinate; we conclude by symmetry that $f(b) = 110{*}$ and $f(c) = 101{*}$. Switching 0 and 1 in the first coordinate yields $f(a), f(b), f(c) = 100{*}, 010{*}, 001{*}$ and restores symmetry. Now no code with one, two, or three 1's in the first three coordinates has distance 1 from each vertex of A. However, if all codes for B have no 1's in the first three coordinates, then we cannot establish distance 2 between any pair of vertices of B.

8.4.14. *Menger's Theorem for edge-disjoint paths in digraphs, from Edmonds' Branching Theorem.* Assume that G is k-edge-connected. Thus at least k edges must be deleted to make some vertex unreachable from another. In particular, at least k edges must be deleted to make some vertex unreachable from x. By Edmonds' Branching Theorem, there is a set of k pairwise edge-disjoint branchings rooted at x. The paths reaching y in these trees are pairwise edge-disjoint. Since x and y were chosen arbitrarily, we have the conclusion of Menger's Theorem: in a k-edge-connected digraph, we can always find k pairwise edge-disjoint x, y-paths.

8.4.15. *The telegraph problem (one-way messages to transmit from each person to every other) requires $2n-2$ message for n people, and this suffices.* Before some person receives all the information, there must be a tree of messages to that person, which requires $n-1$ calls. After that, the remaining $n-1$ people must each receive a message to complete their information.

A tree in and a tree out from the same person completes the transmissions in $2n-2$ messages.

8.4.16. ($\bullet$) Let D be a digraph solving the telegraph problem in which each vertex receives information from each other vertex exactly once. Prove that in D at least $n-1$ vertices hear their own information. For each n, construct such a D in which only $n-1$ vertices hear their own information, but for each $x \neq y$ there is exactly one increasing x, y-path. (Seress [1987])

8.4.17. *The NOHO property.* Let G be a connected graph with $2n-4$ edges having a linear ordering that solves the gossip problem and satisfies NOHO (no increasing cycle).

a) If $n(G) > 8$ and at most two vertices have degree 2, then the graph obtained by deleting the first calls and last calls of vertices in G has 4 components, of which two are isolated vertices and two are caterpillars having the same size. As argued in Claim 3 of Theorem 8.4.23, the set F of first calls is a matching, as is the set L of last calls. Hence the graph M consisting of the remaining "middle" calls has $n - 4$ edges and therefore at least four components.

Let $O(x)$ and $I(x)$ be the trees in the argument of Claim 2 of Theorem 8.4.23, growing the trees that are useful "out from" and "in to" a vertex x. The argument of Claim 2 shows that under the NOHO property, $d(x) - 2$ calls are useless to x, none of which are incident to x.

A call in M can be useful to x only if the component containing it also contains a neighbor of x, because a path to x cannot continue after a last call on it, and a path from x cannot start before a first call on it. Since $e(G) < 2n$, there is a vertex of degree at most 3, so there are at most three nontrivial components in M. Since there are at least four components, at least one is an isolated vertex x. Hence there are at most two nontrivial components (containing the first and last neighbor of x), leaving at least two isolated vertices in M. Since the hypothesis specifies no additional vertices of degree 2 in G, there are exactly four components in M, of which two are isolated vertices. Furthermore, the two nontrivial components must be trees since $e(M) = n - 4$.

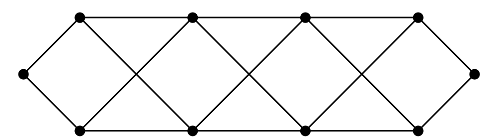
Let x and y be the isolated vertices, with neighbors x_f and y_f in F and neighbors x_l and y_l in L. Since each edge is useful to x, the vertices x_f and y_f are in different components of M. Hence one component is a tree T_1 of paths out of x_f, and the other is a tree T_2 of paths in to x_l. Similarly, one component consists of paths out of y_f and the other is in to y_l.

We claim that y_f cannot lie in T_1. If so, then x_l and y_l both lie in T_2. We claim that this forbids an increasing x, y_l-path. Such a path must start with xx_f, since it cannot continue after xx_l. After x_f, it can only reach T_2 via an edge of L, which it could follow after some increasing path from x_f in T_1. However, this edge of L does not reach y_l, since the last neighbor of y_l is y, and the path cannot continue after a last edge to reach y_l.

Therefore, $x_f, y_l \in V(T_1)$ and $x_l, y_f \in V(T_2)$. The requirement that every edge lies on a path out of x_f and a path into y_l implies that the path from x_f to y_l in T_1 is an increasing path with every edge of T_1 incident to it. In this case T_1 (and similarly T_2) is a caterpillar, and the edges off the spine occur between the incident edges on the spine in the linear ordering of calls. This implies that every two edges in T_1 lie on an increasing path together. Now an edge of F or L joining vertices within one of these components would violate NOHO. After deleting the first and last edges incident to

x and y, the matchings tell us that $n(T_1) = n(T_2)$, so the two nontrivial components are caterpillars of the same size.

b) For even n with $n \geq 4$, there are solutions with $2n - 4$ calls that have the NOHO property. Below is a general construction: first perform the matching consisting of diagonal calls with positive slope, then the top path from left to right and the bottom path from right to left, and finally the matching consisting of diagonals with negative slope. There are many other constructions. ■



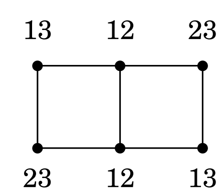
8.4.18. (•) A *NODUP scheme* (NO DUPlicate transmission) is a connected ordered graph that has exactly one increasing path from each vertex to every other.

a) (−) Prove that every NODUP scheme has the NOHO property.

b) Prove that there is no NODUP scheme when $n \in \{6, 10, 14, 18\}$. (Comment: Seress [1986] proved that these are the only even values of n for which NODUP schemes do not exist, constructing them for all other values. For $n = 4k$, West [1982b] constructed NODUP schemes with $9n/4 - 6$ calls, and Seress [1986] proved that these are optimal.)

8.4.19. *Broadcasting can be completed in time $1 + \lceil \lg n \rceil$ in a particular graph with fewer than $2n$ edges.* By having each vertex who knows the information at a given time call a new vertex who does not know it at the next time, broadcasting can be completed from a specified root vertex in $\lceil \lg n \rceil$ steps. Now add edges to make the root adjacent to all other vertices. To broadcast from any other vertex, call the root first, and then finish the job in $\lceil \lg n \rceil$ additional phases. The number of edges in the construction is $2n - 1 - \lceil \lg n \rceil$.

8.4.20. *The graph below is not 2-choosable.* Assign lists as shown. If we put 1 above 2 on the central vertices, then the vertices on the left cannot be properly colored. If we put 2 above 1 on the central vertices, then there is no proper choice for the vertices on the right.



8.4.21. *$K_{k,m}$ is k-choosable if and only if $m < k^k$.* Let the partite sets be $X = \{x_i\}$ of size k and $Y = \{y_j\}$ of size m. For $m \ge k^k$, it suffices to consider $m = k^k$ and a specific collection of k-lists. Let $\{c_{rs}\colon 1 \le r \le k,\ 1 \le s \le k\}$ be a collection of k^2 colors. Let $L(x_i) = \{c_{is}\colon 1 \le s \le k\}$ to x_i. To the vertices of Y, assign the k^k distinct lists obtained by choosing one color with each possible first coordinate. Every choice of colors on X consists of one color with each possible first coordinate. For each such choice, the chosen colors will be precisely the colors in the list for some vertex of Y. No legal color can be chosen for that vertex to complete the coloring.

For $m < k^k$, consider an arbitrary collection of k-lists assigned to the vertices. If some two vertices in X have a common element in their lists, choose that element for them, and choose arbitrarily from the lists for the other vertices of X. This uses at most $k-1$ colors for X, which leaves a color available in the list for each vertex of Y. On the other hand, if the color sets are disjoint, then they can be indexed so that $L(x_i) = \{c_{is}\colon 1 \le s \le k\}$. There are k^k possible choices of one of these colors from each set. Since $m < k^k$, there is at least one such choice that does not occur as a list for vertices of Y. When these colors are chosen for X, for each $y \in Y$ there is a color in $L(y)$ not used on X, and the coloring can be completed.

8.4.22. *Bounds on choosability and edge-choosability.*

$\chi_l(G) \le 1 + \max_{H\subseteq G} \delta(H)$. Order the vertices $v_1, \dots, v_n$ such that v_i is a vertex of minimum degree in the subgraph G_i induced by $v_1, \dots, v_i$ (by selecting the vertices in decreasing order). Consider an arbitrary collection of lists of size $1 + \max_{H\subseteq G} \delta(G)$. Make choices from these lists in the order $v_1, \dots, v_n$. When v_i is considered, there are at most $\delta(G_i)$ neighbors of v_i that have been colored, by the construction of the ordering. Hence there is always a color available in the list for v_i that has not been used on an earlier neighbor.

$\chi_l(G) + \chi_l(\overline{G}) \le n+1$. By part (a), it suffices to show that $\max_{H\subseteq G} \delta(H) + \max_{H\subseteq \overline{G}} \delta(H) \le n-1$. Let H_1 and H_2 be subgraphs of G and $\overline{G}$ achieving the maximums. Let $k_i = \delta(H_i)$. Note that $n(H_i) \ge k_i + 1$. If $k_1 + k_2 \ge n$, then H_1 and H_2 have a common vertex v. Now v must have at least k_i neighbors in H_i, for each i, but only $n-1$ neighbors are available in total.

$\chi_l'(G) \le 2\Delta(G) - 1$. Place the edges in some order. Each edge is incident to at most $2\Delta(G) - 2$ others. If $2\Delta(G) - 1$ colors are available at each vertex, then when we reach it there is always a color available not used on the incident edges colored earlier.

8.4.23. *Every chordal graph G is $\chi(G)$-choosable.* We use the reverse of a simplicial elimination ordering. Consider the vertices in the construction order. Cliques are created only as vertices are added, so the clique number is the maximum k such that a vertex belongs to a clique of order k when added. This also equals the chromatic number, by the greedy coloring with respect to this order. The same greedy coloring algorithm establishes k-choosability. When each vertex v is added, it has at most $k-1$ neighbors already present. Here at most $k-1$ colors from the list allowed for v have already been used on its neighbors, and a color remains that can be chosen from the list for v.

8.4.24. *A connected graph G has an L-coloring from any list assignment L such that $|L(v)| \ge d(v)$ for all v if there is strict inequality for at least one vertex y.* Choose a spanning tree of G, and order the vertices descending away from y, meaning that each vertex other than y has a later neighbor. When we reach a vertex v other than y, we have colored fewer than $d(v)$ of its neighbors, and hence a color remains available in $L(v)$ to use on v. When we reach y, we have colored $d(y)$ neighbors, but an extra color still remains available.

8.4.25. *a) Every graph G has a total coloring with at most $\chi_L'(G) + 2$ colors.* In a total coloring, colored objects have different colors if they are adjacent vertices, incident edges, or an incident vertex and edge.

Let $k = \chi_L'(G) + 2$. Because $\chi_L'(G) \ge \chi'(G) \ge \Delta(G)$, there exists a proper k-coloring f of G. To each edge uv, assign the list $[k] - \{f(u), f(v)\}$. This assigns each edge a list of $\chi_L'(G)$ colors, from which we can choose a proper edge-coloring to complete a total coloring of G.

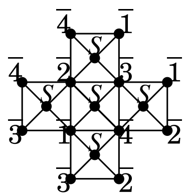
8.4.26. *Non-4-choosable planar graph of order 63.*

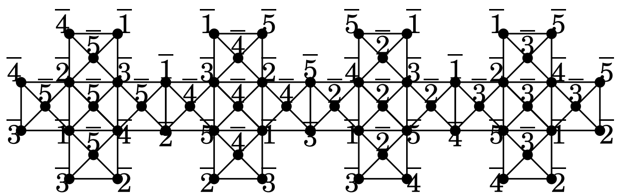
a) With S denoting $[4]$ *and $\bar{i}$ denoting $S - \{i\}$, the given lists for the graph on the left below yield no proper coloring.* The following properties hold for coloring each 4-cycle with distinct lists. (1) the chosen colors cannot be distinct (its center would not be colorable from S). (2) The colors on two consecutive vertices cannot be the colors forbidden from the opposite vertices in the opposite order (those opposite vertices would have to contribute the two remaining colors, violating (1)).

Consider the central 4-cycle C, and view all labels and indices modulo 4. We claim that the vertex in C with list $\bar{i}$ cannot receive color $i+1$. If it does, it forbids $i+1$ from the vertex in C with list $\overline{i-1}$, which by (2) also cannot receive color $i+2$. This leaves only color i for the vertex in C with list $\overline{i-1}$. Repeating the argument leads for each j to color $j+1$ on the vertex in C with list $\bar{j}$, which violates (1).

By making the same argument in the other direction, color $i-1$ on the vertex in C with list $\bar{i}$ would propogate to color i on the vertex in C with list $\overline{i+1}$, again violating (1).

This leaves only the possibility that for each i, color $i+2$ appears on the vertex in C with list $\bar{i}$. This again violates (1).





b) The planar graph G' obtained from G on the right above by adding one vertex with list $\overline{1}$ adjacent to all vertices on the outside face of G has no proper coloring chosen from these lists, where $\overline{i}$ denotes $[5] - \{i\}$. Suppose that G' has such a coloring. When the color chosen for the extra vertex is 5,4,2,3, respectively, the 1st, 2nd, 3rd, or 4th copy of the graph of part (a) in G has lists on its vertices isomorphic to those specified in part (a), via a permutation of the names of the colors. By part (a), there is no way to complete the coloring.

8.4.27. *Equivalence of Dilworth and König–Egerváry Theorems.*

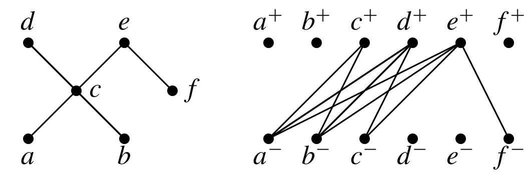
a) Dilworth's Theorem implies the König–Egerváry Theorem. View a bipartite graph G on n vertices as a poset. The vertices of one partite set are maximal elements, the others are minimal, and the edges are cover relations. Chains have one or two elements. Thus every chain-covering of size $n-k$ uses k chains of size 2 and yields a matching of size k in G. Each antichain of size $n-k$ is an independent set in G, and the k remaining vertices are a vertex cover. Hence Dilworth's guarantee of an antichain and a chain-covering of the same size yields a matching and a vertex cover of equal size in G.

b) The König–Egerváry Theorem implies Dilworth's Theorem. Let P be a poset of size n. We apply the König–Egerváry Theorem to a bipartite graph $S(P)$ called the *split* of P. The partite sets of $S(P)$ are $\{x^-\colon x \in P\}$ and $\{x^+\colon x \in P\}$. The edge set is $\{x^-y^+\colon x <_P y\}$.

A matching in $S(P)$ yields a chain-covering in P as follows: if x^-y^+ is in the matching, then y is immediately above x on a chain in the cover. If x^- or x^+ is unmatched, then x is the top or bottom of its chain, respectively. Since each vertex of $S(P)$ appears in at most one edge of the matching, this defines disjoint chains covering P. If the matching has k edges, then the cover has $n-k$ chains, since each added edge links the top of one chain with the bottom of another to form a single chain.

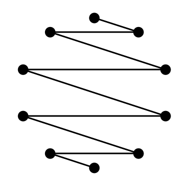
From a minimum vertex cover T of $S(P)$, we obtain an antichain. We show first that T does not use both copies of any element x. By transitivity in P, all of $\{z^-\colon z \in D(x)\}$ is adjacent in $S(P)$ to all of $\{y^+\colon y \in U(x)\}$. Covering the edges of this complete bipartite subgraph requires using all of $\{z^-\colon z \in D(x)\}$ or all of $\{y^+\colon y \in U(x)\}$. Since these are the neighbor sets of x^+ and x^-, respectively, at least one of $\{x^+, x^-\}$ can be omitted from T.

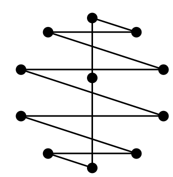
Now let $A = \{x \in P : x^-, x^+ \notin T\}$; we have shown that $|A| = |P| - |T|$. Also, A is an antichain, since a relation between elements of A would yield an edge of $S(P)$ uncovered by T. Thus a minimum cover of size k yields an of equal size yields an antichain and a chain-covering of equal size.



8.4.28. *K_n decomposes into $\lceil n/2 \rceil$ paths.* When n is even, we can use the decomposition into $n/2$ paths that are rotations of the figure on the left below. Each path uses two edges of each "length" around the circle, and each rotation gives a new pair of each length until all n pairs are obtained.

When n is odd, a bit more care is needed. Putting one vertex in the middle yields a decomposition into $(n-1)/2$ cycles, by rotating the figure on the right below. We can kill one short edge from each cycle to make it into a path, choosing always the short edge on the right side of the picture, and these $(n-1)/2$ leftover edges form a path to complete the decomposition. ■





K_n decomposes into $\lfloor n/2 \rfloor$ cycles when n is odd. We rotate the cycle shown in the figure above.

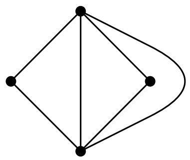
8.4.29. *Decomposition of K_n into spanning connected subgraphs.*

a) If K_n decomposes into k spanning connected subgraphs, then $n \ge 2k$. Each subgraph has at least $n-1$ edges, and they are pairwise edge-disjoint, so $k(n-1) \le \binom{n}{2}$.

b) K_{2k} decomposes into k spanning trees of diameter 3. Such trees are double-stars. Each has $2k-1$ edges, so k of them are needed to cover the $k(2k-1)$ edges of K_{2k}. Partition the vertex set into pairs $\{x_i, y_i\}$ for $i \le i \le n/2$. The ith subgraph consists of x_iy_i along with the edges x_jx_i and y_jy_i for $j < i$ and the edges x_iy_j and y_iy_j for $j > i$.

8.4.30. *Every 2-edge-connected 3-regular simple planar graph decomposes into paths of length 3, as does every simple planar triangulation.* The first statement is a special case of Exercise 3.3.19. For a simple triangulation

G, observe that the dual G^* is 2-edge-connected and 3-regular, so it has a P_4-decomposition. Let e_1', e_2', e_3' be the successive edges in one copy of P_4 in the decomposition. Since G^* is 3-regular, e_1' and e_2' are on the same face, as are e_2' and e_3'. Therefore, the corresponding edges e_1 and e_2 in G are incident, as are e_2 and e_3. Thus e_1, e_2, e_3 form a path or a cycle. It is a cycle if and only if the endpoints of e_1 and e_3 are the same vertex, which requires e_1' and e_3' to bound the same face of G^*. However, this face of G^* also shares boundary edges with the faces corresponding to the endpoints of e_2', since G^* is 3-regular. Therefore, if e_1, e_2, e_3 is a cycle in G, then G must have multiple edges. The triangulation below shows that the prohibition of multiple edges is necessary.



8.4.31. *Theorem 8.4.35 is best possible when $m-1$ divides $n-1$.* Theorem 8.4.35 states that if the number of edges in an n-vertex graph exceeds $m(n-1)/2$, then the circumference exceeds m. We provide an n-vertex graph with exactly $m(n-1)/2$ edges in which the circumference is exactly m. The graph is $(\frac{n-1}{m-1}K_{m-1}) \vee K_1$. Each block is isomorphic to K_m, and every cycle stays within a block, so the circumference is m. The number of edges is $\frac{n-1}{m-1}\binom{m}{2}$, which simplifies to $m(n-1)/2$.

8.4.32. *If G is a graph such that $\overline{G}$ is triangle-free and not a forest, then G has a cycle of length at least $n(G)/2$.* If $\overline{G}$ is triangle-free, then whenever u and v are nonadjacent in G they cannot have a common nonneighbor, and hence $d(u) + d(v) \geq n(G) - 2$. If G is 2-connected, then Theorem 8.4.37 yeilds a cycle of length at least $n(G) - 2$.

If G is not connected, then the prohibition of triangles from $\overline{G}$ implies that G has only two components and that they are complete graphs. Hence one of them has a cycle of length at least $n(G)/2$.

If G is connected and has a cut-vertex v, then $G - v$ again is a disjoint union of two complete graphs. Also v cannot have a nonneighbor in both components. Hence G contains two disjoint complete graphs whose orders sum to $n(G)$, and again there is a cycle of least at least $n(G)/2$.

8.4.33. *Sufficient conditions for spanning cycles in graphs and digraphs.*

Woodall's Theorem implies Ore's Theorem. Ore proved that ($u \not\leftrightarrow v \Rightarrow d(u) + d(v) \geq n(G)$) is sufficient for a spanning cycle in a graph G. Woodall

proved that ($u \not\leftrightarrow v \Rightarrow d^+(u) + d^-(v) \geq n(G)$) is sufficient for a spanning cycle in a digraph G.

Given a graph satisfying Ore's Condition, let G' be the digraph obtained by replacing each edge of G with two opposing edges having the same endpoints. Now $d^+_{G'}(v) = d^-_{G'}(v) = d_G(v)$. Thus Woodall's Condition holds, and Woodall's Theorem implies that G' has a spanning cycle, which yields a spanning cycle in G.

Meyniel's Theorem implies Woodall's Theorem for strict digraphs. Letting $d(u) = d^+(u) + d^-(u)$, Meyniel proved that ($u \not\leftrightarrow v \Rightarrow d(u) + d(v) \leq 2n(G) - 1$) is sufficient for a spanning cycle in a digraph G such that each ordered pair appears at most once as an edge.

Consider a digraph in which each ordered pair of vertices appears at most once as an edge. If Woodall's Condition holds, then when $u \not\leftrightarrow v$ we

$$d(u) + d(v) = d^+(u) + d^-(v) + d^-(u) + d^-(v) \geq 2n(G) > 2n(G) - 1.$$

Thus Meyniel's Condition holds if we also show that the digraph is strongly connected. The holds because when $d^+(u) + d^-(v) \geq n(G) - 1$, there is an edge or a path of length 2 from u to v. Thus Meyniel's Theorem applies, and the digraph has a spanning cycle.

8.4.34. *A strict n-vertex digraph has a spanning path if $d(u)+d(v) \geq 2n-3$ for every pair u, v of distinct nonadjacent vertices.* Given such a digraph, add a vertex w with an edge to and from each of the original vertices. Let G' denote the new digraph, with degree function d'. Now $d'(u) + d'(v) = d(u) + d(v) + 4 \geq 2n + 1 = 2(n + 1) - 1$. Also G' is strongly connected, since each vertex can get to and from w. By Meyniel's Theorem, G' has a spanning cycle. Since this passes through w only once, deleting w leaves a spanning path in G.

8.5. RANDOM GRAPHS

8.5.1. *Expectation.*

a) The expected number of fixed points in a random permutation of $[n]$ is 1. Since there are $(n-1)!$ permutations with element i fixed, the probability that i is fixed is $1/n$. Letting X_i be the indicator variable for element i being fixed, we have $\sum X_i$ as the random variable for the number of fixed points. By linearity, $\mathrm{E}(\sum X_i) = \sum \mathrm{E}(X_i) = n(1/n) = 1$.

b) The expected number of vertices of degree k in a random n-vertex graph with edge probability p is $k\binom{n-1}{k}p^k(1-p)^{n-1-k}$. A vertex has degree k when there are k successes among the $n-1$ trial for its incident edges. The probability of this is $\binom{n-1}{k}p^k(1-p)^{n-1-k}$. Letting X_i be the indicator variable for vertex i having degree k, the expected number of vertices of degree k becomes $k\mathrm{P}(X_i = 1)$, by linearity.

8.5.2. *Always* $1 - p < e^{-p}$ *for* $p > 0$. For $p = 0$, equality holds. Hence it suffices to show that the derivative of e^{-p} exceeds that of $1 - p$ for $p > 0$. We have $(d/dp)e^{-p} = -e^{-p} > -1 = (d/dp)(1-p)$, where the key inequality holds for $p > 0$. (The inequality also holds for $p < 0$, because the terms in the series for e^{-p} are then all positive.)

8.5.3. *The expected number of monochromatic triangles in a random 2-coloring of* $E(K_6)$ *is* $15/4$. When the edges are given red or blue with probability $1/2$ each, independently, the probability that three vertices produce a monochromatic triangle is $1/4$. There are $\binom{6}{3}$ triples where this may occur. By linearity of expectation, the expected number of occurrences is $15/4$, even though the events are not independent.

8.5.4. *Some 2-coloring of the edges of* $K_{m,n}$ *has at least* $\binom{m}{r}\binom{n}{s}2^{1-rs}$ *monochromatic copies of* $K_{r,s}$. We color the edges red or blue with probability $1/2$ each, independently. A particular choice of r vertex in one partite set and s vertices in the other produces a monochromatic copy of $K_{r,s}$ with probability 2^{1-rs}. Since there are $\binom{m}{r}\binom{n}{s}$ ways to choose the vertex sets, by linearity the expected number of copies is $\binom{m}{r}\binom{n}{s}2^{1-rs}$, so some outcome of the experiment is a 2-coloring with that many monochromatic copies of $K_{r,s}$. (Note: The coefficient increases to $\binom{m}{r}\binom{n}{s} + \binom{m}{s}\binom{n}{r}$ if $r \neq s$ and we don't care which partite set contains the bigger part of the subgraph.)

8.5.5. *The statement "*$(1-\varepsilon)n \le f(G_n) \le (1+\varepsilon)n$ *when* $\varepsilon > 0$ *for sufficiently large* n*" is equivalent to "*$f(G_n)/n \to 1$ *as* $n \to \infty$*", written as "*$f(G_n) \le n + o(n)$*".* Let $g(n) = f(G_n)/n$. If $g(n) \to 1$, then for all $\varepsilon > 0$ there exists N such that $n > N$ implies $|g(n) - 1| < \varepsilon$, by the definition of convergence of sequences. The inequality $|g(n) - 1| < \varepsilon$ (for sufficiently large n) is simply the first statement here.

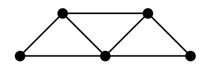
8.5.6. *Probability that the probability that the Hamiltonian closure* $C(G)$ *of a random graph* G *with vertex set* $[5]$ *is complete.* The problem is to determine the fraction of the graphs with vertex set $[5]$ that have complete closure. We describe the graphs, without doing the counting.

If $\delta(G) \le 1$, then $C(G)$ is not complete; a vertex of degree at most 1 never acquires another edge, because every vertex of degree $n(G) - 1$ is already adjacent to it.

If nonadjacent vertices have degree some at least $n(G)$, then all remaining edges are added immediately and the closure is complete.

Therefore, with five vertices, it suffices to consider graphs having nonadjacent vertices of degree 2. Among the remaining graphs, we have $5 \le e(G) \le 7$ for such graphs. All such graphs with five edges are 5-cycles,

which gain no edges. With six edges we can have $K_{2,3}$, which gains one edge and does not become complete, or the union of a 3-cycle and a 4-cycle sharing one edge, whose closure is complete. With seven edges we have only $K_2 \vee \overline{K}_3$ and the graph below; $K_2 \vee \overline{K}_3$ is already closed and gains no edges, but the closure of the graph below is complete.



8.5.7. *If* G *is a graph with* p *vertices,* q *edges, and automorphism group of size* s*, and* $n = (sk^{q-1})^{1/p}$*, then some* k*-coloring of* $E(K_n)$ *has no monochromatic copy of* G. Produce a k-coloring of the edges at random, with each edge receiving each color with probability $1/k$, independently. A particular copy of G in K_n becomes monochromatic with probability $k \cdot 1/k)^q$. On a given set of p vertices, there are $p!/s$ copies of G. If $\binom{n}{p}\frac{p!}{s}k^{1-q} < 1$, then there is an outcome of the experiment in which no copy of G is monochromatic. Since $\binom{n}{p} < n^p/p!$, the desired inequality holds when $n^p < sk^{q-1}$.

8.5.8. *Bipartite subgraphs.*

a) Every graph has a bipartite subgraph with at least half its edges. Select a random vertex subset A by choosing each vertex with probability $1/2$, independently. Each edge has probability $1/2$ of belonging to the cut $[A, \overline{A}]$, since this is the probability that exactly one of its endpoints lies in A. By linearity of expectation, the expected number of edges in the cut is half the total number of edges. The edges in a cut form a bipartite subgraph, so there is a bipartite subgraph with at least half the edges.

b) If G *has* m *edges and* n *vertices, then* G *has a bipartite subgraph with at least* $m\frac{\lceil n/2\rceil}{2\lceil n/2\rceil-1}$ *edges.* Choose A at random from all $\lceil n/2\rceil$-element vertex subsets. The number of these subsets containing exactly one endpoint of a given edge e is $2\binom{n-2}{\lceil n/2\rceil-1}$. Thus e belongs to the cut $[A, \overline{A}]$ with probability $2\binom{n-2}{\lceil n/2\rceil-1}/\binom{n}{\lceil n/2\rceil}$. Since $\binom{n}{\lceil n/2\rceil} = \frac{n}{\lfloor n/2\rfloor}\binom{n-1}{\lceil n/2\rceil} = \frac{n}{\lfloor n/2\rfloor}\,\frac{n-1}{\lceil n/2\rceil}\binom{n-2}{\lceil n/2\rceil-1}$, the probability is $\frac{2\lfloor n/2\rfloor\lceil n/2\rceil}{n(n-1)}$. Since $\frac{n(n-1)}{2\lfloor n/2\rfloor} = 2\lceil n/2\rceil - 1$, linearity of expectation yields $m\frac{\lceil n/2\rceil}{2\lceil n/2\rceil-1}$ as the expected size of the cut, and some cut is at least this large. This fraction of the number of edges is strictly more than .5, so this result improves part (a).

8.5.9. *If in a complete* k*-ary tree with leaves at distance* l *from the root, the vertices fail independently with probability* p*, then the expected number of nodes accessible from the root is* $(1-p)\frac{1-(k-kp)^{l+1}}{1-k+kp}$. There are k^j vertices at depth j (distance j from the root). A vertex at depth j is accessible if and only if it and its ancestors are alive. Thus it is accessible with probability

$(1-p)^{j+1}$. Using linearity of the expectation, the expected number of accessible nodes is $\sum_{j=0}^{l}(1-p)(k-kp)^j$. When $p=1/k$, this is simply $1-p$. Otherwise, the expectation is $(1-p)\frac{1-(k-kp)^{l+1}}{1-k+kp}$.

8.5.10. *The expected number of edges in a matching of size n that are induced by selected k vertices at random is $\frac{n}{2}\frac{k(k-1)}{(n-1)(2n-1)}$.* There are several proofs; using linearity of expectation makes the computations simple. There are $\binom{2n-2}{k-2}$ sets of size k that capture a particular pair of vertices. Hence each edge is captured with probability $\frac{k(k-1)}{(2n-2)(2n-1)}$. By linearity, the expected number of edges in the matching that are captured is $n\frac{k(k-1)}{(2n-2)(2n-1)}$.

8.5.11. *If a graph G has n vertices and m edges, with $m \ge 4n$, then $\nu(G) \ge m^3/[64n^2]$*, where $\nu(G)$ denotes the minimum number of crossings in a drawing of G. Let G have n vertices and m edges, with $m \ge 4n$, and consider a drawing of G in the plane. To obtain a lower bound on $\mathrm{cr}(G)$, we take a random induced subdrawing H, including each vertex independently with probability p. We expect pn vertices and p^2m edges in H. Let Y be the number of edge crossings in the drawing of G that remain in H. We have $E(Y)=p^4\mathrm{cr}(G)$, since all four endpoints of the two edges must be retained to keep the crossing.

Let $X=3n(H)-6+Y-e(H)$. Always $Y \ge e(H)-(3n(H)-6)$, since a planar graph with v vertices has at most $3v-6$ edges, and every edge beyond a maximal plane subgraph of H introduces at least one additional crossing. Thus always $X \ge 0$. We conclude that $E(X) \ge 0$.

By linearity, $E(X)=3np-6+p^4\mathrm{cr}(G)-p^2e(G)$. This yields $3n+p^3\mathrm{cr}(G)-pm>0$. We choose $p=4n/m$, which is feasible since $m \ge 4n$. We now have the inequality $3n+64n^3/m^3\mathrm{cr}(G)>4n$, which yields the desired bound.

8.5.12. *In a random orientation of the vertices of a simple graph G, produced by orienting each edge toward the vertex with higher index in a random permutation, the expected number of sink vertices (outdegree 0) is $\sum_{v\in V(G)}\frac{1}{d(v)+1}$, which ranges from 1 to $n(G)$.*

A vertex is a sink in the resulting orientation if and only if it follows all its neighbors in the permutation. For each vertex v, whether this happens is determined only by whether it is last among the set $N[v]$, which happens with probability $(1+d(v))^{-1}$. By linearity, the expected number of sinks is $\sum_{v\in V(G)}\frac{1}{d(v)+1}$. (Note that the sinks form an independent set, so this is also a lower bound on $\alpha(G)$.)

Given the formula for the expectation, it is minimized by increasing degrees and maximized by reducing degrees, so it is minimized by the complete graph, where the number of sinks is always 1, and it is maximized by

the trivial graph, where every vertex is always a sink. Among connected graphs, it is maximized by the path, where the value is $(n+1)/3$.

In order to have only one sink, the last two vertices in the random permutation must be adjacent. When the last vertex has degree d, then with probability $\frac{n-1-d}{n-1}$ the next-to-last vertex is a nonneighbor of it. Thus the probability of having at least two sinks is at least $\frac{1}{n}\sum_{i=1}^{n}\frac{n-1-d_i}{n-1}$, which simplifies to $1-\frac{1}{n}\frac{1}{n-1}\sum_{i=1}^{n}d_i$. Invoking the Degree-Sum Formula, the probability of having only one sink is at most $e(G)/\binom{n}{2}$.

8.5.13. *Bound on choosability of n-vertex bipartite graphs.*

a) Every k-uniform hypergraph with fewer than 2^{k-1} edges is 2-colorable. Let H be a k-uniform hypergraph with n edges, where $n<2^{k-1}$. Color vertices by X and Y so that each gets color X with probability $1/2$, independently. The probability that a given edge is monochromatic is $2^{-(k-1)}$. Since $n<2^{k-1}$, the probability that some edge is monochromatic is less than 1. Hence some outcome of the experiment is a proper 2-coloring of H.

Since H has n edges and $n<2^{1+\lfloor \lg n\rfloor}$, the hypergraph H is 2-colorable (in a random coloring of a k-uniform hypergraph with fewer than 2^{k-1} edges, the expected number of monochromatic edges is less than 1). A proper 2-coloring of H partitions its vertices into Type X and Type Y.

b) If each vertex of an n-vertex bipartite graph is given a list of more than $1+\lg n$ usable colors, then a proper coloring can be chosen from the lists. Let G be an X,Y-bigraph with n vertices and such a list assignment. Let H be an auxiliary hypergraph whose vertices are the colors in the lists. Each vertex $v\in V(G)$ generates an edge in H consisting of the colors in $L(v)$. We may reduce the sizes of the lists so that H is k-uniform, where $k=2+\lfloor \lg n\rfloor$. Thus $k-1>\lg n$. By part (a), H is 2-colorable; we call the colors Type X and Type Y.

In choosing an L-coloring for G, we must restrict each color to usage in only one partite set. Colors having Type X in the coloring of X will only be used on partite set X; those of Type Y will only be used on Y. Since H was properly 2-colored, each list has colors of both types. If $v\in X$, then we choose a color of Type X from $L(v)$; if $v\in Y$, then we choose a color of Type Y from $L(v)$. Since each color is chosen on only one partite set in G, we have obtained an L-coloring.

8.5.14. *A graph with n vertices and average degree $d\ge 1$ has an independent set with at least $n/(2d)$ vertices.* Note that G has $nd/2$ edges. Let $S\subseteq V(G)$ be generated at random by including each vertex independently with probability p. If S has X vertices and Y edges, then S contains an independent set of size at least $X-Y$, by deleting a vertex of each induced edge. We will choose p to maximize $E(X-Y)$, since there will be an independent set at least that large.

By linearity of expectation, $E(X-Y) = E(X) - E(Y)$. We have $E(X) = np$. Similarly, the probability that a specified edge of G is induced by S is p^2, since both its endpoints must be included, so $E(Y) = p^2nd/2$. Hence $E(X-Y) = np(1-pd/2)$. We choose $p = 1/d$ to maximize this, which is valid since $d \ge 1$, obtaining $E(X-Y) = n/(2d)$.

8.5.15. $ex(n; C_k) \in \Omega(n^{1+1/(k-1)})$. We seek an n-vertex graph with many edges and no k-cycle. We generate a random graph in Model A with some edge probability p. If the expected number $E(Y)$ of k-cycles is much less than the expected number $E(X)$ of edges, then deleting an edge from each k-cycle in some graph where $X-Y$ is large leaves a graph with many edges and no k-cycle.

Given edge probability p, we have $E(X) = \binom{n}{2}p$ and $E(Y) = \binom{n}{k}\frac{1}{2}(k-1)!p^k$. If we can choose p so that $E(Y) < \frac{1}{2}E(X)$, then $\frac{1}{2}E(X)$ will be a lower bound on $ex(n; C_k)$. Using $\binom{n}{k} < n^{k-1}(n-1)/k!$, we have $E(Y) < \frac{1}{2k}(n-1)p(np)^{k-1}$. It suffices to have $\frac{1}{2k}(n-1)p(np)^{k-1}) \le \frac{1}{4}n(n-1)p$, which is implied by $n^{k-2}p^{k-1} \le 1$. Hence we choose $p = n^{(k-2)/(k-1)}$. Now there is a C_k-free graph of size at least $\frac{1}{2}\binom{n}{2}p$, which is asymptotic to $\frac{1}{4}n^{1+1/(k-1)}$.

In the particular case $k = 4$, this lower bound of $\Omega(n^{4/3})$ compares with an upper bound of $O(n^{3/2})$. A graph with m edges contains C_4 if and only if some pair of vertices has two common neighbors. Recall that the counting argument and the convexity of quadratics yield $\binom{n}{2} \ge \sum_{v \in V(G)} \binom{d(v)}{2} \ge n\binom{2m/n}{2}$, and that the resulting quadratic inequality yields $m \le \frac{n}{4}(1+\sqrt{4n-3})$.

8.5.16. $R(k,k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$ *for all* $n \in \mathbb{N}$, *and hence* $R(k,k) > (1/e)(1-o(1))k2^{k/2}$. Generate a random 2-coloring of $E(K_n)$; let X be the resulting number of monochromatic copies of K_k. Each k-set contributes to X with probability $2^{1-\binom{k}{2}}$. Since there are $\binom{n}{k}$ of these sets, $\mathrm{E}(X) = \binom{n}{k}2^{1-\binom{k}{2}}$. Some outcome of the experiment has at most $\mathrm{E}(X)$ bad sets, and deleting a vertex from each such set in such an outcome yields a coloring that establishes the lower bound.

8.5.17. *For* $n \in \mathbb{N}$, *there is a 2-coloring of* $E(K_{m,m})$ *with no monochromatic copy of* $K_{t,t}$ *when* $m = n - \binom{n}{t}^2 2^{1-t^2}$. Generate a random 2-coloring of $E(K_{n,n})$; let X be the resulting number of monochromatic copies of $K_{t,t}$. Each choice of t vertices from each partite set counts with probability 2^{1-t^2}. Since there are $\binom{n}{t}^2$ of these sets, $\mathrm{E}(X) = \binom{n}{t}^2 2^{1-t^2}$. In some outcome of the experiment, X has value at most $\mathrm{E}(X)$, and deleting a vertex (in each partite set) from each monochromatic copy of $K_{t,t}$ in such an outcome yields a coloring that establishes the lower bound.

8.5.18. *Off-diagonal Ramsey numbers.* This problem repeats parts (a) and (b) of Exercise 8.3.20.

8.5.19. *For a fixed graph H and constant edge-probability p, almost every G^p contains H as an induced subgraph.* Let k and l be the number of vertices and edges in H. The probability that a given set of k vertices induces H is $\frac{k!}{A}p^l(1-p)^{\binom{k}{2}-l}$, where A is the number of automorphisms of H; let this probability be q. Since k, l, p, A are all constant, q is a constant. Appearances of H at disjoint sets of vertices are independent. Splitting $[n]$ into n/k disjoint sets, the probability that none of them induce H is $(1-q)^{n/k}$. Since q is constant, this probability tends to 0 as $n \to \infty$.

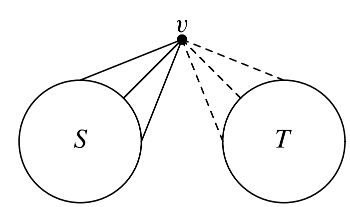
8.5.20. *Common neighbors and nonneighbors.*

a) For constant k, s, t, p, almost every G^p has the following property: for every choice of disjoint vertex sets S and T of sizes s and t, there are at least k vertices that are adjacent to every vertex of S and no vertex of T. Let X be the number of bad choices for S and T in G^p; we need only show that $\mathrm{E}(X) \to 0$. For the ith way to choose $S, T \subseteq V(G)$, define an indicator variable X_i with value 1 when there are fewer than k choices of a vertex v such that $S \subseteq N(v)$ and $T \subseteq \overline{N}(v)$. For $v \notin S \cup T$, failure requires s specified adjacencies and t specified nonadjacencies, so $X_i = 1$ requires more than $n-s-t-k$ failures in $n-s-t$ trials when the failure probability is $p^s(1-p)^t$.

We complete the proof only for $k = 1$; for larger k a binomail tail bound is needed. When $k = 1$, $\mathrm{P}(X_i = 1) = (1-p^s(1-p)^t)^{n-s-t}$. Since $X = \sum X_i$, we count the variables X_i (choices of S, T) by the multinomial coefficient to obtain

$$\mathrm{E}(X) = \binom{n}{s,t,n-s-t}(1-p^s(1-p)^t)^{n-s-t}.$$

For fixed s, t, p, the multinomial coefficient is a polynomial in n. It is bounded by n^{s+t}, while $\mathrm{E}(X_i)$ dies exponentially as $n \to \infty$. The logarithm of the product approaches $-\infty$, and thus $\mathrm{E}(X) \to 0$. ■



b) Almost every G^p is k-connected. If the computation for general k is completed, then it suffices to set $s = 2$ and $t = 0$ to obtain that in almost every graph, every two vertices have k-common neighbors.

c) Almost every tournament has the property that for every choice of disjoint vertex sets S, T of sizes s, t, there are at least k vertices with edges to every vertex of S and from every vertex of T. The argument is essentially the same, using $p = 1/2$ and orienting each edge randomly.

8.5.21. *Random tournaments.*

a) Almost every tournament is strongly connected. This follows by essentially the same computation as part (b): in almost every tournament, for every ordered pair (x, y) of vertices, there is a vertex w such that $x \to w$ and $w \to y$, so every vertex reaches every other. Alternatively, this follows also from the statement of part (b).

b) In almost every tournament, every vertex is a king. The criterion for every vertex being a king is that every vertex be reachable from every other vertex by a path of length at most 2. Let X be the number of ordered pairs of vertices where this fails. For a given pair (x, y) failing to reach y from x by a path of length at most 2 requires that for each other vertex w, the edges xw and wy are not both oriented away from x and toward y. Hence the probability that the ordered pair (x, y) fails is bounded above by $(3/4)^{n-2}$ (the edge xy yields another factor of $1/2$, but this is not important).

Since there are $n(n-1)$ ordered pairs, $E(X) < n^2(3/4)^{n-2}$. The bound tends to 0 as $n \to \infty$, so by Markov's Inequality almost every tournament has no bad pairs and thus has every vertex being a king.

8.5.22. *Edge probability $1/2$ is a sharp threshold for the property that at least half the possible edges of a graph are present.* Let X be the number of edges in G^p. When $p = 1/2$, $E(X) = \frac{1}{2}\binom{n}{2}$. The distribution of X is binomial, and we ask how highly concentrated the distribution is to study the probability of having at least half the edges when we vary p. Although tighter bounds are available, the Chebyshev bound suffices for our purpose. We have $P(|X - E(X)| \ge t) \le V/t^2$, where $V = [E(X^2) - (E(X))^2]$. Direct computation, using the expression of X as a sum of indicator variables, yields $V = Np(1-p)$, where N is the number of trials (here $N = \binom{n}{2}$).

If $p = .5 - \varepsilon$ with ε constant, then $E(X)$ is below $\frac{1}{2}\binom{n}{2}$ by an amount that is quadratic in n. In considering $X \ge \frac{1}{2}\binom{n}{2}$, we are asking for t to be quadratic in n, and the bound on the probability of having at least half the edges tends to 0. Even if we set $p = .5 - c\log n/n$, then still $P(X \ge \frac{1}{2}\binom{n}{2})$ tends to 0. Similarly, if $p = .5 + c\log n/n$, the analogous argument shows that the probability of having at most half the edges goes to 0.

8.5.23. *For $p = 1/n$ and fixed $\varepsilon > 0$, almost every G^p has no component with more than $(1+\varepsilon)n/2$ vertices.* A connected graph with $m+1$ vertices has at least m edges, so it suffices to show that almost every G_p has fewer than $(1+\varepsilon)n/2$ edges. The number of edges is a binomial random

variable; its expectation $\binom{n}{2}p$ equals $(n-1)/2$. The probability that a binomial random variable exceeds its expectation by a constant fraction (here $\varepsilon/2$) is exponentially small in the number of trials. Even so, the weaker Chebyshev bound suffices to show this approaches 0. (We may ignore the .5 in $E(X) = n/2 - .5$ by using a slightly larger choice of ε.) We have $P(|X - E(X)| \ge \varepsilon n/2) \le V/(\varepsilon n/2)^2$, where $V = [E(X^2) - (E(X))^2]$. The expectation of X^2 for $\binom{n}{2}$ independent trials is found by

$$E(X^2) = E(X) + \binom{n}{2}\left(\binom{n}{2} - 1\right)p^2 \sim E(X)^2 = \left(\binom{n}{2}p\right)^2$$

Thus $V = o(n^4p^2) = o(n^2)$. Since the denominator is $\Omega(n^2)$, the ratio bounding the probability approaches 0.

8.5.24. *The smallest connected simple graph that is not balanced is the 5-vertex graph consisting of a kite plus a pendant edge.* If G is unbalanced, then some induced subgraph has larger average degree than G. For the smallest such graph G, we obtain the offending subgraph by deleting one vertex. In Exercise 1.3.44a, we showed that the average degree increases when a vertex x is deleted from an n-vertex graph with average degree a if and only if $d(x) < a/2$ (since $\frac{2e(G-x)}{n-1} = \frac{2[e(G)-d(x)]}{n-1} = \frac{na-2d(x)}{n-1}$).

Since G is connected, every degree is positive. Hence the smallest example will occur by deleting a leaf from a graph with average degree exceeding 2. For average degree exceeding 2, at least four vertices are needed. With four vertices, no graph with at least five edges has a leaf. With five vertices, we need at least six edges in a graph obtained by appending a leaf to a 4-vertex graph with at least five edges. A kite with a pendant edge has this property.

8.5.25. *In terms of the number n of vertices, $n^{-1/\rho(H)}$ is a threshold probability function for the appearance of H as a subgraph of G^p, where $\rho(G) = \max_{G\subseteq H} e(G)/n(G)$.* We extend the second moment argument of Theorem 8.5.23. Let F be a subgraph of H with density $\rho(H)$. This subgraph F is balanced, and the first moment argument in Theorem 8.5.23 shows that if $pn^{\rho(H)} \to 0$, then almost every G^p has no copy of F and hence no copy of H.

To show that $n^{-1/\rho(H)}$ is a threshold probability function for the appearance of H, we must also show that $pn^{\rho(H)} \to \infty$ implies that almost every G^p has a copy of H. An easy modification of the second moment argument in Theorem 8.5.23 (due by Ruciński and Vince) completes the proof.

Let X be the random variable counting the copies of H. We follow the same argument as for balanced graphs, and it suffices to prove that $E(X^2) -> E(X)^2$ when $pn^{1/rho} \to \infty$. The proof of this is the same as for the balanced case Theorem 8.5.23 up to point in the last paragraph where the balance condition is invoked. Replace that portion with the following:

"The desired behavior of $n^{-r}p^{-s}$ is equivalent to $pn^{r/s} \to \infty$. Since s/r is the density of H', we have $s/r \le \rho$. This forces $pn^{r/s} \ge pn^{1/rho} \to \infty$ when $c > 0$."

8.5.26. *Almost every graph (with edge probability p) has the property that for every choice of disjoint vertex sets S, T of size $c\log_{1/(1-p)} n$ with $c > 2$), there is an edge with endpoints in S and T.* (For $p = 1/2$, the formula reduces to $c\lg n$.)

Let X be the number of choices of disjoint sets S and T of this size with no edge between them. By Markov's Inequality, it suffices to show that $E(X) \to 0$ when $c > 2$), because then the probability that Q_k occurs tends to 1. We have $E(X) \sim \frac{1}{k!^2}n^{2k}(1-p)^{k^2}$. Writing this as $c'(n^2(1-p)^k)^k)$, it suffices to have $n^2(1-p)^k < 1$. This requires $k > 2\log_{1/(1-p)} n$. Thus it suffices to choose $c > 2$ in the expression for $|S|$ and $|T|$.

8.5.27. *If $k = \lg n - (2+\varepsilon)\lg\lg n$, then almost every n-vertex tournament has the property that every set of k vertices has a common successor.* The probability that a k-set fails to have a common successor is $(1-2^{-k})^{n-k}$, since this requires that each vertex outside the set is not a common successor. Let X be the number of k-sets with no common successor; we have $\mathrm{E}(X) = \binom{n}{k}(1-2^{-k})^{n-k}$. An upper bound on $\mathrm{E}(X)$ is $\left(\frac{ne}{k}\right)^k e^{-2^{-k}(n-k)}$. If this bound tends to 0 for some choice of k in terms of n, then almost every tournament has the property for this choice of k.

To suggests the appropriate k, we choose k so that $\left(\frac{ne}{k}\right)^k$ grows more slowly than $e^{2^{-k}(n-k)}$. Taking natural logarithms, we want $k(1+\ln n-\ln k) < 2^{-k}(n-k)$. Now taking base-2 logarithms, we want

$$\lg k + \lg\ln n + \lg(1 - \tfrac{(\ln k)-1}{\ln n}) < -k + \lg n + \lg(1 - \tfrac{k}{n}).$$

Roughly speaking, we want $k + \lg k < \lg n - \lg\ln n$. Thus k should be enough less than $\lg n$ that adding $\lg k$ still keeps the value less than $\lg n - \lg\ln n$. Converting from $\ln n$ to $\lg n$ on the right only introduces an additive constant, since the $\ln n$ is inside lg. The ε in the definition of k is more than enough to take care of that.

Setting k as specified above yields $\mathrm{E}(X) \to 0$, and the property almost always holds.

8.5.28. *Transitive subtournaments.*

Every n-vertex tournament has a transitive subtournament with $\lg n$ vertices. We prove by induction on n that every n-vertex tournament has a transitive subtournament with at least $1 + \lfloor \lg n \rfloor$ vertices. The statement holds trivially for $n = 1$.

When $n > 1$, a vertex x with maximum outdegree has outdegree at least $\lfloor n/2 \rfloor$. In the subtournament induced by the successors of x, the induction hypothesis yields a transitive subtournament with at least $1 + \lfloor \lg(\lfloor n/2 \rfloor) \rfloor$ vertices. This equals $\lfloor \lg n \rfloor$ in all cases. Adding x produces a transitive tournament of order $1 + \lfloor \lg n \rfloor$ in the original tournament.

For $c > 1$, almost every tournament has no transitive subtournament with more than $2\lg n + c$ vertices. In the random tournament (each edge is directed toward the lower vertex with probability $1/2$), let X be the number of transitive subtournaments of order k. For each set of k vertices, the possible transitive tournaments correspond to the $k!$ linear orderings of the vertices. Hence $E(X) = \binom{n}{k}k!2^{-\binom{k}{2}}$.

It suffices to show that $E(X) \to 0$ (equivalently, $\lg E(X) \to -\infty$) when k exceeds $1 + 2\lg n$ by any constant. We have $E(X) < n^k 2^{-\binom{k}{2}}$, which we write as $\lg E(X) < k[\lg n - (k-1)/2]$. This bound is a decreasing function of k if $\lg n - (k-1)/2 < 0$. Therefore, we obtain a valid upper bound on $\lg E(X)$ for all larger k if we set $k = 1 + \varepsilon + 2\lg n$.

For further simplification, let $m = 1 + \lg n$, so $k = 2m - 1 + \varepsilon$. The final computation is

$$\lg E(X) < (2m-1+\varepsilon)[m-1-(m-1+\tfrac{\varepsilon}{2})] = (2m-1+\varepsilon)(\tfrac{-\varepsilon}{2}) \to -\infty.$$

8.5.29. *Geometric random variable / Coupon Collector.*

a) Under repeated trials of an experiment with success probability p on each trial, independently, the expected number of the trial when the first success occurs is $1/p$. Let X be the random variable for the trial on which the first success occurs.

Proof 1 (computation). The probability of $X = k$ is $(1-p)^{k-1}p$. By the definition of expectation, behavior of geometric series, and differentiation of convergent series,

$$\mathrm{E}(X) = \sum_{k=1}^{\infty} kp(1-p)^{k-1} = p\frac{d}{dp}\sum_{k=0}^{\infty} -(1-p)^k = p\frac{d}{dp}\frac{-1}{1-(1-p)} = p\frac{1}{p^2} = \frac{1}{p}.$$

Proof 2 (conditional expectation). Let $z = \mathrm{E}(X)$. If the first trial is a failure, then the remainder of the experiment is a repetition of the original experiment. Hence $z = p\cdot 1 + (1-p)\cdot(1+z)$, which yields $pz = p + (1-p) = 1$, so $z = 1/p$.

Proof 3 (linearity). Let X_i be the event that the first i trials fail. Now $X = 1 + \sum_{i=1}^{\infty} X_i$, and $\mathrm{E}(X) = 1 + \sum_{i=1}^{\infty} \mathrm{P}(X_i = 1) = \sum_{i=0}^{\infty}(1-p)^i = 1/p$.

b) Given independent trials with n outcomes, each with probability $1/n$, the expected number of trials to obtain all outcomes is $n\sum_{i=1}^{n} 1/i$. Let X be the number of trials taken to obtain all outcomes. Let X_i be the number of trials after $i-1$ of the outcomes have been obtained, up to and including the trial on which for the first time i outcomes have been obtained. We have $X = \sum_{i=1}^{n} X_i$. When we have obtained $i-1$ of the outcomes, a given trial provides a new outcome with probability $\frac{n-i+1}{n}$. Hence the variable X_i is a geometric random variable with success probability $\frac{n-i+1}{n}$. Using part (a) and linearity of expectation, and reversing the order of summation and letting $j = n+i-i$, we obtain $E(X) = n\sum_{j=1}^{n} 1/j$.

c) A threshold function $m(n)$ for the number of boxes needed to obtain more than k copies of each prize is given by $m(n) = n\ln n + kn\ln\ln n$. Let X be the number of target points hit at most k times. For each $r \in [n]$, the probability that $f^{-1}(r)$ has size j is $\binom{m}{j}p^j(1-p)^{m-j}$, where $p = 1/n$. The probability that $f^{-1}(r)$ has size at most k is the summation of this up to $j = k$. Let this probability be b; thus $E(X) = nb$.

We claim that the contribution to b from terms with $j < k$ is of lower order than the term when $j = k$. Let $\alpha = (1-p)^{m-k+1}$. We bound the sum by a multiple of a geometric sum. If $mp \to \infty$, this yields

$$b = \sum_{j=0}^{k-1}\binom{m}{j}p^j(1-p)^{m-j} \le \alpha\sum_{j=0}^{k-1}(mp)^i = \alpha\frac{(mp)^k-1}{mp-1} \sim \alpha(mp)^{k-1}$$

On the other hand, $\binom{m}{k}p^k(1-p)^{m-k}$ is bounded below by a constant times $\alpha(mp)^k$. Hence $mp \to \infty$ and k constant yields $b \sim \binom{m}{k}p^k(1-p)^{m-k}$.

We want to choose $m(n)$ so that nb approaches 0 or ∞, depending on the choice of a parameter in $m(n)$. Since k is constant, $(1-p)^k \to 1$ and the binomial coefficient in the top term is asymptotic to $m^k/k!$. Thus $b \sim \frac{1}{k!}m^k p^k(1-p)^m$. Also $np^2 \to 0$, so $1-p$ is asymptotic to e^{-p}.

With $m(n) = n\ln n + cn\ln\ln n$, we have $mp = \ln n + c\ln\ln n \sim \ln n$ and $e^{mp} = n(\ln n)^c$. We now compute

$$E(X) = nb \sim n\frac{(mp)^k}{k!e^{mp}} \sim n\frac{(\ln n)^k}{k!n(\ln n)^c} = \frac{1}{k!}(\ln n)^{k-c}.$$

If $c > k+\varepsilon$, then $E(X) \to 0$, and almost always every target point is hit more than k times. If $c < k-\varepsilon$, then $E(X) \to \infty$. The Second Moment Method then will imply that almost always some target point is hit at most k times if we prove that $E(X^2) \sim E(X)^2$.

Let $X = \sum_{r=1}^{n} X_r$, where X_r is the event that $\left|f^{-1}(r)\right| \le k$. The probability that $\left|f^{-1}(r)\right| = i$ and $\left|f^{-1}(s)\right| = j$ is $\binom{m}{i,j,m-i-j}p^i p^j(1-2p)^{m-i-j}$, from the multinomial distribution. Thus sum of this over i, j both at most k equals $E(X_rX_s)$. Again because m grows while k is fixed, the sum is asymptotic to the single term with $i = j = k$. Here the multinomial coefficient is asymptotic to $m^{2k}/(k!k!)$. With the other approximations as above, we have

$$E(X^2) = E(X) + \sum_{r<s} E(X_rX_s) \sim E(X) + n^2b^2 \sim E(X)^2.$$

8.5.30. *The length of the longest constant run in a list of n random heads and tails is $(1+o(1))\lg n$.* Let X be the number of runs of length k in a random list of n flips. A set of k consecutive flips agrees with probability $2\cdot 2^{-k}$. There are $n-k+1$ such sets. Hence $\mathrm{E}(X) = (n-k+1)2^{-k+1}$. Let ε be a fixed small positive constant.

If $k \ge (1+\varepsilon)\lg n$, then $\mathrm{E}(X) \to 0$, which implies that almost every list has no run as long as $(1+\varepsilon)\lg n$.

If $k \le (1-\varepsilon)\lg n$, then $\mathrm{E}(X) \to \infty$. If also $\mathrm{E}(X^2) \to \mathrm{E}(X)^2$, then by the Second Moment Method $\mathrm{P}(X = 0) \to 0$, which implies that almost every list has at least one run as long as $(1-\varepsilon)\lg n$. Since X is the sum of $n-k+1$ indicator variables, we have $\mathrm{E}(X^2) = \mathrm{E}(X) + \sum_{i\ne j} X_iX_j$. When the locations corresponding to X_i and X_j are disjoint, the events $X_i = 1$ and $X_j = 1$ are independent. When they overlap, the probability that both are 1 is bounded by 2^{2k-2}.

The essence of the computation is that almost all of the expectation comes from independent events. When the segments overlap, their starting points differ by less than k. There are at most $2(n-j)$ ordered pairs where the difference in the starting locations is j. As j varies, fewer than $2nk$ ordered pairs of k-segments are overlapping. Hence at least $(n-k+1)(n-k) - 2nk$ ordered pairs of variables satisfy $\mathrm{P}(X_iX_j) = 2^{2(-k+1)}$.

Since we only need the leading behavior of $\mathrm{E}(X^2)$, we compute $\mathrm{E}(X^2) = n2^{-k+1} + n^2 2^{2(-k+1)} - 4nk2^{2(-k+1)} + O(nk) \sim \mathrm{E}(X)^2$.

Comment: This side of the threshold can also be derived by the first moment method. Using $\lfloor n/k\rfloor$ disjoint segments, where $k = (1-\varepsilon)\lg n$, the constancy of these segments are independent events. Each occurs with probability $2\cdot 2^{-k}$, so the probability p that none occurs is $(1-2\cdot 2^{-k})^{\lfloor n/k\rfloor}$. We have

$$p \le (1-2/n^{1-\varepsilon})^{\lfloor n/k\rfloor} < e^{-2/n^{1-\varepsilon}(2n/k)} = e^{-(4/k)n^{\varepsilon}} \to 0,$$

so almost every sequence has a run at least this long.

8.5.31. *With $p = (1-\varepsilon)\log n/n$, almost every graph has at least $(1-o(1))n^{\varepsilon}$ isolated vertices.* Let X be the random variable counting the isolated vertices; we have $\mathrm{E}(X) = n(1-p)^{n-1}$. Let m be a desired threshold, with $m < \mathrm{E}(X)$. By Chebyshev's Inequality,

$$P(X < m) = P(X - \mathrm{E}(X) < m - \mathrm{E}(X)) < P(|X - \mathrm{E}(X)| \ge \mathrm{E}(X) - m) \le \frac{\mathrm{E}(X^2) - \mathrm{E}(X)^2}{(\mathrm{E}(X) - m)^2}.$$

We have $X \ge m$ almost always if m is chosen so this bound approaches 0.

If $p = (1-\varepsilon)\log n/n$ for constant ε, then $\mathrm{E}(X) \sim n^{\varepsilon}$. We have also computed $\mathrm{E}(X^2) \sim n^{2\varepsilon}$; this was what was required of the second moment method to obtain the threshold for disappearance of isolated vertices. Hence $\mathrm{E}(X^2) - \mathrm{E}(X)^2 \in o(\mathrm{E}(X)^2)$, and we may choose $m = (1-\delta)\mathrm{E}(X) =$

$(1-\delta)n^{eps}$ for any $\delta > 0$. It is possible to make m closer to $\mathrm{E}(X)$, but this requires more accurate estimates of $\mathrm{E}(X^2)$ and $\mathrm{E}(X)$, since the leading behavior cancels when $\mathrm{E}(X)^2$ is subtracted from $\mathrm{E}(X^2)$.

8.5.32. *The threshold size k for bad k-sets in G^p, where p is fixed and a k-set S is bad if its vertices have no common neighbor, is $\log_{1/p}\frac{n}{c\ln n}$ with the parameter $c = 1$.* This scenario is obtained from that of Exercise 8.5.20 by setting $t = 0$, redefining s as k, and turning that k into 0, except that $|S|$ is no longer fixed; we seek a threshold. Below the threshold ($c < 1$), k-sets are small enough and leave enough vertices outside so that almost always every k-set has a common neighbor.

Let X be the number of bad k-sets. A vertex v outside a k-set S fails to be a common neighbor with probability $1-p^k$. The probability of having no common neighbor is $(1-p^k)^{n-k}$, so $\mathrm{E}(X) = n(1-p^k)^{n-k} < ne^{-p^k(n-k)}$. If $p^k(n-k) \sim c\ln n$ with $c = 1+\varepsilon$, then $\mathrm{E}(X) \to 0$. Hence we set $p^k = \frac{c\ln n}{n}$, which translates to $k = \log_{1/p}\frac{n}{c\ln n}$.

Since $p^k \to 0$, $1-p^k \sim e^{-p^k}$, and hence $\mathrm{E}(X) \to \infty$ when $k = \log_{1/p}\frac{n}{c\ln n}$ with $c = 1-\varepsilon$. Now the second moment method can be used to show that $\mathrm{P}(X > 0) \to 1$. We need to show $\mathrm{E}(X^2) \sim \mathrm{E}(X)^2$.

Consider the indicator variables for individual k-sets. If X_1 and X_2 correspond to disjoint k-sets, then $\mathrm{E}(X_iX_j) = \mathrm{E}(X_i)^2$. The number of ordered pairs of this sort is the multinomial coefficient $\binom{n}{k,k,n-2k}$, which is asymptotic to $n^{2k}/k!^2$. The terms that come from overlapping k-sets are fewer; the number of them is bounded by a multiple of n^{2k-1}. Since that k in the exponent grows with $\ln n$, there remains work to do, but the idea in the second moment method here is to show that asymptotically all the contribution to $\mathrm{E}(X^2)$ comes from terms that sum to roughly $\mathrm{E}(X)^2$.

8.5.33. *If p is fixed and $k = k(n) \in o(n/\log n)$, then almost every G^p is k-connected.* (sketch) It suffices to show that almost every G^p has the property that any two vertices have k common neighbors. We consider the expected number of vertices failing this. Two vertices fail this with probability $b(n-2, p^2, k-1)$, where $b(m,q,l)$ is the probability of having at most l successes in m independent trials with success probability q.

When $l = o(m)$, $b(m,q,l)$ is bounded by a multiple of the top term in the sum, $\binom{m}{l}q^l(1-q)^{m-l}$ (proof omitted). Applying this enables us to show that $\binom{n}{2}b(n-2,p^2,k-1) \to 0$ when $k \in o(n/\log n)$.

8.5.34. This duplicates Exercise 8.5.31.

8.5.35. A *t-interval* is a subset of $\mathbb{R}$ that is the union of at most t intervals. The *interval number* of a graph G is the minimum t such that G is an intersection graph of t-intervals (each vertex is assigned a set that is the union of at most t intervals). Prove that almost all graphs (edgeprobability

1/2) have interval number at least $(1-o(1))n/(4\lg n)$. (Hint: Compare the number of representations with the number of simple graphs. Comment: Scheinerman [1990] showed that almost all graphs have interval number $(1+o(1))n/(2\lg n)$.) (Erdős–West [1985])

8.5.36. *Threshold for complete matching in random bipartite graph.* Let G be a random labeled subgraph of $K_{n,n}$, with partite sets A, B and independent edge probability $p = (1+\varepsilon)\ln n/n$. Call S a *violated set* if $|N(S)| < |S|$.

a) If $\varepsilon < 0$, then almost surely G has no complete matching. Although the probability that no vertex in A is isolated is $[1-(1-p)^n]^n$, it is not easy to show that this approaches 0 when $\varepsilon < 0$.

If X is the number of isolated vertices in A, then $\mathrm{E}(X) = n(1-p)^n$. With $p = o(1/\sqrt{n})$, this yields $\mathrm{E}(X) \sim ne^{-np} = n^{-\varepsilon}$. Hence $\mathrm{E}(X) \to \infty$ if $\varepsilon < 0$. Because the 0,1-random variables X_i that contribute to X are independent, we have $(\mathrm{E}(X))^2 = \mathrm{E}(X) + n(n-1)\mathrm{E}(X_iX_j) \sim n^2(1-p)^{2n} = \mathrm{E}(X)^2$, and the second moment method yields the claim.

b) If S is a minimal violated set, then $|N(S)| = |S|-1$ and $G[S\cup N(S)]$ is connected. If $|N(S)| < |S|-1$, then $S-x$ is a violated set, for any $x \in S$. If $G[S\cup N(S)]$ is not connected, let $\{S_i\}$ be the partition of S induced by its components. Then $\cup N(S_i) = N(S)$, and $\{N(S_i)\}$ are disjoint, so by the pigeonhole principle some S_i is a violated set.

c) If G has no complete matching, then A or B contains a violated set with at most $\lceil n/2\rceil$ elements. If S is a violated subset of A, then $B-N(S)$ is a violated subset of B. If S is a minimal violated subset of A with more than $n/2$ elements, then $|B-N(S)| \le n-(|S|-1) \le \lceil n/2\rceil$.

d) If $\varepsilon > 0$, then almost surely G has a complete matching. Let X be the number of spanning trees in subgraphs of the form $G[S\cup N(S)]$, where S is a minimal violated subset of A or B having size at most $\lceil n/2\rceil$. By parts (b) and (c), it suffices to show that $\mathrm{E}(X) \to 0$. The number of spanning subtrees of $K_{r,s}$ is $r^{s-1}s^{r-1}$ if $r, s \ge 1$ (see Exercise 2.2.14). Separating the term due to violated sets of size 1, we have

$$\mathrm{E}(X) = 2n(1-p)^n + 2\sum_{k=2}^{\lceil n/2\rceil}\binom{n}{k}\binom{n}{k-1}k^{k-2}(k-1)^{k-1}p^{2k-2}(1-p)^{k(n-k+1)}.$$

As seen in part (a), the expected number of isolated vertices approaches 0. Since $\binom{m}{l} < (me/l)^l$, the summation is bounded by $\sum_{k=2}^{\lceil n/2\rceil}(ne)^{2k-1}k^{-2}p^{2k-2}(1-p)^{k(n+1)/2}$. We can ignore the k^{-2} to obtain an upper bound of $(p^2ne)^{-1}\sum_{k\ge 2}[(pne)^2(1-p)^{(n+1)/2}]^k$. The constant ratio in the geometric series is asymptotic to $[e(1+\varepsilon)\ln n]^2 n^{-(1+\varepsilon)/2}$, which approaches 0 for $\varepsilon > -1$. The geometric series is bounded by $x^2/(1-x)$, which is asymptotic to x^2 when $x \to 0$. For the bound on the summation,

$$(p^2ne)^{-1}(pne)^4(1-p)^{n+1} \sim p^2n^3e^3e^{-np} = e^3(1+\varepsilon)^2(\ln n)^2 n^{-\varepsilon} \to 0.$$

8.5.37. *If* $0 < p < 1$*, and* $k_1, \dots, k_r$ *are nonnegative integers summing to* m*, then* $\prod_{i=1}^r [1-(1-p)^{k_i}] \le [1-(1-p)^{m/r}]^r$. Since the logarithm function is monotone, it suffices to show that $\sum_{i=1}^r \ln[1-(1-p)^{k_i}] \le r\ln[1-(1-p)^{m/r}]$. This reduces to $\frac{1}{r}\sum_{i=1}^r f(k_i) \le f\left(\frac{1}{r}\sum_{i=1}^r k_i\right)$, where $f(x) = \ln[1-(1-p)^x]$. That is, it suffices to show that f is a concave function for $x \ge 0$.

Rewriting $(1-p)^x$ as $e^{x\ln(1-p)}$ makes it easy to differentiate f twice. The value of the second derivative is $-\left(\frac{\ln(1-p)}{(1-p)^{-x}-1}\right)^2$, which is negative. Hence f is concave.

8.5.38. ($\bullet$) *Tail inequality for binomial distribution.* Let $X = \sum X_i'$, where each X_i' is an indicator variable with success probability $P(X_i' = 1) = .5$, so $\mathrm{E}(X) = n/2$. Applying Markov's Inequality to the random variable $Z = (X - \mathrm{E}(X))^2$ yields $P(|Z| \ge t) \le Var(X)/t^2$. Setting $t = \alpha\sqrt{n}$ yields a bound on the tail probability: $P(|X - np| \ge \alpha\sqrt{n}) \le 1/(2\alpha^2)$. Use Azuma's Inequality to prove the stronger bound that $P(|X-np| > \alpha\sqrt{n}) < 2e^{-2\alpha^2}$. (Hint: Let $Y_i' = X_i' - .5$. Let F_i be the knowledge of $Y_1', \dots, Y_i'$, and let $Y_i = \mathrm{E}(Y|F_i)$.)

8.5.39. ($\bullet$) *Bin-packing.* Let the numbers $S = \{a_1, \dots, a_n\}$ be drawn uniformly and independently from the interval $[0,1]$. The numbers must be placed in bins, each having capacity 1. Let X be the number of bins needed. Use Lemma 8.5.36 to prove that $P\big(|X - \mathrm{E}(X)| \ge \lambda\sqrt{n}\big) \le 2e^{-\lambda^2/2}$.

8.5.40. *Azuma's Inequality and the Traveling Salesman Problem.*

a) Azuma's Inequality for general martingales: If $\mathrm{E}(X_i) = X_{i-1}$ *and* $|X_i - X_{i-1}| \le c_i$ *for all* i*, then* $\mathrm{P}(X_n - X_0 \ge \lambda\sqrt{\sum c_i^2}) \le e^{-\lambda^2/2}$. Let $\gamma = \sqrt{\sum c_i^2}$. By translation, we may assume $X_0 = 0$. Markov's Inequality implies $\mathrm{P}(e^{tX_n} \ge e^{t\lambda\gamma}) \le \mathrm{E}(e^{tX_n})/e^{t\lambda\gamma}$ for all $t > 0$. It suffices to prove that $\mathrm{E}(e^{tX_n}) \le e^{t^2\gamma^2/2}$ and then set $t = \lambda/\gamma$. We prove the bound on the expectation by induction on n.

We have

$$\mathrm{E}(e^{tX_n}) = \mathrm{E}(e^{tX_{n-1}}e^{t(X_n - X_{n-1})}) = \mathrm{E}(\mathrm{E}(e^{tX_{n-1}}e^{t(X_n-X_{n-1})}|X_{n-1})) = \mathrm{E}(e^{tX_{n-1}}\mathrm{E}(e^{tY}|X_{n-1})),$$

where $Y = X_n - X_{n-1}$. By hypothesis, $\mathrm{E}(Y) = 0$ and $|Y| \le c_n$. Let Z be the random variable Y/c_n, so $|Z| \le 1$, and let $u = c_n t$. We have $\mathrm{E}(e^{tY}) = \mathrm{E}(e^{uZ}) \le \frac{1}{2}(e^u + e^{-u}) \le e^{u^2/2}$, so the inner expectation is bounded by $e^{t^2c_n^2/2}$. This is a constant, so

$$\mathrm{E}(e^{tX_{n-1}}\mathrm{E}(e^{tY}|X_{n-1})) = e^{t^2c_n^2/2}\mathrm{E}(e^{tX_{n-1}}) = e^{t^2\gamma^2/2},$$

using the induction hypothesis.

b) If Y *is the distance from* $z \in S$ *to the nearest of* n *points chosen uniformly and independently in the unit square* S*, then* $\mathrm{E}(Y) < c/\sqrt{n}$*, for*

some constant c. The probability that a random point x lies in region R equals the area of R. Fixing y, z, $\mathrm{P}(d(x,z) > y) \le \pi y^2/4$, with equality when z is in the corner. Hence the probability that the nearest of n points is farther than y from z is bounded by $(1-\pi y^2/4)^n$. Since $\mathrm{E}(Y) = \int_0^\infty \mathrm{P}(Y \ge y)dy$, we have $\mathrm{E}(Y) \le \int_0^\infty (1-\pi y^2/4)^n dy < \int_0^\infty e^{-n\pi y^2/4}dy = 1/\sqrt{n}$.

c) The smallest length of a tour through a random set of n *points in the unit square is highly concentrated around its expectation.* Let X be the actual length of the optimal tour for the random points $Q = \{p_1, \dots, p_n\}$. Let $F_i = \{p_1, \dots, p_i\}$, and let $X_i = \mathrm{E}(X|F_i)$, so $\{X_i\}$ is a martingale with $X_0 = \mathrm{E}(X)$ and $X_n = X$. Let W be the length of the optimal tour when p_i is omitted from the set. Let $Y = \mathrm{E}(W|F_{i-1})$ and $Y' = \mathrm{E}(W|F_i)$; note that $Y' = Y$, because p_i does not appear in the tour measured by W. Fixing F_i, the expectation of $X - W$ is bounded by $2/\sqrt{(n-i)}$, by part (b), since we can include p_i in the tour by making a detour from the closest point in $Q - F_i$. Hence $0 \le \mathrm{E}(X - W|F_i) \le 2/\sqrt{n} - i$. Since this bound of $2/\sqrt{n} - i$ is valid for any choice of p_i, we also have the same bounds for $\mathrm{E}(X - W|F_{i-1})$. However, $\mathrm{E}(X - W|F_i) = X_i - Y'$, and $\mathrm{E}(X - W|F_{i-1}) = X_{i-1} - Y$. With $Y = Y'$, we have $X_i - X_{i-1} = \mathrm{E}(X - W|F_i) - \mathrm{E}(X - W|F_{i-1})$, and our bounds on these quantities yield $|X_i - X_{i-1}| \le 2/\sqrt{n} - i$. Since the sum of the reciprocals of the first n natural numbers is asymptotic to $\ln n$, part (a) yields $P[X - \mathrm{E}(X) \ge \lambda(2+\varepsilon)\sqrt{\ln n}] \le e^{-\lambda^2/2}$. The same bound holds for the other tail, because part (a) applies also to $-X$.

8.6. EIGENVALUES OF GRAPHS

8.6.1. *Interpretation of cycle space and bond space.* Consider a graph G.

a) The symmetric difference of two even subgraphs G_1 *and* G_2 *is an even subgraph.* At a given vertex v, let S_1 and S_2 be the sets of incident edges in G_1 and G_2, respectively. In $G_1 \triangle G_2$, the edges incident to v are $S_1 \triangle S_2$. We have $|S_1 \triangle S_2| = |S_1 - (S_1 \cap S_2)| + |S_2 - (S_1 \cap S_2)|$. Since $|S_1|$ and $|S_2|$ have the same parity, the sizes of the differences also have the same parity.

b) The symmetric difference of two edge cuts is an edge cut. This is Exercise 4.1.27.

c) Every edge cut shares an even number of edges with every even subgraph. Every even subgraph decomposes into cycles (Proposition 1.2.27), and every cycle crosses every edge cut an even number of times.

Comment: By parts (a) and (b), the cycle space **C** and bond space **B** of a graph G are binary vector spaces. They are subpaces of the space of dimension $e(G)$ whose vectors are the incidence vectors of subsets of the edges. By part (c), they are orthogonal subspaces of this space, since the dot product of the incidence vector of an even subgraph and the incidence vector of an edge cut is 0.

8.6.2. *For a connected graph G with n vertices and m edges, the cycle space* $\mathbf{C}$ *has dimension $m-n+1$, and the bond space* $\mathbf{B}$ *has dimension $n-1$.*

Since the spaces are orthogonal within a space of dimension m, it suffices to show that $\dim \mathbf{C} \ge m-n+1$ and $\dim \mathbf{B} \ge n-1$.

Choose a spanning tree T. Each edge of $E(G)-E(T)$ forms a unique cycle along with edges in T. These cycles are linearly independent in $\mathbf{C}(G)$, since each has a nonzero coordinate outside $E(T)$ that is zero in all other incidence vectors in this set. Hence $\dim \mathbf{C}(G) \ge m-n+1$.

Choose $n-1$ vertices in G. We show that the edge cuts isolating these vertices are linearly independent in $\mathbf{B}$. A nonzero linear combination over $\mathbb{F}_2$ sums a nonempty subset S of these incidence vectors. The resulting coordinate for an edge is even if and only if the edge has an even number of endpoints in S. Edges in $[S,\overline{S}]$ are covered exactly once, and these coordinates remain nonzero. Since S is a nonempty proper subset of the vertices of a connected graph, $[S,\overline{S}] \ne \varnothing$. Hence these vectors are linearly independent, and $\dim \mathbf{B} \ge n-1$.

8.6.3. *a) If in a simple graph G the vertices in $S \subseteq V(G)$ have identical neighborhoods, then 0 is an eigenvalue with multiplicity at least $|S|-1$.* The rows of the adjacency matrix A corresponding to vertices of S are identical. Hence there are at most $n(G)-|S|+1$ linearly independent rows, and the rank is at most $n(G)-|S|+1$, so at least $|S|-1$ eigenvalues are 0.

a) If in a simple graph G the vertices in $S \subseteq V(G)$ have identical closed neighborhoods, then -1 is an eigenvalue with multiplicity at least $|S|-1$. The rows of $A+I$ corresponding to vertices of S are identical, where I is the identity matrix. Hence $A+I$ has 0 as an eigenvalue with multiplicity at least $|S|-1$. However, the spectrum of $A+I$ is shifted up from the spectrum of A by 1, since adding λI to a matrix adds λ to each eigenvalue.

8.6.4. *Counting 3-cycles and 4-cycles using eigenvalues.* Let A be the adjacency matrix of G. Let σ_k be the number of subgraphs of G that are k-cycles. Let L_k and D_k be the sums of the kth powers of the eigenvalues and the vertex degrees, respectively. We have $L_k = \mathrm{Trace} A^k$, by Remark 8.6.2(1) and Proposition 8.6.7. Proposition 8.6.7 also implies that L_k counts the ways to start at a vertex, follow a walk of length k, and end at the same vertex.

$\sigma_3 = \frac{1}{6}L_3$. Every closed walk of length 3 traverses a 3-cycle, and there are six ways to traverse a 3-cycle in in three steps.

$\sigma_4 = \frac{1}{8}L_4 - \frac{1}{4}D_2 - \frac{3}{4}D_1$. A closed walk of length 4 may traverse a 4-cycle, or a path of length 2 (starting at either end or starting in the middle in either direction), or an edge (starting at either end). Hence L_4 counts

the copies of C_4 eight times, the copies of P_3 four times, and the copies of P_2 twice. There are σ_4 copies of C_4, there are $\sum_{v\in V(G)} \binom{d(v)}{2}$ copies of P_3, and there are $\sum_{v\in V(G)} d(v)/2$ copies of P_2. Thus $L_4 = 8\sigma_4 + 2D_2 + 2D_1 + D_1$.

8.6.5. *Deletion formulas for the characteristic polynomial.* We write $\phi(G;\lambda)$ as ϕ_G. For a vertex or edge w in G, let $Z(w)$ denote the set of cycles in G containing w. We use Sachs' formula for the characteristic polynomial: $\phi_G = \sum c_i\lambda^{n-i}$, where $c_i = \sum_{H\in\mathbf{H}_i}(-1)^{k(H)}2^{s(H)}$, where $\mathbf{H}_i$ is the set of i-vertex subgraphs of $V(G)$ whose components are edges or cycles, and $k(H)$ and $s(H)$ denote the number of components and number of cycles of H.

a) $\phi_G = \lambda\phi_{G-v} - \sum_{u\in N(v)} \phi_{G-v-u} - 2\sum_{C\in Z(v)} \phi_{G-V(C)}$. Consider the contributions to ϕ_G in Sachs' formula. The subgraphs avoiding v contribute $\lambda\phi G-v$, since these subgraphs are present in both G and $G-v$, but in G the term where they contribute has an extra factor of λ. The subgraphs having a component that is an edge uv contribute $-\phi_{G-v-u}$, since these subgraphs correspond to subgraphs of $G-u-v$ by adding one component that is an edge. The subgraphs having a cycle through v contribute $-2\sum_{C\in Z(v)} \phi_{G-V(C)}$, since these subgraphs correspond to subgraphs of G by adding the vertex set of the cycle, which adds one component that is a cycle and therefore contributes a factor of 2.

b) $\phi_G = \phi_{G-xy} - \phi_{G-x-y} - 2\sum_{C\in Z(xy)} \phi_{G-V(C)}$. Consider the contributions to ϕ_G in Sachs' formula. The subgraphs avoiding xy contribute $\lambda\phi G-v$, since these subgraphs are present in both G and $G-xy$ and the number of vertices used in computing the exponent is the same. The subgraphs having xy as a component contribute $-\phi_{G-x-y}$, since these subgraphs correspond to subgraphs of $G-x-y$ by adding one component that is an edge. The subgraphs having a cycle through xy contribute $-2\sum_{C\in Z(v)} \phi_{G-V(C)}$, by the same reasoning as for the last term in part (a).

8.6.6. *Characteristic polynomial for paths and cycles.*

a) Recurrence. Using Exercise 8.6.5, deleting an endpoint of P_n yields $\phi_{P_n} = \lambda\phi_{P_{n-1}} - \phi_{P_{n-2}}$, with $\phi_{P_0} = 1$ and $\phi_{P_1} = \lambda$. Deleting an edge from C_n yields $\phi_{C_n} = \phi_{P_n} - \phi_{P_{n-2}} - 2$.

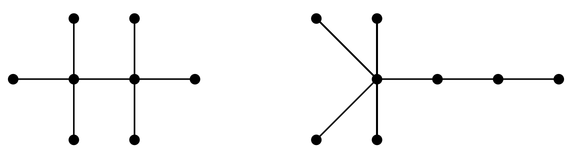
(•) *b) Without solving the recurrence, prove that $\{2\cos(2\pi j/n)\colon 0 \le j \le n-1\}$ are the eigenvalues of C_n.* This is a matter of designing the appropriate eigenvectors and checking the multiplication.

c) Eigenvalues of C_n^2. If G^2 is obtained from a k-regular graph G by making vertices at distance 2 adjacent, then $A(G^2) = A^2(G) + A(G) - kI$. If x is an eigenvector of $A(G)$ with associated eigenvalue λ, then $A(G^2)x = (\lambda^2+\lambda-k)x$, so x is an eigenvector of $A(G)$ with associated eigenvalue $\lambda^2+\lambda-k$.

8.6.7. *When G is a tree, the coefficient of λ^{n-2k} in the characteristic polynomial is $(-1)^k\mu_k(G)$, where $\mu_k(G)$ is the number of matchings of size k.* By

Corollary 8.6.6, the coefficient c_i of λ^{n-i} in the characteristic polynomial is $\sum(-1)^{k(H)}2^{s(H)}$, where the summation is over all i-vertex subgraphs for which every component is an edge or a cycle, $k(H)$ is the number of these components, and $s(H)$ is the number of cycles. In a tree T, there are no cycles, so $c_i = (-1)^{i/2}\mu_{i/2}(T)$.

Nonisomorphic "co-spectral" 8-vertex trees that both have characteristic polynomial $\lambda^8 - 7\lambda^6 + 9\lambda^4$. We seek two trees on eight vertices that have nine matchings of size 2 and no larger matchings (as trees, they automatically have seven matchings of size 1). The trees appear below. (Comment: As $n \to \infty$, almost no trees are uniquely determined by their spectra.)



8.6.8. *If T is a tree, then $\alpha(T)$ is the number of nonnegative eigenvalues of T.* Let T be an n-vertex tree. In any graph, the vertices outside a maximum independent set form a minimum vertex cover, so $\alpha(T) = n(T) - \beta(T)$, where $\beta(T)$ is the vertex cover number.

Since the characteristic polynomial has the form $\prod(\lambda - \lambda_i)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues, the degree of the last nonzero term is the multiplicity of 0 as an eigenvalue. By Exercise 8.6.7, the coefficient of $n-2k$ is nonzero if and only if T has a matching of size k. Hence the least degree of a nonzero term is $n - 2\alpha'(T)$. Since T is bipartite, this equals $n - 2\beta(T)$, which equals $\alpha(T) - \beta(T)$. Since $\alpha(T) + \beta(T) = n$, there remain $2\beta(T)$ nonzero eigenvalues, which are split equally between positive and negative values, since T is bipartite. We conclude that there are exactly $\alpha(T)$ nonnegative eigenvalues.

8.6.9. *The eigenvalues of a graph with n vertices and m edges are bounded by $\sqrt{2m(n-1)/n}$.* Applying the Cauchy-Schwarz Inequality to the vector of eigenvalues other than the maximum yields

$$\left(\sum_{i=2}^{m} \lambda_i\right)^2 \le (n-1)\left(\sum_{i=2}^{m} \lambda_i^2\right).$$

Using $\sum \lambda_i = 0$ on the left and $\sum \lambda_i^2 = 2e$ on the right converts this to $(-\lambda_1)^2 \le (n-1)(2e - \lambda_1^2)$, which is equivalent to $\lambda_1 \le \sqrt{2e(n-1)/n}$.

8.6.10. *The eigenvalues of the cartesian product of graphs G and H are the sums of eigenvalues of G and H.* Let $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ be the eigenvalues of G and H, with adjacency matrices A and B, respectively. The entry in row (i,j) and column (r,s) of C, the adjacency matrix of $A(G \square H)$, is $b_{j,s}$ if $i = r$ and $a_{i,r}$ if $j = s$; otherwise it is 0.

Let u and v be eigenvectors for eigenvalues λ and μ of G and H, respectively. Let w be the vector indexed by $[m]\times[n]$ that is defined by $w_{i,j} = u_i v_j$. In Cu, coordinate (i,j) is $\sum_r \sum_s c_{(i,j),(r,s)} w_{r,s}$. The terms in the sum are 0 except when $i = r$ or $j = s$. Note that $c_{(i,j),(i,j)} = 0$. We thus obtain

$$\begin{aligned}
(Cw)_{i,j} &= \sum_s c_{(i,j),(i,s)} u_i v_s + \sum_r c_{(i,j),(r,j)} u_r v_j - c_{(i,j),(i,j)} u_i v_j \\
&= u_i \sum_s b_{j,s} v_s + v_j \sum_r a_{i,r} u_r = u_i (Bv)_j + v_j (Au)_i \\
&= u_i \mu v_j + \lambda v_j \lambda u_i = (\mu + \lambda) u_i v_j = (\mu + \lambda) w_{i,j}
\end{aligned}$$

This computation shows that w is an eigenvector associated with eigenvalue $\mu + \lambda$. Furthermore, if u and u' are two linearly independent eigenvectors of A associated with λ, then the resulting w and w' are linearly independent eigenvectors of C. Thus the eigenvalues of C are given by the list of all $\lambda_i + \mu_j$ such that $1 \le i \le m$ and $1 \le j \le n$.

The eigenvalues of the k-dimensional hypercube Q_k range from k to $-k$, with $k - 2r$ being an eigenvalue with multiplicity $\binom{k}{r}$, for $0 \le r \le k$. The claim holds by inspection for $k = 1$, where $Q_k = K_2$. For $k > 1$, express Q_k as $Q_{k-1} \square K_2$. Since the eigenvalues of K_2 are 1 and -1, each with multiplicity 1, each eigenvalue μ for Q_{k-1} becomes eigenvalues $\mu + 1$ and $\mu - 1$ for Q_k. Thus the multiplicity of $k - 2r$ in the spectrum of Q_k is the sum of the multiplicities of $k - 2r - 1$ and $k - 2r + 1$ in the spectrum of Q_{k-1}. Using the induction hypothesis, the multiplicity is $\binom{k-1}{r} + \binom{k-1}{r-1}$, which by the binomial recurrence equals $\binom{k}{r}$.

8.6.11. (•) Compute the spectrum of the complete p-partite graph $K_{m,\ldots,m}$. (Hint: Use the expression $A(\overline{G}) = J - I - A(G)$ for the adjacency matrix of the complement.)

8.6.12. *If the characteristic polynomial of G is $x^8 - 24x^6 - 64x^5 - 48x^4$, then $G = K_{2,2,2,2}$.* The degree is $n(G)$, and the coefficient of $x^{n(G)-2}$ is $-e(G)$. Since $\binom{8}{2} = 28$, we obtain G by deleting four edges from K_8. Since the coefficient of $x^{n(G)-3}$ is -2 times the number of triangles (Corollary 8.6.6), our graph has 32 triangles. In K_n there are 56 triangles, and deleting an edge kills six triangles. If we kill 24 triangles by deleting four edges, then we must not kill a triangle twice, which means that the four deleted edges are pairwise disjoint. Now $K_{2,2,2,2}$ is the only graph satisfying all these requirements.

8.6.13. (!) Prove that G is bipartite if G is connected and $\lambda_{\max}(G) = -\lambda_{\min}(G)$.

8.6.14. *The squashed-cube dimension (Definition 8.4.12) of a graph G is at least the maximum of the number of positive eigenvalues and the number* *of negative eigenvalues of the matrix $R(G)$ whose whose (i,j)th entry is $d_G(v_i,v_j)$.* Note that $R(K_n)=J-I$, whose eigenvalues are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$. Hence the squashed cube dimension of K_n is at least $n-1$, and equality follows from the construction in Example 8.4.13.

To prove the eigenvalue bound, we encode the distances in a quadratic form. Let $x=(x_1,\dots,x_n)$, and let $h(x)=\sum_{i,j}d_G(v_i,v_j)x_ix_j=x^TR(G)x$. The combinatorial part of the argument recomputes this sum by accumulating contributions from coordinates in a squashed-cube embedding.

Consider an encoding f, with $f(v_i)=(f_1(v_i),\dots,f_N(v_i))$. Let V_m^α denote the set of indices i such that $f_m(v_i)=\alpha$. Coordinate m contributes 1 to $d(f(v_i),f(v_j))$ if and only if $i\in V_m^0$ and $j\in V_m^1$, or vice versa. For each such unit contribution, we have a contribution of $x_ix_j+x_jx_i$ to $h(x)$. Hence the grouping by coordinates yields $h(x)=2\sum_{m=1}^N\left(\sum_{i\in V_m^0}x_i\right)\left(\sum_{j\in V_m^1}x_j\right)$. We have rewritten the quadratic form as a sum of N products of linear combinations. Now Sylvester's Law of Inertia (Lemma 8.6.14) states that expressing a quadratic form as a sum of N products of linear combinations of the variables requires $N\ge r$, where r is the maximum of the number of negative and number of positive eigenvalues of $R(G)$.

8.6.15. (!) The *Laplacian matrix* Q of a graph G is $D-A$, where D is the diagonal matrix of degrees and A is the adjacency matrix. The *Laplacian spectrum* is the list of eigenvalues of Q.

a) Prove that the smallest eigenvalue of Q is 0.

b) Prove that if G is connected, then eigenvalue 0 has multiplicity 1.

c) Prove that if G is k-regular, then $k-\lambda$ is a Laplacian eigenvalue if and only if λ is an ordinary eigenvalue of G, with the same multiplicity.

8.6.16. Given that $\lambda_{\max}(M)+\lambda_{\min}(M)\le\lambda_{\max}(P)+\lambda_{\max}(R)$ for any real symmetric matrix M partitioned as $\left(\begin{smallmatrix}P & Q\\ Q^T & R\end{smallmatrix}\right)$ with P, R square:

a) If A is a real symmetric matrix partitioned into t^2 submatrices $A_{i,j}$ such that the diagonal submatrices A_{ii} are square, then

$$\lambda_{\max}(A)+(t-1)\lambda_{\min}(A)\le\textstyle\sum_{i=1}^m\lambda_{\max}(A_{i,i}).$$

Let $P=A_{1,1}$, and let R be the matrix obtained by deleting the first row and column of blocks. By the Interlacing Theorem, $\lambda_{\min}(A)\le\lambda_{\min}(R)\le\lambda_{\max}(R)\le\lambda_{\max}(R)$. The desired identity is trivial for $t=1$. For $t>1$, we apply induction. Using the given identity and then the induction hypothesis and the Interlacing Theorem,

$$\begin{aligned}\lambda_{\max}(A)+\lambda_{\min}(A)&\le\lambda_{\max}(A_{1,1})+\lambda_{\max}(R)\\&\le\lambda_{\max}(A_{1,1})+\textstyle\sum_{i=2}^t\lambda_{\max}(A_{i,i})-(t-2)\lambda_{\min}(R)\\&\le\lambda_{\max}(A_{1,1})+\textstyle\sum_{i=2}^t\lambda_{\max}(A_{i,i})-(t-2)\lambda_{\min}(A).\end{aligned}$$

b) $\chi(G)\ge 1+\lambda_{\max}(G)/(-\lambda_{\min}(G))$ for nontrivial G. Partition the vertices into the $\chi(G)$ color classes of an optimal coloring. With the vertices ordered by color classes, the diagonal submatrices of the adjacency matrix are identically 0, so their eigenvalues are all 0. If G is nontrivial, then the eigenvalues are not all 0, so $\lambda_{\min}(G)<0$, since the sum is 0. Now part (a) yields $\lambda_{\max}(G)+(\chi(G)-1)\lambda_{\min}(G)\le 0$, which becomes the desired inequality upon solving for $\chi(G)$.

c) $\lambda_1(G)+3\lambda_n(G)\le 0$ for planar graphs. Using the Four Color Theorem to set $\chi(G)\le 4$ in part (b) yields the claim.

8.6.17. *The number of spanning trees in $K_{m,m}$ is m^{2m-2}.* Since $K_{m,m}$ is m-regular with $2m$ vertices, Theorem 8.6.28 applies and yields $\tau(K_{m,m})=(2m)^{-1}\prod_{i=2}^{2m}(m-\lambda_i)$, where $m,\lambda_2,\dots,\lambda_{2m}$ are the eigenvalues in nonincreasing order. Since the spectrum is Spec $(K_{m,m})=\left(\begin{smallmatrix}m & 0 & -m\\ 1 & 2m-2 & 1\end{smallmatrix}\right)$ (Example 8.6.3), we obtain $\tau(K_{m,m})=(2m)^{-1}(m-0)^{2m-2}(2m)^1=m^{2m-2}$.

8.6.18. *If the columns of a matrix sum to **0**, then the cofactors obtained from deletion of a fixed row of A are all equal.* Let A be such a matrix. The cofactor $b_{i,j}$ is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting row i and column j of A. The definition of the determinant by expansion along rows of A yields $A(\mathrm{Adj}\,A)=(\det A)I$ for all A.

If rank $(A)<n-1$, than all cofactors are 0. Otherwise, rank $(A)=n-1$ and $\det A=0$. Now $A\mathrm{Adj}\,A=0$ implies that every column of Adj A is in the null-space of A. Since every row-sum of A is 0, we have $(1,\dots,1)^T$ in the null-space. Since rank $A=n-1$, every vector in the null-space is a multiple of this. Hence the columns of Adj A are constant-valued, meaning that the cofactors in each row of A are the same.

8.6.19. *Binet-Cauchy Formula:* $\det AB=\sum_{S\in\binom{[n]}{m}}\det A_S\det B_S$, *where A and B are $n\times m$ and $m\times n$ matrices, A_S consists of the columns of A indexed by S, and B_S consists of the rows of B indexed by S.*

Consider the matrix equation $DE=F$ below. Since each of these matrices has $n+m$ rows and columns, the usual product rule applies.

$$\begin{pmatrix}I_m & 0\\ A & I_n\end{pmatrix}\begin{pmatrix}-I_m & B\\ A & 0\end{pmatrix}=\begin{pmatrix}-I_m & B\\ 0 & AB\end{pmatrix}$$

Since D is lower triangular with diagonal 1, $\det D=1$. Expanding the determinant of F along the first m columns yields $\det F=(-1)^m\det AB$. Thus it suffices to prove that $\det E=(-1)^m\sum_{S\in\binom{[n]}{m}}\det A_S\det B_S$.

We use the permutation definition of $\det E$. Each nonzero product of $m+n$ elements must use an element from each row of A and an element from each column of B. The remaining $m-n$ elements lie on the diagonal of the upper left block. Thus the columns of the n elements from A have the same indices as the rows of the n elements from B. Furthermore, each term in the expansion of $\det A_S$ is multiplied by each term in the expansion of $\det B_S$ to obtain contributions to $\det E$.

It remains only to consider the signs of the contributions. Two obtain the positions on the main diagonal of A_S and the main diagonal of B_S, we apply n row interchanges from the positions on the main diagonal of E. For other terms in $\det A_S$, the sign corresponds to the parity of the permutation of rows within A_S, and similarly for B_S. Hence we obtain $\det A_S \det B_S$ times $(-1)^n$ for the inital permutation time $(-1)^{m-n}$ for the elements on the diagonal of $-I_m$.

8.6.20. *The incidence matrix of a simple graph G is totally unimodular if and only if G is bipartite.* (The incidence matrix has two $+1$s in each column; a matrix is *totally unimodular* if every square submatrix has determinant in $\{0,1,-1\}$. Let $S=\{0,1,-1\}$.)

Proof 1 *Sufficiency*. Given that G is bipartite, we prove by induction on k that the determinant of every k-by-k submatrix of the incidence matrix $M(G)$ is in S. This is certainly true for $k=1$, since the entries in $M(G)$ are 0 or 1. Suppose $k>1$, and let A be a k-by-k submatrix. Every column of $M(G)$ has two nonzero entries, so every column of A has at most two. If a column of A is 0, then $\det A=0$. If a column of A has one nonzero entry, which must be 1, then expanding the determinant down that column expresses $\det A$ and ±1 times the determinant of a $(k-1)$-by-$(k-1)$ matrix, which by the induction hypothesis is in S. Hence $\det A\in S$.

Finally, suppose that every column of A has two 1s. Each row of A corresponds to a vertex; weight the rows by $+1$ if the corresponding vertices belong to one partite set, -1 if they belong to the other. Since G is bipartite, every edge contains a vertex of each partite set, and hence with this weighting each column of A sums to 0, and $\det A=0$.

Necessity. If G is not bipartite, then G contains an odd cycle C of length $2k+1$. Consider the rows and columns of $M(G)$ corresponding to the vertices and edges of C. Permuting the rows and columns of this submatrix A may change the sign of $\det A$ but not its magnitude. With $v_1,\ldots,v_{2k+1}$ being the names of the vertices on the cycle in order, permute the rows of A into this order, and permute the columns of A into the order $v_1v_2, v_2v_3,\ldots,v_{2k+1}v_1$. Now A has 1s in positions (i,i) for $1\le i\le 2k+1$, positions $(i+1,i)$ for $1\le i\le 2k$, and position $(1,2k+1)$, with the other positions 0. If we expand the determinant of this matrix along the first row, we have only two nonzero terms. One is a subdeterminant with 1s only on the main diagonal and the first subdiagonal, and the other is a subdeterminant with 1s only on the main diagonal and the first superdiagonal. Because these come from expansion in columns 1 and $2k+1$, which are both odd, they have the same sign. Since each is a triangular matrix with 1s on the diagonal, we have $|\det A|=2$, and $M(G)$ is not totally unimodular. The expansion is illustrated below for $k=2$.

$$\begin{vmatrix}1&0&0&0&1\\1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\end{vmatrix}=\begin{vmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{vmatrix}+\begin{vmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{vmatrix}$$

Proof 2 (induction on $e(G)$). If $e(G)=0$, then the determinant of the empty matrix is 0, and G has no odd cycle. If $e(G)>1$, then consider an arbitrary edge xy of G. The induction hypothesis states that $G-xy$ is bipartite if and only if $M(G-xy)$ is totally unimodular. If $M(G)$ is totally unimodular, then the submatrix $M(G-xy)$ is totally unimodular; hence $G-xy$ is bipartite whenever $xy\in E(G)$. Hence G is bipartite unless every edge of G belongs to every odd cycle of G. This happens only when G itself is an odd cycle; in this case $|\det M(G)|=2$, as discussed above.

Conversely, if G is bipartite, then every $G-xy$ is bipartite, and the induction hypothesis guarantees every $M(G-xy)$ is totally unimodular. Hence $M(G)$ is totally unimodular unless some submatrix A using all the columns has determinant outside $\{-1,0,+1\}$. Since $M(G-xy)$ is totally unimodular, expansion on the column indexed by xy forces both 1s in this column to appear in A. Since xy is arbitrary, every 1 in $M(G)$ appears in A. Now the fact that G is bipartite allows us to weight the rows corresponding to one partite set with $+1$ and those corresponding to the other partite set with -1 to obtain a linear dependence among the rows, yielding $\det A=0$.

8.6.21. *If G is an (n,k,c)-magnifier with vertices $v_1,\ldots,v_n$, and H is the X,Y-bigraph with $X=\{x_1,\ldots,x_n\}$ and $Y=\{y_1,\ldots,y_n\}$ such that $x_iy_j\in E(H)$ if and only if $i=j$ or $v_iv_j\in E(G)$, then H is an $(n,k+1,c)$-expander.* We verify the properties of an $(n,k+1,c)$-expander. The construction yields $d_H(x_i)=d_H(y_i)=d_G(v_i)$, so $\Delta(G)\le k$ yields $\Delta(H)\le k+1$. For $S\subseteq X$ with $|S|\le n/2$, let $S'=\{v_i\in V(G)\colon x_i\in S\}$. We have

$$|N_H(S)|=|S|+\left|N_G(S')-S'\right|\ge(1+c)\,|S|>(1+c(1-|S|/n)\,|S|.$$

8.6.22. *Existence of expanders of linear size.* An (n,α,β,d)-**expander** is an A,B-bigraph G with $|A|=|B|=n$, $\Delta(G)\le d$, and $|N(S)|\ge\beta\,|S|$ whenever $|S|\le\alpha n$.

a) If X is the size of the union of d k-subsets of $[n]$ chosen at random, then $P(X\le l)\le\binom{n}{l}(l/n)^{kd}$. If $X\le l$, then all the k-sets are confined to some

l-set. By multiplying the probability of this occurrence for a particular l by $\binom{n}{l}$, we obtain a loose upper bound (the events for distinct l-sets are not disjoint). For a particular l-set, we bound the probability it contains any selected k-set in the sequence: $\binom{l}{k}/\binom{n}{k} = \prod_{i=0}^{k-1} \frac{l-i}{n-i} \le (l/n)^k$. We need this event to happen d times.

b) If $\alpha\beta < 1$, then there is a constant d such that, for sufficiently large n, an (n, α, β, d)-expander exists. We generate an A, B-bigraph G by choosing d random complete matchings, discarding extra copies of edges. Hence $\Delta(G) \le d$. We claim that G fails to be an expander with probability less than 1. Let S be a *violated set* if $|N(S)| \le \beta\,|S|$. Let E be the event that a violated set exists; we bound $P(E)$ by a quantity that is less than 1 when n is sufficiently large. The d random matchings provide d random k-sets as neighbors of S when $|S| = k$. By (a), when $|S| < \alpha n$ we have

$$\mathrm{P}(E) < \sum_{k=1}^{\alpha n} \binom{n}{k}\binom{n}{\beta k}\left(\frac{\beta k}{n}\right)^{kd} < \sum_{k=1}^{\alpha n} \left(\frac{ne}{k}\right)^{k}\left(\frac{ne}{\beta k}\right)^{\beta k}\left(\frac{\beta k}{n}\right)^{kd}$$
$$= \sum_{k=1}^{\alpha n}\left[e^{1+\beta}\beta\left(\frac{\beta k}{n}\right)^{d-\beta-1}\right]^k < \sum_{k\ge 1}\left[e^{1+\beta}\beta(\alpha\beta)^{d-\beta-1}\right]^k .$$

If $\alpha\beta < 1$, then we can choose d to make the constant ratio in the geometric series as small as desired. We choose d so $e^{1+\beta}\beta(\alpha\beta)^{d-\beta-1} < \frac{1}{2}$.

c) Conclude the existence of k such that n, k, c-expanders exist for all sufficiently large n. We prove this when $c < 1$. Choose $\alpha = 1/2$ and $\beta = 1 + c$, so $\alpha\beta < 1$. By part (b), there is a constant d such that for sufficiently large n there exists an X, Y-bigraph with $|X| = |Y| = n$ that is an (n, α, β, d)-expander. For $S \subseteq X$ with $|S| \le n/2$, we have $|N(S)| \ge \beta\,|S| = (1 + c)\,|S| > (1 + c(1 - |S|/n))\,|S|$.

8.6.23. *Triangle-free graphs in which every two nonadjacent vertices have exactly two common neighbors.*

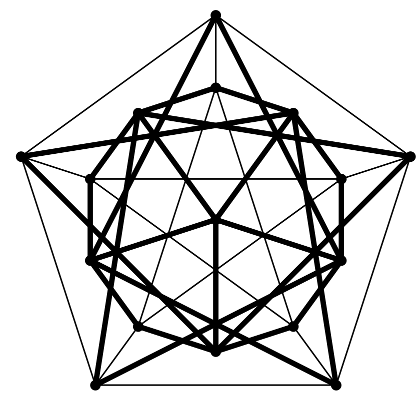
If k is the degree of a vertex in an n-vertex graph G of this sort, then $n = 1 + \binom{k+1}{2}$. For every pair of neighbors of x, there is exactly one nonneighbor of x that they have as a common neighbor. Conversely, every nonneighbor of x has exactly one pair of neighbors of x in its neighborhood, because these are its common neighbors with x. This establishes a bijection between the pairs in $N(x)$ and the nonneighbors of x. Counting x, $N(x)$, and $\overline{N}(x)$, we have $n(G) = 1 + k + \binom{k}{2} = 1 + \binom{k+1}{2}$. Since this argument holds for every $x \in V(G)$, we conclude that G is k-regular. (This is Exercise 1.3.33.)

G is strongly regular. By definition, $\lambda = 0$ (triangle-free) and $\mu = 2$, so G is strongly regular with parameters $(k, 0, 2)$ and $1 + \binom{k+1}{2}$ vertices.

The parameter k must be 1 more than the perfect square of an integer m that is not a multiple of 4. Setting $n - 1 = \binom{k+1}{2}$ and $\lambda = 0$ and $\mu = 2$

in the integrality conditions for strongly regular graphs (Theorem 8.6.33) shows that the two numbers $\frac{1}{2}\left(\binom{k+1}{2} \pm \frac{k(k+1)-2k}{\sqrt{4+4(k-2)}}\right)$ must be integers. The formulas simplify to $\frac{k}{4}(k+1\pm\sqrt{k-1})$. Since these are integers, $k-1$ must be a perfect square. With $m = \sqrt{k-1}$, the numbers $(m^2+1)(m^2+2\pm m)$ must be multiples of 4. This is impossible if m is a multiple of 4.

Examples. The values of k satisfying the necessary conditions are 1, 2, 5, 10, 26, etc. For $k = 1$, we have the degenerate example K_2. For $k = 2$, the graph is C_4. For $k = 5$, the 16-vertex graph is known as the Clebsch graph shown below; deleting any closed neighborhood yields the Petersen graph. For $k = 10$, a realization is known using combinatorial designs; it is called the Gewirtz graph.



8.6.24. *The Petersen graph; spectrum and application to decomposition of K_{10}.* The Petersen graph is regular of degree 3; any pair of adjacent vertices have no common neighbor, while every pair of non-adjacent vertices have one common neighbor. Hence the Petersen graph is strongly regular with $k = 3$, $\lambda = 0$, $\mu = 1$. It eigenvalues are therefore $3, r, s$ where $r+s = \lambda-\mu = -1$ and $rs = -(k-\mu) = -1$, so $r = 1$ and $s = -2$. The multiplicities a and b of r and s satisfy $k + ar + bs = 0$ and $1 + a + b = n$, so $(a, b) = (5, 4)$, and the spectrum is $\binom{3,1,-2}{1,5,\ 4}$.

Without using strong regularity, the following ad hoc argument also yields the spectrum. Consider the $3 + 3^2$ walks of length 1 or 2 that begin at a specified vertex v. Each other vertex is the other end of one of these, and v is the other end of three of them. Hence $P^2 + P = 2I + J$, where J is the all-1 matrix. Factoring $P^2 + P - 2I$ and multiplying $P - 3I$ by both this and J yields $(P-3I)(P-I)(P+2I) = 0$. Hence $(\lambda-3)(\lambda-1)(\lambda+2)$ is the minimum polynomial of P, and P has eigenvalues $3, 1, -2$. To determine the multiplicities a, b, c, use the fact that for every $j \ge 0$, $a \cdot 3^j + b \cdot 1^j + c \cdot (-2)^j = \text{trace} P^j$. For $j = 0$, $j = 1$, and $j = 2$, traceP^j is 10, 0, and 30, respectively, and the resulting three equations in three unknowns yield $(a, b, c) = (1, 5, 4)$.

If K_{10} can be factored into three disjoint copies of the Peterson graph, then we can write $J - I = P_1 + P_2 + P_3$, where P_1, P_2, P_3 are adjacency matrices for the Petersen graph, under various numberings of the vertices by $1, \ldots, 10$. The vector $\overline{1}$ of all ones is an eigenvector for each P_i, and for each P_i there is a five-dimensional space of eigenvectors with eigenvalue 1 that is orthogonal to $\overline{1}$. Since the orthogonal complement of $\overline{1}$ has nine dimensions, P_1 and P_2 have a common eigenvector wv with eigenvalue 1. Being orthogonal to $\overline{1}$, its coordinates sum to 0. Letting both sides of the decomposition operate on it yields $-wv = Jwv - Iwv = \sum P_i wv = wv + wv + P_3 wv$. However, this says that wv is an eigenvector of P_3 with eigenvalue -3, which is impossible. (This result is a special case of a theorem of J. Bosák that no complete graph with fewer than 12 vertices has a decomposition into three spanning subgraphs of diameter 2.)

8.6.25. *If $F = G \square H$, where G and H are simple graphs of order at least 2, and every two nonadjacent vertices in F have exactly two common neighbors, then G and H are complete graphs.* Given distinct vertices $u, v \in V(G)$ and $x, y \in V(H)$, consider the vertices (u, x) and (v, y) in F. By the definition of the cartesian product, these vertices are nonadjacent. Hence they have two common neighbors. The only possible common neighbors are (u, y) and (v, x). The resulting 4-cycle implies that $uv \in E(G)$ and $xy \in E(H)$. Since these vertices were chosen arbitrarily, G and H are complete graphs.

8.6.26. The *subconstituents* of a graph are the induced subgraphs of the form $G[U]$, where $v \in V(G)$ and $U = N(v)$ or $U = \overline{N[v]}$. Vince [1989] defined G to be *superregular* if G has no vertices or if G is regular and every subconstituent of G is superregular. Let $\mathbf{S}$ be the class consisting of $\{aK_b\colon a, b \ge 0\}$ (disjoint unions of isomorphic cliques), $\{K_m \square K_m\colon m \ge 0\}$, C_5, and the complements of these graphs.

a) Every graph in $\mathbf{S}$ is superregular and every disconnected superregular graph is in $\mathbf{S}$. Each $G \in \mathbf{S}$ is regular and vertex-transitive; so it suffices to consider any $x \in V(G)$. By induction on $a+b$, we have superregularity for $G = aK_b$, since $G[N(x)] = K_{b-1}$ and $G[\overline{N}(x)] = (a-1)K_b$. For $G = K_m^2$, we also apply induction, since $G[N(x)] = 2K_{m-1}$ and $G[\overline{N}(x)] = K_{m-1}^2$. Finally, for $G = C_5$, $G[N(x)] = 2K_1$ and $G[\overline{N}(x)] = K_2$.

Now suppose that G is superregular and disconnected. If some component of G is not a complete graph, then we may choose vertices x, y, z such that y has distance two from x and z belongs to another component. This implies $d_{G[\overline{N}(x)]}(y) < k = d_{G[\overline{N}(x)]}(z)$, which contradicts the regularity of $G[\overline{N}(x)]$. If every component of G is a clique, then regularity of G requires equal sizes. (Comment: In fact, every superregular graph is in $\mathbf{S}$, but the complete inductive proof of this requires several pages (Maddox [1996], West [1996])

b) Every superregular graph is strongly regular. If x and y are nonadjacent, then t-regularity of $G[\overline{N}(x)]$ implies that x and y have $k-t$ common neighbors. We have noted that adjacent pairs have s common neighbors. Hence G is strongly regular, with parameters $\lambda = s$ and $\mu = k - t$.

8.6.27. *Automorphisms and eigenvalues.*

a) A permutation σ is an automorphism of G if and only if the permutation matrix corresponding to σ commutes with the adjacency matrix of G; that is, $PA = AP$. Say that P is defined by letting position (j, i) be 1 if $\sigma(i) = j$. That is, $Pe_i = e_j$, where e_k is the kth canonical basis vector. Multiplication by P permutes rows, moving row i to become row j if $P_{j,i} = 1$. The inverse of a permutation matrix moves the rows back again. In order to move row j to become row i, we need $P^{-1}_{i,j} = 1$. That is, we have argued that $P^T P = I$, so $P^{-1} = P^T$.

Multiplication by a permutation matrix Q on the right permutes columns, moving column i to column j if $Q_{i,j} = 1$. To accomplish the renaming by σ on the adjacency matrix, we want to move row i to row j and column i to column j whenever $\sigma(i) = j$. This is accomplished by multiplying by P on the left and by P^T on the right. Thus $PAP^T = A$. Since we have argued that $P^T = P^{-1}$, we have $PA = AP$.

b) If x is an eigenvector of G for an eigenvalue λ of multiplicity 1, and P is the permutation matrix for an automorphism of G, then $Px = \pm x$. Part (a) yields $APx = PAx = P\lambda x = \lambda Px$. Thus Px is also an eigenvector for A with eigenvalue λ. Since λ has multiplicity 1, Px is a multiple of x. Since P merely permutes the entries in x, it cannot change the largest magnitude that appears in x. Hence the new multiple of x must be $\pm x$.

c) When every eigenvalue of G has multiplicity 1, every automorphism of G is an involution (repeating it yields the identity). Part (b) yields $P^2 x = x$ whenever x is an eigenvalue of multiplicity 1. If this is true for every eigenvalue, then when we express any vector w as a linear combination of eigenvalues, we obtain $P^2 w = w$. If $P^2 w = w$ for every vector w, then P^2 is the identity, and hence σ is an involution.

8.6.28. *Every graph has an odd dominating set, meaning a set whose intersection with every closed neighborhood has odd size.* The phrasing in the text, based on Problem 10197 in the *American Mathematical Monthly* (1992), is that lightswitch s_i changes the status of light l_j if and only if s_j changes l_i. Let G be the n-vertex graph with vertices v_i and v_j adjacent if and only switches s_i and s_j affect lights l_j and l_i.

Let S be the set of switches flipped an odd number of times; the flips of other vertices have no effect. Since also s_i changes l_i and l_i begins off, l_i is on at the end if and only if its closed neighborhood in G has an odd number of vertices in S. Thus, the problem is equivalent to finding an odd dominating set S. This form has a combinatorial proof by Gallai (see Lovász [1979], Exercise 5.17).

The proof using linear algebra is much shorter. Let B be the augmented adjacency matrix $A(G) + I$ (add 1 to each diagonal entry). If x is the incidence vector of S in a binary vector space, then Bx is the incidence vector of the set of lights on after flipping the switches at S (since arithmetic is modulo 2). The problem is to show that $1 \in T$, where $T = \{Bx \in \mathbb{Z}_2^n\}$. We prove the more general statement that if B is a symmetric binary matrix, with vector u along the diagonal, then $u \in T$. (see Problem 798, *Nieuw Archief voor Wiskunde* (4) 9 (1991), 117-118) The solution given in the *Monthly* (1993, p. 806) is as follows.

We show that every vector orthogonal to T is also orthogonal to u. This implies that u is in the orthogonal complement of the orthogonal complement of T and hence is in T itself. Thus we want to show that if $\sum_{i=1}^n x_i B_{i,j} = 0$ for all j, then $\sum_{i=1}^n x_i u_i = 0$, where all computation is modulo 2.

Multiplying the vector 0 by x yields $\sum_{j=1}^n \sum_{i=1}^n x_i B_{i,j} x_j = 0$. By the symmetry of B, the off-diagonal entries contribute nothing, and we obtain $\sum_{i=1}^n x_i^2 B_{i,i} = 0$. Since $x_i^2 = x_i$ in binary, this reduces to $0 = \sum_{i=1}^n x_i B_{i,i} = \sum_{i=1}^n x_i u_i$, which completes the proof.